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Least-energy Solutions to a Non-autonomous Semilinear Problem with Small Diffusion Coefficient *

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Abstract

Least-energy solutions of a non-autonomous semilinear problem with a small diffusion coefficient are studied in this paper. We prove that the solutions will develop single peaks as the diffusion coefficient approaches 0. The location of the peaks is also considered in this paper. It turns out that the location of the peaks is determined by the non-autonomous term of the equation and the type of the boundary condition. Our results are based on fine estimates of the energies of the solutions and some non-existence results for semilinear equations on half spaces with Dirichlet boundary condition and some decay conditions at infinity.

1 Introduction

This work is devoted to the least-energy solutions of a non-autonomous semi-linear problem with a small diffusion coefficient. Considering

$$\begin{cases} \epsilon^2 \Delta u - u + K(x) u^p = 0 \text{ in } \Omega \\ B(u) = 0 \end{cases} \tag{1.1}$$

where

- Ω is a smooth bounded domain in R^N,
- $K(x) > 0$ in $\overline{\Omega}$ is a C^α function,
- $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $p > 1$ if $N = 2$,
- $B(u)$ is the boundary operator which is either Dirichlet, i.e. $B(u) = u|_{\partial\Omega}$, or Neumann, i.e. $B(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$, and

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- ϵ is a small parameter,

we would like to understand the behavior of positive least-energy solutions to (1.1) as ϵ tends to 0.

The problem is motivated by some pattern formation problems in biology. Keller and Siegel in [6] proposed a model to describe the chemotactic aggregation stage of cellular slime molds. Let $u_1(x,t)$ be the population of amoebae at place x and time t and let $u_2(x,t)$ be the concentration of the chemical. Then a simplified Keller-Siegel system may be written as

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \Delta u_1 - \chi \nabla (u_1 \nabla \phi(u_2)) \text{ in } \Omega \times [0,T) \\[2ex] \dfrac{\partial u_2}{\partial t} = D_2 \Delta u_2 + k(x, u_1, u_2) \text{ in } \Omega \times [0,T) \\[2ex] \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,T) \\[2ex] u_1(x,0) = u_1^{(0)}(x) > 0, \ u_2(x,0) = u_2^{(0)}(x) > 0 \text{ on } \overline{\Omega}. \end{cases} \tag{1.2}$$

We refer to [6] for more information about this model. If we take $\phi(u_2) = \log u_2$ and $k(x, u_1, u_2) = -au_2 + b(x)u_1$ with $b(x) > 0$ on $\overline{\Omega}$, and consider steady states of the above system, then the system for steady states is reduced to (1.1) with the Neumann boundary condition. We refer to C. Lin, W.-M. Ni and I. Takagi [8] for the details of the derivation. Numerical experiments indicate that (1.2) possesses stable steady states with spiky patterns when D_2 is small. On the other hand, A. Gierer and H. Meinhardt discussed a activator-inhibitor problem in [4], yielding the so-called Gierer-Meinhardt system. Numerical experiments there also indicate point-condensation phenomena. Under certain conditions the Gierer-Meinhardt system can be reduced to (1.1) too. We again refer to [8] for the derivation. In this paper we shall show that the least-energy solutions of (1.1) with either the Dirichlet boundary condition or the Neumann boundary condition develop spiky pattern of single peak as ϵ approaches 0.

The autonomous case of (1.1), i.e. $K(x) = 1$, with the Neumann boundary condition has been studied by C. Lin, W.-M. Ni and I. Takagi in a series of papers [8], [10] and [11]. They proved that the least-energy solutions possesses a single peak on the boundary of Ω as ϵ tends to 0. The problem of locating these peaks is finally settled by W.-M. Ni and I. Takagi in [11] where they proved that the peaks will approach a point on the boundary of Ω which assumes the maximum of the mean curvature of $\partial\Omega$. W.-M. Ni, X. Pan and I. Takagi also considered the critical case, i.e. $p = \frac{N+2}{N-2}$, in [9] and [12] where they showed that the least-energy solutions will blow up at the boundary as ϵ tends to 0.

Here we shall focus on the non-autonomous term $K(x)$ and see how it will affect the shape of the least-energy solutions and the location of the peaks. We shall see that when $K(x)$ is nontrivial the location of peaks is indeed dominated by $K(x)$. Therefore in most situations the effect of $K(x)$ overrides the effect of the geometry of Ω.

Let H be the Hilbert space $W_0^{1,2}(\Omega)$ if $B(u) = u|_{\partial\Omega}$ takes the Dirichlet boundary condition, or $W^{1,2}(\Omega)$ if $B(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ takes the Neumann boundary condition. Define energy

$$J_\epsilon :\ H \to R$$

by

$$J_\epsilon(v) = \frac{1}{2}\int_\Omega [\epsilon^2|\nabla v|^2 + v^2]dx - \frac{1}{p+1}\int_\Omega K(x)v_+^{p+1}dx \tag{1.3}$$

where $v_+(x) = \max(v(x), 0)$. Choose $e \in H$, $e \geq 0$, $e \neq 0$, such that $J_\epsilon(e) = 0$. Then the well known mountain-pass lemma asserts that there is a nontrivial positive critical point, i.e. a solution of (1.1), u_ϵ of J_ϵ with positive critical value c_ϵ such that

$$c_\epsilon := \inf_{h\in\Gamma}\max_{t\in[0,1]} J_\epsilon(h(t)) = J_\epsilon(u_\epsilon) \tag{1.4}$$

where Γ is the class of all paths $h(t)$ in H connecting 0 and e, i.e. $h(0) = 0$ and $h(1) = e$. We refer to [1] for the mountain-pass lemma. It turns out in Proposition 2.1 that u_ϵ does not depend on the choice of e and u_ϵ achieves the least energy among all nontrivial critical points of J_ϵ, so we call these mountain-pass solutions least-energy solutions.

To investigate the asymptotic behavior of u_ϵ, we start with the energy of u_ϵ. Once we obtain the asymptotic behavior of the energy of u_ϵ in Proposition 3.1 and 3.2 with the aid a so-called ground state profile (see (2.3)), we shall prove for the Dirichlet problem

Theorem 1.1 *Letting u_ϵ be a least-energy solution of (1.1) with the Dirichlet boundary condition, we have*

1. *There exist positive constants C_1 and C_2 independent of ϵ such that*

$$C_1 \leq \|u_\epsilon\|_{L^\infty(\Omega)} \leq C_2.$$

2. *For ϵ small enough u_ϵ has only one local maximum point P_ϵ with*

$$\lim_{\epsilon\to 0}\epsilon^{-1}\mathrm{dist}(P_\epsilon, \partial\Omega) = \infty.$$

3. *If P is a limit point of $\{P_\epsilon\}$ as $\epsilon \to 0$, then*

$$K(P) = \max_{x\in\overline{\Omega}} K(x).$$

The story for the Neumann problem seems more complicated. Depending on

$$\max_{x\in\overline{\Omega}} K(x) > 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x)$$

or

$$\max_{x\in\overline{\Omega}} K(x) < 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

we have

Theorem 1.2 *Let u_ϵ be a least-energy solution of (1.1) with the Neumann boundary condition. Assuming*

$$\max_{x\in\overline{\Omega}} K(x) > 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

we have the following:

1. *There exist positive constants C_1 and C_2 independent of ϵ such that*

$$C_1 \le \|u_\epsilon\|_{L^\infty(\Omega)} \le C_2.$$

2. *For ϵ small enough u_ϵ possesses only one local maximum point P_ϵ which stays away from the boundary of Ω as $\epsilon \to 0$.*
3. *Every limit point of $\{P_\epsilon\}$ as $\epsilon \to 0$ must be a maximum point of $K(x)$ in the interior of Ω.*

Theorem 1.3 *Let u_ϵ be a least-energy solution of (1.1) with the Neumann boundary condition. Assuming*

$$\max_{x\in\overline{\Omega}} K(x) < 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

we have the following:

1. *There exist positive constants C_1 and C_2 independent of ϵ such that*

$$C_1 \le \|u_\epsilon\|_{L^\infty(\Omega)} \le C_2.$$

2. *For ϵ small enough u_ϵ possesses only one local maximum point on the boundary of Ω,*
3. *Every limit point of $\{P_\epsilon\}$ as $\epsilon \to 0$ must be a maximum point of $K(x)$ restricted on $\partial\Omega$.*

We shall also see the shape of u_ϵ in the proofs of these theorems. I would like to mention that in the works [13] [14] of J. Wei and myself spiky patterns have also appeared in some different problems with the parameters in the exponents.

This article is organized as follows: After some preliminaries in section 2, we shall describe the energy of u_ϵ in section 3. Then we shall prove Theorem 1.1 in section 4 and Theorem 1.2 and 1.3 in section 5.

2 Preliminaries

We first give an alternative description of the least energy solutions to (1.1) which implies that the mountain-pass solutions have the least energy among all nontrivial critical points of J_ϵ. The proof of the equivalence is identical with the proof of Lemma 3.1 in [10]. We include it here for completeness.

Lemma 2.1 *Letting u_ϵ be a mountain-pass solution of (1.1) with either the Dirichlet boundary condition or the Neumann boundary condition, then we have that $J_\epsilon(u_\epsilon)$ does not depend on the choice of e where $e \geq 0$ with $e \not\equiv 0$ and $J_\epsilon(e) = 0$ is the base point in the mountain-pass lemma. Furthermore, $J_\epsilon(u_\epsilon)$ is the least positive critical value of J_ϵ and characterized by*

$$J_\epsilon(u_\epsilon) = \inf\{M[v] : v \in H, v \not\equiv 0 \text{ and } v \geq 0 \text{ in } \Omega\} \tag{2.1}$$

where

$$M[v] = \sup_{t \geq 0} J_\epsilon(tv) = \max_{t \geq 0} J_\epsilon(tv).$$

Proof. Let $v \in H$ be non-negative with $|\{x : \ v(x) > 0\}| > 0$. Putting

$$h_\epsilon(t) := J_\epsilon(tv)$$

for $t > 0$. We note that $h_\epsilon(t)$ has a unique positive critical point.

Now fix a non-negative function $e \not\equiv 0$ in H with $J_\epsilon(e) = 0$ and apply the mountain-pass lemma to obtain the critical value c_ϵ. Let u_ϵ be a critical point of J_ϵ with $J_\epsilon(u_\epsilon) = c_\epsilon$. Since $u_\epsilon > 0$ and $J'_\epsilon(u_\epsilon) = 0$, it is clear that $M[u_\epsilon] = c_\epsilon$ and hence

$$c_\epsilon \geq \inf\{M[v] : v \in H, v \not\equiv 0 \ \textit{and} \ v \geq 0 \ \textit{in} \ \Omega\}. \tag{2.2}$$

Suppose that the strict inequality holds in (2.2). Then there is a non-negative function $v_* \not\equiv 0$in H such that

$$M[v_*] < c_\epsilon.$$

¿From the above observation we know that there is $T_* > 0$ such that $e_* := T_* v_*$ satisfies $J_\epsilon(e_*) = 0$. Consider the set $V^+ := \{\lambda e + \mu e_* : \ \lambda, \ \mu \geq 0\}$ and the two dimensional subspace V of H spanned by e and e_*. Let S be a circle with radius R so large that for $R > \max(\|e\|, \|e_*\|)$ and $J_\epsilon \leq 0$ on $S \cap V^+$. Let γ be the path consisting of the line segment with endpoints 0 and $Re_*/\|e_*\|$, the circular arc $S \cap V^+$ and the line segment with endpoints $Re/\|e\|$ and e. Clearly γ belongs to Γ defined in (1.4). It is also easy to see that, along γ, J_ϵ is positive only on the line segment joining 0 and e_*. Hence by the definition of $M[v_*]$ one finds that

$$\max_{v \in \gamma} J_\epsilon(v) = M[v_*] < c_\epsilon$$

which is inconsistent with (1.4). Thus the inequality in (2.2) becomes equality. Since any nontrivial critical point of J_ϵ must be positive by the Hopf lemma, the characterization (2.1) shows that c_ϵ is the least positive critical value of J_ϵ and the proof is now complete. □

It will be shown later that for most $K(x)$ the unique solution w to the following ground state equation

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } R^N \\ u > 0,\ \lim_{|x|\to\infty} u(x) = 0 \text{ and } \nabla u(0) = 0 \end{cases} \tag{2.3}$$

serves as an asymptotic profile for u_ϵ. We collect some well-known facts about w. The proofs of these facts can be found in M. Kwong [7].

Proposition 2.2 *1. (2.3) has a unique solution in $W^{1,2}(R^N)$.*

2. w is spherical symmetric: $w(z) = w(r)$ with $r = |z|$ and $\frac{dw}{dr} < 0$ for $r > 0$.

3. w and its derivatives decay exponentially at infinity, i.e. there exit positive constants C and μ such that $|D^\alpha w| \le Ce^{-\mu|z|}$ for all $z \in R^N$ with $|\alpha| \le 1$.

We also adapt a non-existence result of M. Esteban, and P. Lions [2] for some unbounded domains with a sketch of their proof for completeness.

Proposition 2.3 *Let Ω be a unbounded smooth domain in R^N so that there exits $X \in R^N, |X| = 1$, such that $(\nu(x), X) \ge 0$ and $(\nu(x), X) \not\equiv 0$ where $\nu(x)$ is the outward unit normal vector at $x \in \partial\Omega$. (Notice: Half space $(R^N)^+$ satisfies the condition about Ω). Suppose u satisfies*

$$\Delta u + f(u) = 0$$

in Ω, $u \in C^2(\overline{\Omega})$, $u = 0$ on $\partial\Omega$ where f is a locally Lipschitz function on R^N with $f(0) = 0$. If, in addition, $\nabla u \in L^2(\Omega), F(u) \in L^1(\Omega)$ where $F(t) = \int_0^t f(s)ds$, then we have necessarily $u \equiv 0$ in Ω.

Proof. Multiplying the equation by $\frac{\partial u}{\partial x_i}$ and integrating it by parts over $\Omega \cap B_R$ where B_R is the ball centered at origin with radius R, we obtain

$$\begin{aligned} 0 &= \int_{\Omega\cap B_R} [\Delta u + f(u)] \frac{\partial u}{\partial x_i} \\ &= \int_{\Omega\cap B_R} [\frac{\partial F(u)}{\partial x_i} - (\nabla u, \nabla \frac{\partial u}{\partial x_i})] + \int_{\partial(\Omega\cap B_R)} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial \nu} \end{aligned}$$

where ν denotes the outer normal of the domain.

Let $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_N(x))$. Integrating by parts and $\nabla u = \frac{\partial u}{\partial \nu}$ on $\partial\Omega$ yield

$$\int_{\Omega\cap B_R} \frac{\partial F(u)}{\partial x_i} = \int_{\partial(\Omega\cap B_R)} F(u)\nu_i(x) = \int_{\Omega\cap\partial B_R} F(u)\frac{x_i}{|x|};$$

$$\int_{\Omega\cap B_R}(\nabla u,\nabla\frac{\partial u}{\partial x_i})=\frac{1}{2}\int_{\partial(\Omega\cap B_R)}|\nabla u|^2\nu_i(x);$$

$$\int_{\partial(\Omega\cap B_R)}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial \nu}=\int_{\partial B_R\cap\Omega}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial \nu}+\int_{\partial\Omega\cap B_R}|\nabla u|^2\nu_i(x).$$

Hence

$$-\frac{1}{2}\int_{\partial\Omega\cap B_R}|\nabla u|^2\nu_i(x)=\int_{\Omega\cap\partial B_R}[F(u)\frac{x_i}{x}-\frac{1}{2}|\nabla u|^2\frac{x_i}{x}+\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial \nu}].$$

We then obtain

$$|\int_{\partial\Omega\cap B_R}|\nabla u|^2\nu_i(x)|\le\int_{\partial B_R\cap\Omega}[|F(u)|+\frac{3}{2}|\nabla u|^2].$$

Since $\nabla u\in L^2(\Omega), F(u)\in L^1(\Omega)$,

$$\int_0^\infty\int_{\partial\Omega\cap B_R}|\nabla u|^2\nu_i(x)dsdR\le\int_0^\infty\int_{\partial B_R\cap\Omega}[F(u)+\frac{3}{2}|\nabla u|^2]$$

$$=\int_\Omega[F(u)+\frac{3}{2}|\nabla u|^2]<\infty.$$

So there exists $\{R_j\}_{j=0}^\infty$ such that as $j\to\infty$, $R_j\to\infty$ and

$$\int_{\partial\Omega\cap B_{R_j}}|\nabla u|^2\nu_i(x)\to 0.$$

Now we have

$$\lim_{j\to\infty}\int_{\partial\Omega\cap B_{R_j}}(\nu(x),X)|\nabla u|^2=0.$$

This implies that $\nabla u=0$ on a open subset of $\partial\Omega$ by our assumption on Ω. Applying the standard theory of unique continuation, we conclude $u\equiv 0$ on Ω. □

3 On the Energy of u_ϵ

We start to investigate the energy of u_ϵ. For the Dirichlet problem we have

Proposition 3.1 *Let u_ϵ be a least-energy solution of (1.1) with the Dirichlet boundary condition. Then*

$$\lim_{\epsilon\to 0}\epsilon^{-N}J_\epsilon(u_\epsilon)=[\max_{x\in\overline{\Omega}}K(x)]^{-\frac{2}{p-1}}I(w)$$

where

$$I(w)=\frac{1}{2}\int_{R^N}[|\nabla w|^2+w^2]dx-\frac{1}{p+1}\int_{R^N}w^{p+1}dx$$

and w is the ground state solution defined in (2.3).

Proof. Fix a point $O \in \Omega$ to be determined later. Let r_1, $r_2 > 0$ such that $B_1 = B_{r_1}(O) \subset \Omega \subset B_2 = B_{r_2}(O)$. Define

$$\underline{J_\epsilon} : \ W_0^{1,2}(B_1) \to R$$

by

$$\underline{J_\epsilon}(v) = \frac{1}{2}\int_{B_1} [\epsilon^2|\nabla v|^2 + v^2]dx - \frac{1}{p+1}\int_{B_1} [\min_{x\in B_{r_1}} K(x)]v_+^{p+1}dx.$$

Also define

$$\overline{J_\epsilon} : \ W_0^{1,2}(B_2) \to R$$

by

$$\overline{J_\epsilon}(v) = \frac{1}{2}\int_{B_2} [\epsilon^2|\nabla v|^2 + v^2]dx - \frac{1}{p+1}\int_{B_2} [\max_{x\in\overline{\Omega}} K(x)]v_+^{p+1}dx.$$

Through trivial extension, we may write

$$W_0^{1,2}(B_1) \subset W_0^{1,2}(\Omega) \subset W_0^{1,2}(B_2).$$

Therefore with the aid of Lemma 2.1 we conclude

$$\overline{J_\epsilon}(\overline{u}_\epsilon) \le J_\epsilon(u_\epsilon) \le \underline{J_\epsilon}(\underline{u}_\epsilon) \tag{3.1}$$

where $\underline{u}_\epsilon$ is a least-energy critical point of $\underline{J_\epsilon}$ and $\overline{u}_\epsilon$ is a least-energy critical point of $\overline{J_\epsilon}$. Therefore $\underline{u}_\epsilon$ solves

$$\begin{cases} \epsilon^2\Delta u - u + [\min_{x\in B_{r_1}} K(x)]u^p = 0 \\ u|_{\partial B_1} = 0 \end{cases}$$

and $\overline{u}_\epsilon$ solves

$$\begin{cases} \epsilon^2\Delta u - u + [\max_{x\in\overline{\Omega}} K(x)]u^p = 0 \\ u|_{\partial B_2} = 0. \end{cases}$$

Now we can focus on $\underline{J_\epsilon}(\underline{u}_\epsilon)$ and $\overline{J_\epsilon}(\overline{u}_\epsilon)$. Let $\underline{w}_\epsilon = \underline{u}_\epsilon(\epsilon x + O)$, $\overline{w}_\epsilon = \overline{u}_\epsilon(\epsilon x + O)$. Observe that $\underline{w}_\epsilon$ solves

$$\begin{cases} \Delta u - u + [\min_{x\in B_{r_1}} K(x)]u^p = 0 \text{ in } B_{\epsilon^{-1}r_1} \\ u|_{\partial B_{\epsilon^{-1}r_1}} = 0 \end{cases} \tag{3.2}$$

and $\overline{w}_\epsilon$ solves

$$\begin{cases} \Delta u - u + [\max_{x\in\overline{\Omega}} K(x)]u^p = 0 \text{ in } B_{\epsilon^{-1}r_2} \\ u|_{\partial B_{\epsilon^{-1}r_2}} = 0. \end{cases} \tag{3.3}$$

It is well known that any positive solution to each of the above two equations is radially symmetric, (see [3]).

Claim:

$$\underline{w}_\epsilon \to [\min_{x\in B_{r_1}} K(x)]^{-\frac{1}{p-1}} w;$$

$$\overline{w}_\epsilon \to [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{1}{p-1}} w$$

in $W^{1,2}(R^N)$ and $C^{2,\alpha}_{loc}(R^N)$.

We shall only prove the convergence for $\underline{w}_\epsilon$; the other part follows in the same way. From the characterization in Lemma 2.1 we conclude that

$$I_\epsilon(\underline{w}_\epsilon) := \frac{1}{2}\int_{B_{\epsilon^{-1}r_1}} [|\nabla \underline{w}_\epsilon|^2 + \underline{w}_\epsilon^2]dx - \frac{1}{p+1}\int_{B_{\epsilon^{-1}r_1}} [\min_{x\in B_{r_1}} K(x)]\underline{w}_\epsilon^{p+1}dx$$

is non-increasing in ϵ as ϵ tends to 0. Since $\underline{w}_\epsilon$ is a solution of (3.2), we have, by multiplying (3.2) by $\underline{w}_\epsilon$ and integrating by parts,

$$\int_{B_{\epsilon^{-1}r_1}} [|\nabla \underline{w}_\epsilon|^2 + \underline{w}_\epsilon^2]dx = \int_{B_{\epsilon^{-1}r_1}} [\min_{x\in B_{r_1}} K(x)]\underline{w}_\epsilon^{p+1}dx.$$

Hence

$$I_\epsilon(\underline{w}_\epsilon) = (\frac{1}{2} - \frac{1}{p+1})\int_{B_{\epsilon^{-1}r_1}} [|\nabla \underline{w}_\epsilon|^2 + \underline{w}_\epsilon^2]dx. \tag{3.4}$$

¿From the fact that $I_\epsilon(\underline{w}_\epsilon)$ is decreasing in ϵ and (3.4) we conclude that $\{\underline{w}_\epsilon\}$ is bounded in $W^{1,2}(R^N)$. A standard boot-strapping argument shows that $\{\underline{w}_\epsilon\}$ is bounded in $C^{2,\alpha}_{loc}(R^N)$. For the sake of completeness we include the boot-strapping argument here.

Since $\{\underline{w}_\epsilon\}$ is bounded in $W^{1,2}(R^N)$, also in $L^{2^*}(B_r)$ for all $r > 0$ where 2^* denotes the Sobolev conjugate of 2, therefore $\{\underline{w}_\epsilon\}$ is bounded in $W^{2,p_1}(B_{r-1})$ with $p_1 = 2^*/p > 1$ by the interior elliptic L^p regularity theory (see, for example, [5]). Sobolev embedding theorem implies that $\{\underline{w}_\epsilon\}$ is bounded in $L^{p_1^*}(B_{r-1})$. If we use the above argument successively, we can show that after $k = k(N, p)$ times, $\{\underline{w}_\epsilon\}$ is bounded in $W^{2,p_k}(B_{r-k})$ with $p_k > N/2$. By the Sobolev embedding theorem $\{\underline{w}_\epsilon\}$ is bounded in $C^\alpha(B_{r-k})$ for some $\alpha > 0$. Now applying the interior Schauder estimate, we conclude that $\{\underline{w}_\epsilon\}$ is bounded in $C^{2,\alpha}(B_{r-k})$, hence bounded in $C^{2,\alpha}_{loc}(R^N)$.

Passing to a subsequence if necessary, we have

$$\underline{w}_\epsilon \to \underline{w}'$$

in $C^{2,\alpha}_{loc}(R^N)$ where $\underline{w}'$ solves

$$\begin{cases} \Delta u - u + [\min_{x\in B_{r_1}} K(x)]u^p = 0 \\ u > 0,\ \lim_{|x|\to\infty} u(x) = 0,\ \nabla u(0) = 0. \end{cases}$$

Notice that here we say $\underline{w}' > 0$ because

$$\underline{w}_\epsilon(0) \geq \left(\frac{1}{[\min_{x\in B_{r_1}} K(x)]}\right)^{\frac{1}{p-1}}$$

by the maximum principle. From the uniqueness result in [7], we have

$$\underline{w}' = [\min_{x\in B_{r_1}} K(x)]^{-\frac{1}{p-1}} w;$$

hence

$$\underline{w}_\epsilon \to [\min_{x\in B_{r_1}} K(x)]^{-\frac{1}{p-1}} w \text{ in } C^{2,\alpha}_{loc}(R^N).$$

To show

$$\underline{w}_\epsilon \to [\min_{x\in B_{r_1}} K(x)]^{-\frac{1}{p-1}} w \text{ in } W^{1,2}(R^N),$$

we need only to show

$$\overline{\lim_{\epsilon\to 0}} \|\underline{w}_\epsilon\|_{W^{1,2}(R^N)} \leq \|[\min_{x\in B_{r_1}} K(x)]^{-\frac{1}{p-1}} w\|_{W^{1,2}(R^N)}.$$

Using Lemma 2.1 again with the test function

$$[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w - [\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w(\epsilon^{-1}r_1),$$

we have

$$I_\epsilon(w_\epsilon) \leq M[[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w - [\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w(\epsilon^{-1}r_1)] =$$

$$\max_{t>0}\{\frac{t^2}{2}\int_{B_{\epsilon^{-1}r_1}} [|\nabla[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w|^2 + |[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w$$

$$-[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w(\epsilon^{-1}r_1)|^2]dx$$

$$-\frac{t^{p+1}}{p+1}\int_{B_{\epsilon^{-1}r_1}} [\min_{x\in B_{r_1}} K(x)]|[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w$$

$$-[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w(\epsilon^{-1}r_1)|^{p+1}dx\}.$$

It is easy to see that the maximum is obtained at

$$t = t_\epsilon = \left[\frac{\int_{B_{\epsilon^{-1}r_1}} [|\nabla w|^2 + |w - w(\epsilon^{-1}r_1)|^2]}{\int_{B_{\epsilon^{-1}r_1}} |w - w(\epsilon^{-1}r_1)|^{p+1}}\right]^{1/(p-1)}.$$

By Proposition 2.2 (3), we see

$$\lim_{\epsilon\to 0} t_\epsilon = [\frac{\int_{R^N}[|\nabla w|^2 + w^2]}{\int_{R^N} w^{p+1}}]^{1/(p-1)} = 1.$$

Using Proposition 2.2 (3) again, we have

$$\lim_{\epsilon\to 0} M[[\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w - [\min_{x\in B_1} K(x)]^{-\frac{1}{p-1}} w(\epsilon^{-1} r_1)] = [\min_{x\in B_{r_1}} K(x)]^{-\frac{2}{p-1}} I(w);$$

hence

$$\overline{\lim_{\epsilon\to 0}} I_\epsilon(\underline{w}_\epsilon) \le [\min_{x\in B_{r_1}} K(x)]^{-\frac{2}{p-1}} I(w),$$

and by (3.4)

$$\overline{\lim_{\epsilon\to 0}} \|\underline{w}_\epsilon\|_{W^{1,2}(R^N)} \le \|[\min_{x\in B_{r_1}} K(x)]^{-\frac{1}{p-1}} w\|_{W^{1,2}(R^N)}.$$

So we have proved the claim.

It follows from the claim that

$$\lim_{\epsilon\to 0} \epsilon^{-N} \underline{J}_\epsilon(\underline{u}_\epsilon) = [\min_{x\in B_{r_1}} K(x)]^{-\frac{2}{p-1}} I(w); \tag{3.5}$$

$$\lim_{\epsilon\to 0} \epsilon^{-N} \overline{J}_\epsilon(\overline{u}_\epsilon) = [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w). \tag{3.6}$$

Therefore (3.1) implies

$$[\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w) \le \liminf_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon)$$

$$\le \limsup_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) \le [\min_{x\in B_{r_1}} K(x)]^{-\frac{2}{p-1}} I(w). \tag{3.7}$$

Now if we choose $O = O_n \in \Omega$ such that

$$O_n \to P \in \overline{\Omega}$$

where $\max_{x\in\overline{\Omega}} K(x) = K(P)$ and choose $r_1 = r_{1n} = \frac{1}{n}\text{dist}(O_n, \partial\Omega) \to 0$, then as $n \to \infty$, (3.7) implies

$$\lim_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) = [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w).$$

□

Next we turn our attention to the Neumann problem. We prove

Proposition 3.2 *Letting u_ϵ be a least-energy solution of (1.1) with the Neumann boundary condition, then we have the following:*

1. If

$$\max_{x\in\overline{\Omega}} K(x) > 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

then

$$\overline{\lim_{\epsilon\to 0}}\epsilon^{-N} J_\epsilon(u_\epsilon) \le [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w).$$

2. If

$$\max_{x\in\overline{\Omega}} K(x) < 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

then

$$\overline{\lim_{\epsilon\to 0}}\epsilon^{-N} J_\epsilon(u_\epsilon) \le \frac{1}{2}[\max_{x\in\partial\Omega} K(x)]^{-\frac{2}{p-1}} I(w).$$

Proof of Part 1. Let $P \in \Omega$ such that $K(P) = \max_{x\in\overline{\Omega}} K(X)$. Notice that in this case maximum of K must be achieved in Ω. Define

$$w_\epsilon = u_\epsilon(\epsilon x + P)$$

for $x \in \Omega_\epsilon = \{x \in R^N : \ \epsilon x + P \in \Omega\}$. Notice that Ω_ϵ is expanding toward R^N as $\epsilon \to 0$. Let $w_*(x) = [K(P)]^{-\frac{1}{p-1}} w$. Define

$$I_\epsilon(v) = \frac{1}{2}\int_{\Omega_\epsilon} [|\nabla v|^2 + v^2] - \frac{1}{p+1}\int_{\Omega_\epsilon} K(\epsilon x + P) v^{p+1}.$$

Hence

$$M_\epsilon(w_*) := \max_{t>0} I_\epsilon(t w_*)$$

$$= \max_{t>0}\{\frac{t^2}{2}\int_{\Omega_\epsilon} [|\nabla w_*|^2 + w_*^2] - \frac{t^{p+1}}{p+1}\int_{\Omega_\epsilon} K(\epsilon x + P) w_*^{p+1}\}.$$

Observe that the maximum for t is obtained at

$$t_\epsilon = [\frac{\int_{\Omega_\epsilon} |\nabla w_*|^2 + w_*^2}{\int_{\Omega_\epsilon} K(\epsilon x + P) w_*^{p+1}}]^{-\frac{1}{p-1}} \to 1$$

by Proposition 2.2 (3). Using Proposition 2.2 (3) again, we have, with the aid of the Lebesgue domination convergence theorem,

$$\lim_{\epsilon\to 0} M_\epsilon(w_*) = \frac{1}{2}\int_{R^N} [|\nabla w_*|^2 + w_*^2] - \frac{1}{p+1}\int_{R^N} K(P) w_*^{P+1}$$

$$= [K(P)]^{-\frac{2}{p-1}} I(w).$$

But by Lemma 2.1, we have

$$\epsilon^{-N} J_\epsilon(u_\epsilon) = I_\epsilon(u_\epsilon(\epsilon x + P))$$

$$\leq M_\epsilon(w_*) \to [K(P)]^{-\frac{2}{p-1}} I(w),$$

so we have

$$\limsup_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) \leq [K(P)]^{-\frac{2}{p-1}} I(w) = [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w).$$

Proof of Part 2. In this case we shall construct a test function on the boundary of Ω. Let $P \in \partial\Omega$ such that $K(P) = \max_{x\in\partial\Omega} K(x)$. Without the loss of generality, we can assume $P = 0$. Then we introduce a map Φ from a neighborhood of $P = 0$ to a neighborhood of 0 flattening the boundary of Ω around 0. We may suppose

$$\Phi :\ \mathcal{U} \to (R^N)^+$$

where $\mathcal{U}$ is a neighborhood of 0 in Ω and $(D\Phi)_{ij}(0) = \delta_{ij}$. Then we let η be a smooth cut-off function in $(R^N)^+$ which is 0 outside $\Phi(\mathcal{U})$ and 1 in a neighborhood of the origin in $(R^N)^+$. Let

$$u^*_\epsilon(x) = \begin{cases} [\max_{x\in\partial\Omega} K(x)]^{-\frac{1}{p-1}} w(\epsilon^{-1}\eta(\epsilon\Phi(x))\Phi(x)), & \text{if } x \in \mathcal{U} \\ 0 \text{ in } \Omega\backslash\mathcal{U}. \end{cases}$$

We now define the test function

$$w^*_\epsilon(x) = u^*_\epsilon(\epsilon x)$$

in Ω_ϵ where $\Omega_\epsilon = \{x \in R^N :\ \epsilon x \in \Omega\}$. Then as in the proof of part 1, define

$$I_\epsilon(v) = \frac{1}{2}\int_{\Omega_\epsilon} [|\nabla v|^2 + v^2] - \frac{1}{p+1}\int_{\Omega_\epsilon} K(\epsilon x + P) v^{p+1}.$$

Hence

$$M_\epsilon(w^*_\epsilon) := \max_{t>0} I_\epsilon(t w^*_\epsilon)$$

$$= \max_{t>0}\{\frac{t^2}{2}\int_{\Omega_\epsilon} [|\nabla w^*_\epsilon|^2 + (w^*_\epsilon)^2] - \frac{t^{p+1}}{p+1}\int_{\Omega_\epsilon} K(\epsilon x + P)(w^*_\epsilon)^{p+1}\}.$$

A careful analysis, like the formula (3.10) in the proof of Proposition 3.3 [10], of w^*_ϵ shows that

$$t_\epsilon = [\frac{\int_{\Omega_\epsilon} |\nabla w^*_\epsilon|^2 + (w^*_\epsilon)^2}{\int_{\Omega_\epsilon} K(\epsilon x + P)(w^*_\epsilon)^{p+1}}]^{-\frac{1}{p-1}}$$

$$\to [\frac{\int_{(R^N)^+} |[K(P)]^{-\frac{1}{p-1}} \nabla w|^2 + ([K(P)]^{-\frac{1}{p-1}} w)^2}{\int_{(R^N)^+} K(P)([K(P)]^{-\frac{1}{p-1}} w)^{p+1}}]^{-\frac{1}{p-1}} = 1$$

as $\epsilon \to 0$ where t_ϵ assumes the maximum of $I_\epsilon(t w^*_\epsilon)$. Then we conclude with the aid of the analysis of w^*_ϵ that

$$\lim_{\epsilon\to 0} M_\epsilon(w^*_\epsilon)$$

$$= \frac{1}{2}\int_{(R^N)^+} [|[K(P)]^{-\frac{1}{p-1}}\nabla w|^2 + ([K(P)]^{-\frac{1}{p-1}} w)^2]$$

$$-\frac{1}{p+1}\int_{(R^N)^+} ([K(P)]^{-\frac{2}{p-1}} w^{p+1}$$

$$= \frac{1}{2}[K(P)]^{-\frac{2}{p-1}} I(w).$$

But by Lemma 2.1, we have

$$\epsilon^{-N} J_\epsilon(u_\epsilon) = I_\epsilon(u_\epsilon(\epsilon x + P))$$

$$\leq M_\epsilon(w^*_\epsilon) \to \frac{1}{2}[K(P)]^{-\frac{2}{p-1}} I(w),$$

so we have

$$\limsup_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) \leq \frac{1}{2}[K(P)]^{-\frac{2}{p-1}} I(w) = \frac{1}{2}[\max_{x\in\partial\Omega} K(x)]^{-\frac{2}{p-1}} I(w).$$

□

Remark 3.3 *The inequalities in the conclusions of Proposition 3.2 are indeed equalities. See Remark 5.1 after the proof of Theorem 1.3.*

4 Proof of Theorem 1.1

Proof of Part 1. Let P_ϵ be a local maximum of u_ϵ Then the maximum principle implies

$$u_\epsilon(P_\epsilon) \geq [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{1}{p-1}}.$$

¿From Proposition 3.1 we know $w_\epsilon(x) = u_\epsilon(\epsilon x + P_\epsilon)$ is bounded in $W^{1,2}(R^N)$. Then the standard boot-strapping argument plus the elliptic regularity implies that w_ϵ is uniformly bounded in R^N; hence u_ϵ is uniformly bounded in Ω. This proves part 1. □

Proof of Part 2. Suppose $\epsilon_n^{-1}\text{dist}(P_{\epsilon_n}, \partial\Omega) < R$ for a sequence ϵ_n with $\epsilon_n \to 0$ and $R > 0$ independent of ϵ_n. We denote P_{ϵ_n} by P_n for simplicity. Let $Q_n \in \partial\Omega$ such that $\text{dist}(P_n, Q_n) = \text{dist}(P_n, \partial\Omega)$. Take Φ_n to be a map flattening $\partial\Omega$ around Q_n. So Φ_n maps a neighborhood of Q_n to a neighborhood of 0 in $(R^N)^+ := \{x : x_N > 0\}$. We may further assume $Q_n \to Q$. Then $u'_n(x) = u_n(\Phi_n^{-1}x)$ solves

$$\frac{\epsilon^2}{2} a_{ij}^{(n)} D^{ij} u + b_i^{(n)} D^i u - u + K(\Phi^{-1}(x))u^p = 0$$

in a neighborhood of 0 in $(R^N)^+$ where $a_{ij}^{(n)}$ and $b_i^{(n)}$ depend on n but their C^α norm can be bounded uniformly from both below and above and the ellipticity of a_{ij} are also bounded from below and above. Indeed we make Φ_n appropriate so that $a_{ij}^{(n)}(0) = \delta_{ij}$ and $b_i^{(n)}(0) = 0$. See [10] section 4 for details.

Then $w_n(x) = u_n'(\epsilon x)$ solves

$$a_{ij}^{(n)}(\epsilon x)D^{ij}u + b_i^{(n)}(\epsilon x)D^i u - u + K(\Phi^{-1}(x))u^p = 0$$

in an expanding domain of $(R^N)^+$. Letting $n \to \infty$, we see $w_n \to w'$ in $C^{2,\alpha}_{loc}(R^N)^+)$ where w' solves

$$\begin{cases} \Delta u - u + K(Q)u^p = 0 \text{ in } (R^N)^+ \\ u = 0 \text{ on } \{x = (x_1, ..., x_N) \in R^N : \ x_N = 0\}. \end{cases}$$

We observe that since $\epsilon_n^{-1}\mathrm{dist}(P_{\epsilon_n}, \partial\Omega) < R$, $\epsilon_n^{-1}\Phi(P_n)$ stays in $B_R(0)$. But w_n gets a local maximum at $\epsilon_n^{-1}\Phi(P_n)$, so

$$w_n(\epsilon_n^{-1}\Phi(P_n)) \geq [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{1}{p-1}};$$

hence $w' \not\equiv 0$. However since w' has finite $W^{1,2}((R^N)^+)$ norm, $w' \equiv 0$ by Proposition 2.3. We reached a contradiction, so we have proved

$$\lim_{\epsilon\to 0} \epsilon^{-1}\mathrm{dist}(P_\epsilon, \partial\Omega) = \infty. \tag{4.1}$$

Assume u_{ϵ_n} has two local maximum points P_n and P_n' for a sequence $\epsilon_n \to 0$. We first claim

$$\lim_{n\to\infty} \epsilon_n^{-1}\mathrm{dist}(P_n, P_n') = \infty. \tag{4.2}$$

Again passing to a subsequence if necessary we suppose that there exist $\{P_n\}$, $\{P_n'\}$ and R independent of n such that

$$\epsilon_n^{-1}\mathrm{dist}(P_n, P_n') \leq R. \tag{4.3}$$

As before we set $w_n(x) = u_{\epsilon_n}(\epsilon x + P_n)$. Because $w_n(x)$ solves

$$\begin{cases} \Delta u - u + K(\epsilon_n x + P_n)u^p = 0 \text{ in } \Omega_n \\ u|_{\Omega_n} = 0 \end{cases}$$

where

$$\Omega_n \to R^N$$

in the obvious sense, we conclude with the aid of part 1 and Proposition 3.1 that

$$w_n \to [K(P)]^{-\frac{1}{p-1}} w \tag{4.4}$$

in $C^{2,\alpha}_{loc}(R^N)$ where P is a limit point of $\{P_n\}$. Since $[K(P)]^{-\frac{1}{p-1}}w$ has only one critical point at 0 which is non-degenerate, w_n can not have any other critical point except 0 in B_R. This contradicts (4.3), so we have proved (4.2).

Now consider the energy of w_n. Defining energy I_n for w_n by

$$I_n(v) = \frac{1}{2}\int_{\Omega_n} [|\nabla v|^2 + v^2]dx - \frac{1}{p+1}\int_{\Omega_n} v_+^{P+1}dx,$$

we have

$$I_n(w_n) = (\frac{1}{2} - \frac{1}{p+1})\|w_n\|^2_{W^{1,2}(\Omega_n)}$$

$$\geq (\frac{1}{2} - \frac{1}{p+1})(\|w_n\|^2_{W^{1,2}(B_1)} + \|w_n\|^2_{W^{1,2}(B_2)})$$

where $B_1 = B_{r_n}((P_n)$, $B_2 = B_{r_n}(P_n')$ and $r_n = \frac{\mathrm{dist}(P_n, P_n')}{2\epsilon_n}$. Since $r_n \to \infty$, repeating the construction leading to (4.4) for both P_n and P_n', we have

$$w_n(\epsilon_n^{-1}x + P_n)|_{B_1} \to [K(P)]^{-\frac{1}{p-1}}w;$$

$$w_n(\epsilon_n^{-1}x + P_n')|_{B_2} \to [K(P')]^{-\frac{1}{p-1}}w$$

in $C^{2,\alpha}_{loc}(R^N)$ where P' is a limit point of P_n'. Hence

$$\liminf_{n\to\infty} \|w_n\|^2_{W^{1,2}} \geq 2\|[K(P)]^{-\frac{1}{p-1}}w\|^2_{W^{1,2}}.$$

Therefore

$$\liminf_{n\to\infty} \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n}) = \liminf_{n\to\infty} I_n(w_n)$$

$$\geq 2(\frac{1}{2} - \frac{1}{p})\|w\|^2_{W^{1,2}} = 2[\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-2}}I(w)$$

which contradicts Proposition 3.1. The proof of part 2 is now complete. □

Proof of Part 3. Suppose that there is a sequence $\{P_{\epsilon_n}\}$ of $\{P_\epsilon\}$ with

$$P_n := P_{\epsilon_n} \to P$$

where $K(P) < \max_{x\in\overline{\Omega}} K(x)$. Consider

$$w_n(x) := u_{\epsilon_n}(\epsilon_n x + P_n)$$

in $\Omega_n = \{x \in R^N : \ \epsilon_n x + P_n \in \Omega\}$. ¿From part (2) of this theorem, we know

$$\Omega_n \to R^N.$$

Then a boot-strapping argument again shows that

$$w_n \to w'$$

in $C^{2,\alpha}_{loc}(R^N)$ and w' is a positive solution of

$$\Delta u - u + K(P)u^p = 0$$

which decays at infinity. Therefore the uniqueness result in [7] implies that $w' = [K(P)]^{-\frac{1}{p-1}} w$. Hence

$$w_n \to [K(P)]^{-\frac{1}{p-1}} w$$

weakly in $W^{1,2}(R^N)$ by the Fatou's lemma, so

$$\liminf_{n\to\infty} \epsilon^{-N} J_{\epsilon_n}(u_{\epsilon_n}) = \liminf_{n\to\infty} (\frac{1}{2} - \frac{1}{p+1}) \|w_n\|^2_{W^{1,2}(R^N)}$$

$$\geq (\frac{1}{2} - \frac{1}{p+1}) \|[K(P)]^{-\frac{1}{p-1}} w\|^2_{W^{1,2}(R^N)} = [K(P)]^{-\frac{2}{p-1}} I(w)$$

$$> [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w).$$

But this is inconsistent with Proposition 3.1, so we have proved part 3. □

Remark 4.1 *If function $K(x)$ has only one local maximum point P on the boundary of Ω, then the peaks of least-energy solutions have to converge to P by Theorem 1.1 (3). But from Theorem 1.1 (2), we also know that those peaks converge to P* slowly.

5 Proof of Theorem 1.2 and 1.3

Proof of Theorem 1.2.

The proof of Part 1 is identical with the proof of part 1 of Theorem 1.1. One applies the maximum principle to get a lower bound for u_ϵ and the boot-strapping argument to get an upper bound for u_ϵ. We leave the details to reader.

To prove part 2, we first prove that P_ϵ stays away from the boundary of Ω. Suppose $P_{\epsilon_n} \to P \in \partial\Omega$ for a sequence ϵ_n. We consider two cases. First assume

$$\lim_{\epsilon_n\to 0} \epsilon_n^{-1} \mathrm{dist}(P_{\epsilon_n}, P) = \infty.$$

Then with the aid of a boot-strapping argument as in the proof of Proposition 3.1, we know

$$u_{\epsilon_n}(\epsilon_n x + P_{\epsilon_n}) \to [K(P)]^{-\frac{1}{p-1}} w$$

in $C^{2,\alpha}_{loc}(R^N)$. Hence

$$\liminf_{\epsilon_n\to 0} \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n}) \geq [K(P)]^{-\frac{2}{p-1}} I(w) > [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w)$$

which contradicts Proposition 3.2. Here the last strict inequality follows from the assumption

$$\max_{x\in\overline{\Omega}} K(x) > 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x).$$

Next we assume

$$\epsilon_n^{-1}\mathrm{dist}(P_{\epsilon_n}, P) \leq R$$

for some $R > 0$ independent of ϵ_n. Again by the boot-strapping technique, we know

$$u_{\epsilon_n}(\epsilon_n x + P_{\epsilon_n}) \to [K(P)]^{-\frac{1}{p-1}} w$$

in $C^{2,\alpha}_{loc}((R^N)^+)$. Hence

$$\liminf_{\epsilon_n \to 0} \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n}) \geq \frac{1}{2}[K(P)]^{-\frac{2}{p-1}} I(w) > [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w)$$

which again contradicts Proposition 3.2. The last strict inequality is exactly the assumption

$$\max_{x\in\overline{\Omega}} K(x) > 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x).$$

Therefore all local maximum of u_ϵ stays away from the boundary of Ω.

Now we show that u_ϵ has only one local maximum for ϵ small enough. Suppose P_{ϵ_n} and P'_{ϵ_n} are two local maximum for a subsequence $\{\epsilon_n\}$. As in (4.2), we know

$$\lim_{\epsilon_n \to 0} \epsilon_n^{-1}\mathrm{dist}(P_{\epsilon_n}, P'_{\epsilon_n}) = \infty.$$

If

$$\lim_{\epsilon_n \to 0} \epsilon_n^{-1}\mathrm{dist}(P_{\epsilon_n}, P'_{\epsilon_n}) = \infty,$$

we can proceed as in the proof of Theorem 1.1 part 2. Consider the energy of u_{ϵ_n} and conclude

$$\liminf_{\epsilon_n \to 0} J_{\epsilon_n}(u_{\epsilon_n}) \geq 2[\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w)$$

which contradicts Proposition 3.2. So we proved part 2.

Finally we let $P \in \Omega$ be a limit point of $\{P_{\epsilon_n}\}$. Then since

$$u_{\epsilon_n}(\epsilon_n x + P_{\epsilon_n}) \to [K(P)]^{-\frac{1}{p-1}} w$$

in $C^{2,\alpha}_{loc}((R^N)^+)$, we have by the Fatou lemma

$$\liminf_{\epsilon_n \to 0} J_{\epsilon_n}(u_{\epsilon_n}) \geq [K(P)]^{-\frac{2}{p-1}} I(w). \tag{5.1}$$

It follows from Proposition 3.2 that

$$K(P) = \max_{x\in\overline{\Omega}} K(x).$$

□

Proof of Theorem 1.3.

The proof of this theorem is similar to the proof of Theorem 2.1 [10], so we shall be very sketchy. The proof of part 1 is again identical with the proof of part 1 of Theorem 1.1. We omit it.

To prove part 2, we first show that for a local maximum P_ϵ of u_ϵ

$$\epsilon^{-1}\mathrm{dist}(P_\epsilon, \partial\Omega) < R$$

for some R independent of ϵ. If this is not the case, we can prove that

$$u_{\epsilon_n}(\epsilon_n x + P_{\epsilon_n}) \to [K(P)]^{-\frac{1}{p-1}} w$$

for a sequence $\{u_{\epsilon_n}\}$ in $C^{2,\alpha}_{loc}(R^N)$ as in the proof of Theorem 1.1 part 2 where P is a limit point of $\{P_{\epsilon_n}\}$. Therefore as in the proof of Theorem 1.1 part 2 we have

$$\liminf_{\epsilon_n \to 0} \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n}) \geq [K(P)]^{-\frac{2}{p-1}} I(w)$$

which contradicts Proposition 3.2.

Now we need to know that any local maximum point P_ϵ must be on the boundary of Ω if ϵ is small enough. We can follow Step 2 of the proof of Theorem 2.1 [10]. The basic idea to prove this fact is to show that after a diffeomorphism

$$u_\epsilon(\epsilon x + P^*_\epsilon) \to [K(P)]^{-\frac{1}{p-1}} w$$

in $C^{2,\alpha}_{loc}((R^N)^+)$ where $P^*_\epsilon \in \partial\Omega$ such that $\mathrm{dist}(P_\epsilon, P^*_\epsilon) = \mathrm{dist}(P_\epsilon, \partial\Omega)$. Therefore since 0 is the only critical point of $[K(P)]^{-\frac{1}{p-1}} w$ which is non-degenerate, P^*_ϵ has to be the only critical point of u_ϵ if ϵ is small. So we conclude $P^*_\epsilon = P_\epsilon$, i.e. $P_\epsilon \in \partial\Omega$.

Next we show that u_ϵ has only one local maximum if ϵ is small enough. Assume that P_{ϵ_n} and P'_{ϵ_n} are two local maximum of u_{ϵ_n} with $\epsilon_n \to 0$. Then as in the proof of Theorem 2.1 [10] we can show that

$$\liminf_{\epsilon_n \to 0} \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n}) \geq 2\frac{1}{2}[K(P)]^{-\frac{2}{p-1}} I(w) = [K(P)]^{-\frac{2}{p-1}} I(w)$$

which again contradicts Proposition 3.2. Finally let P be a limit point of $\{P_\epsilon\}$. Assume

$$P_{\epsilon_n} \to P \in \partial\Omega$$

as $n \to \infty$. Then after a diffeomorphism

$$u_{\epsilon_n}(\epsilon_n x + P_{\epsilon_n}) \to [K(P)]^{-\frac{1}{p-1}} w$$

in $C^{2,\alpha}_{loc}((R^N)^+)$. Therefore by the Fatou's lemma

$$\liminf_{\epsilon_n \to 0} \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n}) \geq \frac{1}{2}[K(P)]^{-\frac{2}{p-1}} I(w). \tag{5.2}$$

Then Proposition 3.2 implies

$$K(P) = \max_{x\in\partial\Omega} K(x).$$

□

Remark 5.1 *(5.1) and (5.2) show that in the Neumann problem when*

$$\max_{x\in\overline{\Omega}} K(x) > 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

$$\liminf_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) \geq [\max_{x\in\overline{\Omega}} K(x)]^{-\frac{2}{p-1}} I(w),$$

and when

$$\max_{x\in\overline{\Omega}} K(x) < 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x),$$

$$\liminf_{\epsilon\to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) \geq \frac{1}{2}[\max_{x\in\partial\Omega} K(x)]^{-\frac{2}{p-1}} I(w).$$

So the inequalities in Proposition 3.2 are actually equalities with $\overline{\lim}$ *replaced by* lim.

Remark 5.2 *The border line case* $\max_{x\in\overline{\Omega}} K(x) = 2^{\frac{p-1}{2}} \max_{x\in\partial\Omega} K(x)$ *for the Neumann problem seems quite interesting to me. I don't know if the geometry of* Ω *will come out and affect the location of the peaks in this case.*

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