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On Critical Points of p Harmonic Functions in the Plane *

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Abstract

We show that if u is a p harmonic function, $1 < p < \infty$, in the unit disk and equal to a polynomial P of positive degree on the boundary of this disk, then ∇u has at most $\deg P - 1$ zeros in the unit disk.

In this note we prove the following theorem.

Theorem 1 *Given $p, 1 < p < \infty$, let u be a real valued weak solution to*

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \tag{*}$$

in $D = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\} \subset \mathbf{R^2}$ with $u = P$ on ∂D where P is a real polynomial in x_1, x_2 of degree $m \geq 1$. Then ∇u has at most $m - 1$ zeros in D counted according to multiplicity.

In (*), $\nabla\cdot$ denotes the divergence operator while ∇u denotes the gradient of u. The above theorem answers a question in the affirmative first posed by D. Khavinson in connection with determining the extremal functions for certain linear functionals in the Bergman space of p th power integrable analytic functions on D, $1 < p < \infty$. We note that the differential operator in (*) is often called the p Laplacian and it is well known (see [GT]) that solutions to this equation are infinitely differentiable (in fact real analytic) at each point where $\nabla u \neq 0$ while (*) is degenerate elliptic at each point where $\nabla u = 0$. The above theorem appears to be the first of its kind to establish independent of p and the structure constants for the p Laplacian, a bound (m - 1) for the number of points in D where (*) degenerates. Because of this independence we conjecture that our theorem also remains true for $p = \infty$ and the so called ∞ Laplacian (see [BBM] or [J] for definitions). Finally we remark that in [Al] a result, in the same spirit as ours, is obtained for smooth linear equations whose matrix of coefficients has determinant one.

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Proof of main theorem.

Consider the strong solutions, $v = v(\cdot, \epsilon, p)$, to

$$\nabla \cdot ((\epsilon + |\nabla v|^2)^{\frac{p}{2}-1} \nabla v) = 0 \tag{**}$$

in D, with $v = P$ on ∂D. We note that (**) implies

$$Lv = (p-2) \sum_{j,k=1}^{2} v_{x_j x_k} v_{x_j} v_{x_k} + (\epsilon + |\nabla v|^2)\, \Delta v = 0 \tag{0}$$

at each point of D. Here Δ denotes the Laplacian. From (0) and elliptic theory it follows that $v(\cdot, \epsilon)$ is unique and infinitely differentiable in the closed unit disk ($v \in C^\infty(\bar{D})$). Indeed this statement follows easily from Schauder's theorem (see [GT], ch 6) and induction once $C^{1,\alpha}$ regularity of v in $\bar{D}$ is established (for $C^{1,\alpha}$ regularity of v see [L]).

Next we introduce complex notation. Let $z = x_1 + ix_2$, $i = \sqrt{-1}$, and put $g_z = \frac{1}{2}(g_{x_1} - ig_{x_2})$, $g_{\bar{z}} = \frac{1}{2}(g_{x_1} + ig_{x_2})$. as usual and note from (0) as in [GT, ch 11, section 2] or [IM], that if $f(z) = f(z, \epsilon, p) = v_z(z)$, then f is quasiregular in D with $k = |1 - 2/p|$. That is f is a sense preserving mapping of D and

$$|f_{\bar{z}}| \;\leq\; |1 - 2/p|\, |f_z| \tag{1}$$

at each point of D. From the factorization theorem for quasiregular mappings (see [A, ch V]) we find that $f = g \circ h$ where g is analytic in $h(D)$ and h is a QC mapping of $\mathbf{R^2}$ onto itself (i.e. a quasiregular homeomorphism of $\mathbf{R^2}$). Using this factorization, the argument principle for analytic functions, and C^1 smoothness of f in $\bar{D}$, it follows that we can calculate the number of zeros of f counted according to multiplicity inside a contour $\Gamma \subset \bar{D}$ with $f \neq 0$ on Γ (i.e the number of zeros of g counted according to multiplicity inside $h(\Gamma)$) by calculating

$$(2\pi i)^{-1} \int_\Gamma \frac{d \log f(z(t))}{dt}\, dt \tag{2}$$

where $\log f$ denotes a continuous branch of the logarithm of f on Γ and we assume $z = z(t)$ is a piecewise smooth parametrization of Γ . Now we can write x_1, x_2 in terms of $z, \bar{z}$ in the usual way and thus regard P as a function of $z, \bar{z}$. If $z = e^{i\theta}, \theta$ real, we note first that $\bar{z} = z^{-1}$ and second that

$$P_\theta(z) = izP_z - i\bar{z}P_{\bar{z}}$$

is identically equal to a rational function of degree at most $2m$ on ∂D. To construct Γ let $z_j = e^{i\theta_j}, j = 1, 2, \dots n$ be the distinct zeros of $\frac{\partial P}{\partial \theta}$ on ∂D. From our note we have $n \leq 2m$. For small $\delta > 0$ let $D(z_j, \delta) = \{z : |z - z_j| < \delta\}$ for $1 \leq j \leq n$. Then for δ small enough, clearly $\partial D \setminus \cup_{i=1}^n D(z_j, \delta)$ consists of n closed arcs, say $\cup_{i=1}^n \gamma_i$, oriented counterclockwise, as seen from the origin.

Let C_j be the arc of $\partial D(z_j,\delta)$ that lies inside the unit circle for $1 \le j \le n$ oriented counterclockwise as seen from the origin. We put $\Gamma = (\cup C_j) \cup (\cup \gamma_j)$. and shall show that the integral in (2) is $\le m-1$. To this end, let $\gamma \in \{\gamma_j\}$ and note that if $z = e^{i\theta}$, then $P_\theta = 2\ \mathrm{Re}\ (izv_z)$. Since P_θ does not change sign on γ it follows that the image of γ under $zf = zv_z$ lies inside a halfplane whose boundary contains 0. Thus a continuous argument of zf can change by at most π on γ and so

$$\left| \ \mathrm{Re}\ \left[(2\pi i)^{-1} \int_\gamma \frac{d \log[z(t)f(z(t))]}{dt}\, dt \right] \right| \le 1/2. \tag{3}$$

Next we consider $C_k \in \{C_j\}$. Recall that $v \in C^\infty(\bar{D})$. If $v_z(z_k) \ne 0$ then clearly

$$\left| (2\pi i)^{-1} \int_{C_k} \frac{d \log[z(t)f(z(t))]}{dt}\, dt \right| \to 0 \tag{4}$$

as $\delta \to 0$. Otherwise, let $l > 1$ be the largest positive integer such that all homogeneous Taylor polynomials of $v - v(z_k)$ about z_k of degree less than l are identically 0 and let Q be the homogeneous Taylor polynomial of degree l about z_k corresponding to $v - v(z_k)$. Using (0) and continuity of the derivatives of v in $\bar{D}$ we see that for $z \in D \cap D(z_k,\delta)$

$$0 = Lv(z) = O(|z-z_k|^{3l-4}) + \epsilon\, \Delta Q(z) \tag{5}$$

as $z \to z_k$, Now ΔQ is either a homogeneous polynomial of degree $l-2$ or $\Delta Q \equiv 0$. Dividing (5) by $|z-z_k|^{l-2}$ and taking a limit as $z \to z_k$ we conclude that the second possibility must occur. Thus Q is harmonic and so $Q = \ \mathrm{Re}\ [c(z-z_k)^l]$ for some complex c. From this fact we conclude first that for a continuous branch of $\log f$ on C_k we have

$$\log(izf(z)) = \log[izQ_z(z)] + o(1), \ \text{ as } \delta \to 0 \text{ for } z \in C_k,$$

where the $o(1)$ term is independent of $z \in C_k$. Second we conclude

$$(2\pi i)^{-1} \int_{C_k} \frac{d \log[z(t)f(z(t))]}{dt}\, dt \to -(l-1)/2 \tag{6}$$

as $\delta \to 0$. Since the integral in (2) must be a nonnegative integer we see from (3) and (6) that for δ sufficiently small

$$(2\pi i)^{-1} \int_{\Gamma} \frac{d \log[f(z(t))]}{dt}\, dt \ \le\ m-1 \tag{7}$$

since there are at most $2m$ members of $\{\gamma_j\}$ and the argument of z changes by 2π as we go around Γ.

Finally, v, v_z considered as functions of ϵ converge uniformly on compact subsets of D to u, u_z, for a fixed p as $\epsilon \to 0$. These facts follow from the uniqueness of u as a solution to the p Laplacian and $C^{1,\alpha}$ regularity of u, v (with constants independent of ϵ). Moreover from (1) it follows that u_z is quasiregular in D with $k = |1 - 2/p|$ (again see [IM] for these facts). From these observations, (7), and another winding number argument we find that if $u_z \neq 0$ on $\{z : |z| = r\}$ for some $r, 0 < r < 1$, then u_z has at most $m - 1$ zeros in $\{z : |z| < r\}$. Hence our theorem is true. □

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