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Quasireversibility Methods for Non-Well-Posed Problems *

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Abstract

The final value problem,

$$\begin{cases} u_t + Au = 0\,, & 0 < t < T \\ u(T) = f \end{cases}$$

with positive self-adjoint unbounded A is known to be ill-posed. One approach to dealing with this has been the method of quasireversibility, where the operator is perturbed to obtain a well-posed problem which approximates the original problem. In this work, we will use a quasi-boundary-value method, where we perturb the final condition to form an approximate non-local problem depending on a small parameter α. We show that the approximate problems are well posed and that their solutions u_α converge on $[0,T]$ if and only if the original problem has a classical solution. We obtain several other results, including some explicit convergence rates.

1 Introduction

Let A be a self-adjoint operator on a Hilbert space H such that $-A$ generates a compact contraction semi-group on H. We consider the problem of finding a $u : [0,T] \longrightarrow H$ such that

$$\begin{cases} u'(t) + Au(t) = 0\,, & 0 < t < T \\ u(T) = f \end{cases} \qquad (FVP)$$

for some prescribed final value f in H. Such problems are not well posed, that is, even if a unique solution exists on $[0,T]$ it need not depend continuously on the

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final value f. One method for approaching such problems is quasi reversibility, introduced by Lattes and Lions in the 1960's. The idea is to replace (FVP) with an approximate problem which is well posed, then use the solutions of this new problem to construct approximate solutions to (FVP). In the original method of quasi reversibility [2] Lattes and Lions approximate (FVP) with

$$\begin{cases} v'_\alpha(t) + Av_\alpha(t) - \alpha A^2 v_\alpha(t) = 0\,, \quad 0 < t < T \\ v_\alpha(T) = f\,, \end{cases}$$

where the operator A is replaced by a perturbation, in this case by $A - \alpha A^2$. For each $\alpha > 0$, they use the *initial value* $u_0 = v_a(0)$ in

$$\begin{cases} u'_\alpha(t) + Au_\alpha(t) = 0\,, \quad 0 < t < T \\ u_\alpha(0) = v_\alpha(0)\,. \end{cases}$$

Finally they show that the $u_\alpha(T)$ converge to f as α tends to zero. The method does not consider $u(t)$ for $t < T$ and the operator carrying f into $v_\alpha(0)$ has large norm for small α (on the order of $e^{\frac{c}{\alpha}}$)[3].

In [6], Showalter approximates (FVP) with

$$\begin{cases} v'_\alpha(t) + \alpha A v'_\alpha(t) + Av_\alpha(t) = 0\,, \quad 0 < t < T \\ v_\alpha(T) = f \end{cases}$$

and as above for each $\alpha > 0$, uses the initial value $u_0 = v_\alpha(0)$ in

$$\begin{cases} u'_\alpha(t) + Au_\alpha(t) = 0\,, \quad 0 < t < T \\ u_a(0) = v_\alpha(0)\,. \end{cases}$$

The solutions u_a are shown to approximate (FVP) in the sense that $u_\alpha(T)$ converges to f as α tends to zero. Also the $u_\alpha(t)$ are shown to converge to the solution $u(t)$ of (FVP) if and only if such exists, but again the norm of the function carrying f to $v_\alpha\,(0)$ is quite large for small α.

Miller [3] addresses this problem of large norm by finding optimal perturbations of the operator A. He states that it should be possible to make the norm on the order of $\frac{c}{\alpha}$ rather than $exp(\frac{c}{\alpha})$ and derives conditions on the perturbation $f(A)$ to achieve *best possible* results. As in the methods above he approximates (FVP) with

$$\begin{cases} v'(t) + f(A)v(t) = 0\,, \quad 0 < t < T \\ v(T) = f \end{cases}$$

and again solves the problem forward using $v(0)$ as an initial condition. Miller calls this *stabilized quasi reversibility*.

Finally Showalter [7] addresses a more general problem in a different way. He approximates the problem

$$\begin{cases} u'(t) + Au(t) - Bu(t) = 0\,, \quad 0 < t < T \\ u(0) = f\,. \end{cases}$$

with

$$\begin{cases} u'(t) + Au(t) - Bu(t) = 0\,, \quad 0 < t < T \\ \qquad u(0) + \alpha u(T) = f\,. \end{cases}$$

He calls this the *quasi-boundary-value method*, and he suggests that this method gives a better approximation than many other quasireversibility type methods. In this work we study this method to approximate (FVP) and prove results analogous to the ones stated in [7]. We note that (FVP) is a special case of the problem studied in [7]. However, ore results are proved directly and this allows us to obtain explicit estimates for the convergence rate of the approximations.

2 Perturbing the final conditions

We approximate (FVP) with the *quasi-boundary value problem*

$$\begin{cases} u'(t) + Au(t) = 0\,, \quad 0 < t < T \\ \alpha u(0) + u(T) = f\,. \end{cases} \tag{$QBVP$}$$

One superficial advantage of this method is that there is no need to *solve forward* here. More importantly, the error introduced by small changes in the final value f is not exponential, but of the order $\frac{1}{\alpha}$. We will show that this problem is well posed for each $\alpha > 0$, and that the approximations u_α are stable. We show that $u_\alpha(T)$ converges to f as α goes to zero and that the values $u_\alpha(t)$ converge on $[0, T]$ if and only if (FVP) has a solution.

In the following, assume that H is a separable Hilbert space and A is as above and that 0 is in the resolvent set of A. Let $S(t)$ be the compact contraction semi-group generated by $-A$. Since A^{-1} is compact, there is an orthonormal eigenbasis ϕ_n for H and eigenvalues $\frac{1}{\lambda_n}$ of A^{-1} such that $A^{-1}\phi_n = \frac{1}{\lambda_n}\phi_n$. Then the eigenvalues of $-A$ are $-\lambda_n$ and those for $S(t)$ are $e^{-t\lambda_n}$ (and possibly zero) [5]. In particular, for each positive α, $\alpha I + S(T)$ is invertible. Also, if $u = \sum_{i=1}^{\infty} a_i\phi_i$, then $S(T)u = \sum_{i=1}^{\infty} e^{-T\lambda_i} a_i\phi_i$ and

$$(S(T)u, u) = \sum_{i=1}^{\infty} e^{-T\lambda_i} a_i^2 \geq 0\,.$$

From this accretive type condition we obtain

$$\| \left(\alpha I + S(T)\right)^{-1} \| \leq \frac{1}{\alpha}\,.$$

It is useful to know exactly when (FVP) has a solution. The following lemma answers this question.

Lemma 1 *If $f = \sum_{i=1}^{\infty} b_i\phi_i$, then (FVP) has a solution if and only if $\sum_{i=1}^{\infty} b_i^2 e^{2T\lambda_i}$ converges.*

Proof. If $\sum_{i=1}^{\infty} b_i^2 e^{2T\lambda_i}$ converges, we merely define $u(t) = \sum_{i=1}^{\infty} e^{(T-t)\lambda_i} b_i \phi_i$. Let u be a solution to (FVP). Then $u(0)$ has an eigenfunction expansion $u = \sum_{i=1}^{\infty} a_i \phi_i$, and

$$S(T)u = \sum_{i=1}^{\infty} e^{-T\lambda_i} a_i \phi_i = f = \sum_{i=1}^{\infty} b_i \phi_i \,.$$

This implies that $e^{-T\lambda_i} a_i = b_i$ and thus $a_i = b_i e^{T\lambda_i}$. Since $u(0)$ is in H, we have $||u||^2 = \sum_{i=1}^{\infty} a_i^2 < \infty$ and we are done. $\square$

We wish to show that our approximate problem is well-posed and the following gives us what we need.

Definition. Define $u_\alpha(t) = S(t)(\alpha I + S(T))^{-1} f$, for f in H, $\alpha > 0$ and t in $[0, T]$.

Theorem 1 *The function $u_\alpha(t)$ is the unique solution of (QBVP) and it depends continuously on f.*

Proof. Since $(\alpha I + S(T))^{-1} f$ is in the domain of A, it is clear that u_α is a classical solution of the differential equation. Furthermore,

$$\begin{aligned} \alpha u_\alpha(0) + u_\alpha(T) &= \alpha(\alpha I + S(T))^{-1} f + S(T)(\alpha I + S(T))^{-1} f \\ &= (\alpha I + S(T))(\alpha I + S(T))^{-1} f = f. \end{aligned}$$

To see the continuous dependence of u_α on f, compute

$$\begin{aligned} &\|S(t)(\alpha I + S(T))^{-1} f_1 - S(t)(\alpha I + S(T))^{-1} f_2\| \\ &= \|S(t)(\alpha I + S(T))^{-1}(f_1 - f_2)\| \\ &\leq \frac{1}{\alpha}\|f_1 - f_2\| \,. \end{aligned}$$

Uniqueness follows from the fact that any solution v must satisfy $v(0) = (\alpha I + S(T))^{-1} f$ and the uniqueness of solutions to the forward problem. $\square$

We make two observations at this point which will be useful later. First, from the above it is clear that $\|u_\alpha(t)\| \leq \frac{1}{\alpha}\|f\|$. Secondly, if $u = \sum_{i=1}^{\infty} a_i \phi_i$, then $(\alpha I + S(T))u = \sum_{i=1}^{\infty} (\alpha + e^{-T\lambda_i}) a_i \phi_i$ and

$$(\alpha I + S(T))^{-1} u = \sum_{i=1}^{\infty} \frac{a_i}{\alpha + e^{-T\lambda_i}} \phi_i \,.$$

Theorem 2 *For all f in H, $\alpha > 0$, and t in $[0, T]$ we have that*

$$\|u_\alpha(t)\| \leq \alpha^{\frac{t-T}{T}} \|f\| \,.$$

Proof. If $f = \sum_{i=1}^{\infty} b_i \phi_i$, we have

$$\begin{aligned}
\|u_\alpha(t)\|^2 &= \sum_{i=1}^{\infty} e^{-2t\lambda_i} b_i^2 \left(\alpha + e^{-T\lambda_i}\right)^{-2} \\
&\le \sum_{i=1}^{\infty} e^{-2t\lambda_i} b_i^2 \left[\left(\alpha + e^{-T\lambda_i}\right)^{\frac{t}{T}} \left(\alpha + e^{-T\lambda_i}\right)^{1-\frac{t}{T}}\right]^{-2} \\
&\le \sum_{i=1}^{\infty} b_i^2 \left(\alpha^{1-\frac{t}{T}}\right)^{-2} \\
&= \left(\alpha^{\frac{t-T}{T}}\right)^2 \sum_{i=1}^{\infty} b_i^2
\end{aligned}$$

and we are done. □

Theorem 3 *For all f in H, $||u_\alpha(T) - f||$ tends to zero as α tends to zero. That is $u_\alpha(T)$ converges to f in H.*

Proof. If $f = \sum_{i=1}^{\infty} b_i \phi_i$, then

$$\begin{aligned}
\|u_\alpha(T) - f\|^2 &= \|S(T)(\alpha I + S(T))^{-1} f - f\|^2 \\
&= \alpha^2 \|(\alpha I + S(T))^{-1} f\|^2
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \alpha^2 b_i^2 \left(\alpha + e^{-T\lambda_i}\right)^{-2}.$$

Fix $\epsilon > 0$. Choose N so that $\sum_{i=N}^{\infty} b_i^2 < \epsilon/2$. Thus

$$\begin{aligned}
\|u_\alpha(T) - f\|^2 &< \sum_{i=1}^{N} \alpha^2 b_i^2 \left(\alpha + e^{-T\lambda_i}\right)^{-2} + \frac{\epsilon}{2} \\
&\le \alpha^2 \sum_{i=1}^{N} b_i^2 e^{2\lambda_i T} + \frac{\epsilon}{2}.
\end{aligned}$$

Now let α be such that $\alpha^2 < \epsilon \left(2 \sum_{i=1}^{N} b_i^2 e^{2\lambda_i T}\right)^{-2}$ and we are done. □

Theorem 4 *For all f in H, (FVP) has a solution u if and only if the sequence $u_\alpha(0)$ converges in H. Furthermore, we then have that $u_\alpha(t)$ converges to $u(t)$ as α tends to zero uniformly in t.*

Proof. Assume that $\lim_{\alpha\downarrow 0} u_\alpha(0) = u_0$ exists. Let $u(t) = S(t)u_0$. Since $\lim_{\alpha\downarrow 0} u_\alpha(T) = f$,

$$\lim_{\alpha\downarrow 0} \|u(t) - u_\alpha(t)\| = \|S(t)u_0 - u_\alpha(t)\|$$

$$
\begin{aligned}
&= \lim_{\alpha\downarrow 0} \|S(t)\left(u_0 - (\alpha I + S(T))^{-1} f\right)\| \\
&\le \lim_{\alpha\downarrow 0} \|u_0 - (\alpha I + S(T))^{-1} f\| \\
&= \lim_{\alpha\downarrow 0} \|u_0 - u_\alpha(0)\| = 0\,.
\end{aligned}
$$

Thus, $u(T) = f$ and $u(t) = S(t)u_0$ solves (FVP). We also see that $u_\alpha(t)$ converges to $u(t)$ uniformly in t.

Now let us assume that $u(t)$ is the solution to (FVP). Let $\epsilon > 0$ and $f = \sum_{i=1}^\infty b_i\phi_i$. From Lemma 1 we have that $\|u(0)\|^2 = \sum_{i=1}^\infty b_i^2 e^{2T\lambda_i}$. Choose N so that $\sum_{i=N}^\infty b_i^2 e^{2T\lambda_i} < \frac{\epsilon}{2}$. Let $\alpha, \gamma > 0$. Then

$$
\begin{aligned}
\|u_\alpha(0) - u_\gamma(0)\|^2 &= \|(\alpha I + S(T))^{-1} f - (\gamma I + S(T))^{-1} f\| \\
&= \|\sum_{i=1}^\infty \left(\frac{1}{\alpha + e^{-T\lambda_i}} - \frac{1}{\gamma + e^{-T\lambda_i}}\right) b_i\phi_i\| \\
&= \sum_{i=1}^\infty (\gamma-\alpha)^2 \left(\alpha\gamma + (\alpha+\gamma)e^{-T\lambda_i} + e^{-2T\lambda_i}\right)^{-2} b_i^2 \\
&= \sum_{i=1}^N (\gamma-\alpha)^2 \left(\alpha\gamma + (\alpha+\gamma)e^{-T\lambda_i} + e^{-2T\lambda_i}\right)^{-2} b_i^2 \\
&\quad + \sum_{i=N+1}^\infty (\gamma-\alpha)^2 \left(\alpha\gamma + (\alpha+\gamma)e^{-T\lambda_i} + e^{-2T\lambda_i}\right)^{-2} b_i^2 \\
&\le \sum_{i=1}^N (\gamma-\alpha)^2 e^{4T\lambda_i} b_i^2 + \sum_{i=N+1}^\infty \left(\frac{\gamma-\alpha}{\alpha+\gamma}\right)^2 b_i^2 e^{2T\lambda_i} \\
&\le \sum_{i=1}^N (\gamma-\alpha)^2 e^{4T\lambda_i} b_i^2 + \frac{\epsilon}{2}\,.
\end{aligned}
$$

Now if we choose $\delta > 0$ so that $\delta^2 < \epsilon\left(\sum_{i=1}^N e^{4T\lambda_i} b_i^2\right)^{-1}$ and require that α and γ be less than δ, we have that

$$
\|u_\alpha(0) - u_\gamma(0)\|^2 < \epsilon\,.
$$

We therefore have that $\{u_\alpha(0)\}$ is Cauchy and thus converges. From the first part of the theorem, we have that $u_\alpha(t)$ converges to $u(t)$ uniformly in t. □

We end this paper with a result that gives explicit convergence rates in the case that (FVP) is soluble for some positive final time.

Theorem 5 *If $f = \sum_{i=1}^\infty b_i\phi_i$ is in H and there exists an $\epsilon > 0$ so that $\sum_{i=1}^\infty b_i^2 e^{\epsilon\lambda_i T}$ converges, then $\|u_\alpha(T) - f\|$ converges to zero with order $\alpha^\epsilon\epsilon^{-2}$.*

Proof. Let ϵ be in $(0,2)$ such that $\sum_{i=1}^{\infty} b_i^2 e^{\epsilon\lambda_i T}$ is finite and let k be in $(0,2)$. Fix a natural number n. Define

$$g_n(\alpha) = \frac{\alpha^k}{(\alpha + e^{-\lambda_n T})^2}.$$

Differentiating with respect to α yields

$$g_n'(\alpha) = \alpha^{k-1}\frac{(k-2)\alpha + ke^{-T\lambda_n}}{(\alpha + e^{-\lambda_n T})^3}.$$

Thus $g_n'(\alpha) = 0$ when either $\alpha = 0$ or

$$\alpha = \frac{k}{2-k}e^{-T\lambda_n}.$$

Since $g_n(\alpha) > 0$, $g_n(0) = 0$, and $\lim_{\alpha\to\infty} g_n(\alpha) = 0$ we have that $\alpha_0 = \frac{k}{2-k}e^{-T\lambda_n}$ is the critical value at which g_n achieves its maximum. Thus we have the inequality

$$g_n(\alpha) \le \frac{\left(\frac{k}{2-k}\right)^k e^{-kT\lambda_n}}{(\alpha_0 + e^{-\lambda_n T})^2}.$$

We now calculate

$$\begin{aligned}
\|u_\alpha(T) - f\|^2 &= \sum_{n=1}^{\infty} b_n^2 \alpha^2 (\alpha + e^{-\lambda_n T})^{-2} = \alpha^{2-k}\sum_{n=1}^{\infty} b_n^2 g_n(\alpha) \\
&\le \alpha^{2-k}\sum_{n=1}^{\infty} b_n^2 \left(\frac{k}{2-k}\right)^k e^{-kT\lambda_n}(\alpha_0 + e^{-\lambda_n T})^{-2} \\
&\le \alpha^{2-k}\sum_{n=1}^{\infty} b_n^2 \left(\frac{k}{2-k}\right)^k e^{(2-k)T\lambda_n}(\alpha_0^2 + 2\alpha_0 e^{\lambda_n T} + 1)^{-1} \\
&\le \alpha^{2-k}\sum_{n=1}^{\infty} b_n^2 \left(\frac{k}{2-k}\right)^k e^{(2-k)T\lambda_n} \\
&= \alpha^{2-k}\left(\frac{k}{2-k}\right)^k \sum_{n=1}^{\infty} b_n^2 e^{(2-k)T\lambda_n}.
\end{aligned}$$

If we choose $k = 2 - \epsilon$ we arrive at

$$\|u_\alpha(T) - f\|^2 \le \left(\frac{2}{\epsilon}\right)^2 \alpha^\epsilon \sum_{n=1}^{\infty} b_n^2 e^{\epsilon T\lambda_n} = c\alpha^\epsilon \epsilon^{-2}.$$

$\square$

If we assume that $\sum_{i=1}^{\infty} b_i^2 e^{(2+\epsilon)\lambda_i T}$ converges, working as above, we have that

$$\|u_\alpha(0) - u(0)\|^2 = \alpha^{2-k}\sum_{n=1}^{\infty} b_n^2 g_n(\alpha) e^{2T\lambda_n}$$

$$\leq \quad \alpha^{2-k} \sum_{n=1}^{\infty} b_n^2 \left(\frac{k}{2-k} \right)^k e^{(4-k)T\lambda_n} .$$

As above, letting $k = 2 - \epsilon$, we arrive at the following.

Corollary 1 *If* $f = \sum_{i=1}^{\infty} b_i \phi_i$ *is in* H *and there exists an* $\epsilon > 0$ *so that* $\sum_{i=1}^{\infty} b_i^2 e^{(2+\epsilon)\lambda_i T}$ *converges, then* $\|u_\alpha(t) - u(t)\|$ *converges to zero with order* $\alpha^\epsilon \epsilon^{-2}$ *uniformly in* t.

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