

A RADÓ TYPE THEOREM FOR p -HARMONIC FUNCTIONS IN THE PLANE

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ABSTRACT. We show that if $u \in C^1(\Omega)$ satisfies the p -Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in $\Omega \setminus \{x: u(x) = 0\}$, then u is a solution to the p -Laplacian in the whole $\Omega \subset \mathbb{R}^2$.

Throughout this paper we let Ω be an open set in \mathbb{R}^n , $n \geq 2$ and $1 < p < \infty$ a fixed number. The divergence form differential operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is called the p -Laplacian, and a continuous function $u \in W_{\text{loc}}^{1,p}(\Omega)$ is termed to be p -harmonic in Ω if

$$\Delta_p u = 0$$

in Ω in the sense of distributions.

In this note we prove the following theorem:

1. Theorem. *Let $n = 2$ and $u \in C^1(\Omega)$. If u is p -harmonic in $\Omega \setminus \{x: u(x) = 0\}$, then u is p -harmonic in Ω .*

Theorem 1 can be regarded as an extension of Radó's classical theorem on analytic functions that states that a continuous complex valued function is analytic as soon as it is analytic outside the preimage of a point. For ordinary harmonic functions (i.e. $p = 2$) in any dimension this extension is due to Král [5]. We also discuss the nonlinear case in higher dimensions, but so far we have not been able to treat it completely. Our argument in the plane relies on the intimate connection between quasiregular mappings and planar p -harmonic functions; therefore it cannot be generalized for higher dimensions nor for more general equations in the plane.

Clearly the conclusion of the theorem fails to hold if we only assumed that u be Lipschitz (for example, let u be the positive part of the first coordinate of

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x). Moreover, for general quasilinear equations with measurable coefficients the assumption that $u \in C^1(\Omega)$ does not make sense, for then solutions are usually only Hölder continuous. Removability of the zero sets was studied in [4] for more general nonlinear equations; there the focus was on the rate of the decay, not on the smoothness properties.

2. The singular set. We first show that in the plane the set where the gradient vanishes is removable for C^1 -smooth p -harmonic functions.

3. Proposition. *Let $n = 2$ and $u \in C^1(\Omega)$. If u is p -harmonic in*

$$\Omega \setminus \{x \in \Omega : \nabla u(x) = 0\},$$

then u is p -harmonic in Ω .

Proof. Let $E = \{x \in \Omega : \nabla u(x) = 0\}$. It is well known (see [1] or [7]) that the complex gradient $f = u_{x_1} - i u_{x_2} = (\partial_1 u, -\partial_2 u)$ of the p -harmonic function u is quasiregular in $\Omega \setminus E$, i.e. $f \in W_{\text{loc}}^{1,2}(\Omega \setminus E)$ and

$$\|f'(x)\|^2 \leq (p-1 + \frac{1}{p-1})J(x, f)$$

for a.e. $x \in \Omega \setminus E$; here $\|f'(x)\|$ is the sup-norm of the formal derivative $f'(x)$ and $J(x, f)$ is the Jacobian determinant of f . Since f is continuous in Ω , Radó's theorem for quasiregular mappings [4, 14.47] ensures us that f is indeed quasiregular in the whole Ω . Since nonconstant quasiregular mappings are discrete (see [6, Ch. IV] or [9, Thm II.6.3]), we are free to assume that E is a discrete set in Ω . In particular, the 1-dimensional measure of E is zero, whence E is removable for Lipschitz continuous p -harmonic functions by [2, 4.5]. Thus u is p -harmonic in Ω .

4. Remarks. a) If $1 < p \leq 2$, we can give a more elementary proof without appealing to the deep result on discreteness of quasiregular mappings. Indeed, the proof of Radó's theorem for quasiregular mappings reveals to us that the set E above is of 2-capacity zero, hence of p -capacity zero if $1 < p \leq 2$; thereby E is removable for bounded p -harmonic functions. The case $p > 2$ is different, for then not even single points are removable for bounded p -harmonic functions in general.

b) If $n \geq 3$ and $p \neq 2$, then it is not known whether the set $\{\nabla u = 0\}$ can contain interior points, even if we knew *a priori* that u is nonconstant and p -harmonic in Ω .

5. Nonvanishing gradient. Next we treat the case where the gradient of u does not vanish. It appears to be simpler than the general case and our proof works in all dimensions. We are going to employ a change of variables argument. For the reader's convenience we state a removability result for slightly more general equations of the p -Laplacian type:

6. Lemma. *Suppose that $\mathcal{A}: \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous mapping such that*

$$\mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p$$

with equivalence constants independent of x . If H is a hyperplane in \mathbb{R}^n and $u \in C^1(\Omega)$ is a function that satisfies the equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \quad (7)$$

in $\Omega \setminus H$, then u verifies equation (7) in Ω .

Lemma 6 was proven by Martio in [8, 2.22] for $p = n$, but his proof can be extended verbatim to cover all values of p .

We now return to our issue and show:

8. Proposition. *Suppose that $u \in C^1(\Omega)$ is such that $\nabla u \neq 0$ in Ω . If u is p -harmonic in $\Omega \setminus \{x: u(x) = 0\}$, then u is p -harmonic in Ω .*

Proof. Since the problem is local we may assume that $|\nabla u|$ is bounded away from zero in Ω . Hence the set

$$S = \{x \in \Omega: u(x) = 0\}$$

is a regular C^1 -hypersurface. By localizing further, if necessary, we find an open neighborhood G of S in Ω and a bilipschitz diffeomorphism g from G onto an open set V in \mathbb{R}^n such that S is mapped into the hyperplane

$$H = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n: x_n = 0\}.$$

If $f = g^{-1}$, then the pull-back of the p -Laplacian under f is $\operatorname{div} \mathcal{A} = 0$, where

$$\mathcal{A}(x, \xi) = J(x, f) f'(x)^{-1} |f'(x)^{-1*} \xi|^{p-2} f'(x)^{-1*} \xi;$$

here B^* stands for the transpose of the matrix B . Then it is easily checked that \mathcal{A} satisfies the assumptions of Lemma 6 (cf. [3, 14.90]). Moreover, we have that u is p -harmonic in G if and only if $v = u \circ f$ satisfies the equation

$$\operatorname{div} \mathcal{A}(x, \nabla v) = 0 \quad (9)$$

in V (see [3, 13.2 & 14.92]). Since v satisfies equation (9) in $V \setminus H$ we conclude from Lemma 6 that v is a solution of (9) in V . Consequently, u is p -harmonic in G and hence in Ω as desired.

Proof of Theorem 1. By Proposition 8 u is p -harmonic in

$$\Omega \setminus \{x: u(x) = 0 \text{ and } \nabla u(x) = 0\}.$$

Hence the theorem follows from Proposition 3.

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