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POSITIVE SOLUTIONS FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Solutions that are positive with respect to a cone are obtained for the boundary value problem, $u^{(n)} + a(t)f(u) = 0$, $u^{(i)}(0) = u^{(n-2)}(1) = 0$, $0 \le i \le n-2$, in the cases that f is either superlinear or sublinear. The methods involve application of a fixed point theorem for operators on a cone.

1. Introduction

We are concerned with the existence of solutions for the two-point boundary value problem,

$$u^{(n)} + a(t)f(u) = 0, \quad 0 < t < 1, \tag{1}$$

$$u^{(i)}(0) = u^{(n-2)}(1) = 0, \quad 0 \le i \le n-2, \tag{2}$$

where

(A) $f : [0,\infty) \to [0,\infty)$ is continuous, and
(B) $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval.

We remark that, if $u(t)$ is a nonnegative solution of (1), (2), then $u^{(n-2)}(t)$ is concave on $[0,1]$.

Specifically, our aim is to extend the work of Erbe and Wang [10] to obtain solutions of (1), (2), that are positive with respective to a cone, in the cases when, either (i) f is superlinear, or (ii) f is sublinear; that is, in the respective cases when, either (i) $f_0 = 0$ and $f_\infty = \infty$, or (ii) $f_0 = \infty$ and $f_\infty = 0$, where

$$f_0 = \lim_{x\to 0^+} \frac{f(x)}{x} \text{ and } f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}.$$

In the case that $n = 2$, the boundary value problem (1), (2) arises in applications involving nonlinear elliptic problems in annular regions; see [1], [2], [12], [19]. Applications of (1), (2) can also be made to singular boundary value problems as in [3], [6], [8], [13], [16], [18], as well as to extremal point characterizations for

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boundary value problems in [9], [14], [17]. In these applications, frequently, only solutions that are positive are useful. The results herein are also somewhat related to those obtained in [5] and [11].

Our arguments for establishing the existence of solutions of (1), (2) involve concavity properties of solutions that are used in defining a cone on which a positive integral operator is defined. A fixed point theorem due to Krasnosel'skii [15] is applied to yield a positive solution of (1), (2).

In Section 2, we present some properties of a Green's function which will be used in defining the positive operator. We also state the fixed point theorem from [15]. In Section 3, we provide an appropriate Banach space and cone in order to apply the fixed point theorem yielding solutions of (1), (2) in both the superlinear and sublinear cases.

2. Some preliminaries

In this section, we state a theorem due to Krasnosel'skii, an application of which will yield in the next section a positive solution of (1), (2). The mapping to which we apply this fixed point theorem will include an integral whose kernel, $G(t,s)$, is the Green's function for

$$\begin{gathered} -y^{(n)} = 0, \\ y^{(i)}(0) = y^{(n-2)}(1) = 0, \quad 0 \le i \le n-2. \end{gathered} \tag{3}$$

Eloe [7] has shown that, for $0 \le i \le n-2$,

$$\frac{\partial^i}{\partial t^i} G(t,s) > 0 \text{ on } (0,1) \times (0,1), \tag{4}$$

as well as the fact that the function

$$K(t,s) = \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) \tag{5}$$

is the Green's function for

$$\begin{gathered} -y'' = 0, \\ y(0) = y(1) = 0. \end{gathered} \tag{6}$$

We note that

$$K(t,s) = \begin{cases} t(1-s), & 0 \le t < s \le 1, \\ s(1-t), & 0 \le s < t \le 1, \end{cases} \tag{7}$$

from which it is straightforward that

$$K(t,s) \le K(s,s), 0 \le t, s \le 1, \tag{8}$$

and a nice argument in [10] shows that

$$K(t,s) \ge \frac{1}{4} K(s,s),\ \frac{1}{4} \le t \le \frac{3}{4},\ 0 \le s \le 1. \tag{9}$$

The existence of solutions of (1), (2) is based on an application of the following fixed point theorem [15].

Theorem 1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then T has a fixed point in $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3. Existence of solutions

We are now ready to apply Theorem 1. We remark that $u(t)$ is a solution of (1), (2) if, and only if,

$$u(t) = \int_0^1 G(t,s)a(s)f(u(s))ds, \quad 0 \le t \le 1.$$

For our construction, we let

$$\mathcal{B} = \{x \in C^{(n-2)}[0,1] \mid x^{(i)}(0) = 0, \quad 0 \le i \le n-3\},$$

with norm, $\|x\| = |x^{(n-2)}|_\infty$, where $|\cdot|_\infty$ denotes the supremum norm on $[0,1]$. Then $(\mathcal{B}, \|\cdot\|)$ is a Banach space.

Remark 1. *We note that, for each $x \in \mathcal{B}$,*

$$|x^{(i)}|_\infty \le \|x\|, \quad 0 \le i \le n-2. \tag{10}$$

We will seek solutions of (1), (2) which lie in a cone, $\mathcal{P}$, defined by

$$\mathcal{P} = \{x \in \mathcal{B} \mid x^{(n-2)}(t) \ge 0 \text{ on } [0,1], \text{ and } \min_{\frac14 \le t \le \frac34} x^{(n-2)}(t) \ge \frac{1}{4}\|x\|\}.$$

Remark 2. *We note here that, if $x \in \mathcal{P}$, then $x^{(i)}(t) \ge 0$ on $[0,1]$ and*

$$x^{(i)}(t) \ge \frac{1}{4}\|x\| \frac{(t-\frac14)^{n-i-2}}{(n-i-2)!}$$

on $[\frac14, \frac34]$, $0 \le i \le n-2$. As a consequence

$$x^{(i)}(t) \ge \frac{1}{(n-i-2)!4^{n-i-1}}\|x\|$$

on $[\frac12, \frac34]$, $0 \le i \le n-2$.

Theorem 2. *Assume that conditions (A) and (B) are satisfied. If, either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(i.e., f is superlinear), or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(i.e., f is sublinear),*

then (1), (2) has at least one solution in $\mathcal{P}$.

Proof. We begin by defining an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(u(s))\,ds, \quad u \in \mathcal{P}, \tag{11}$$

and we seek a fixed point of T in the cone $\mathcal{P}$ for the respective cases of f superlinear and f sublinear.

Before dealing with these cases, we make a few observations. First, if $u \in \mathcal{P}$, it follows from (8) that

$$\begin{aligned} (Tu)^{(n-2)}(t) &= \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)a(s)f(u(s))\,ds \\ &= \int_0^1 K(t,s)a(s)f(u(s))\,ds \\ &\le \int_0^1 K(s,s)a(s)f(u(s))\,ds, \end{aligned}$$

so that

$$\|Tu\| = |(Tu)^{(n-2)}|_\infty \le \int_0^1 K(s,s)a(s)f(u(s))\,ds.$$

In fact,

$$\|Tu\| = \int_0^1 K(s,s)a(s)f(u(s))\,ds. \tag{12}$$

Next, if $u \in \mathcal{P}$, it follows from (9) and (12) that

$$\begin{aligned} \min_{\frac14 \le t \le \frac34} (Tu)^{(n-2)}(t) &= \min_{\frac14 \le t \le \frac34} \int_0^1 K(t,s)a(s)f(u(s))\,ds \\ &\ge \frac14 \int_0^1 K(s,s)a(s)f(u(s))\,ds \\ &\ge \frac14 \|Tu\|. \end{aligned}$$

Moreover, properties of $G(t,s)$ give that $(Tu)^{(n-2)}(t) \ge 0$, so that $Tu \in \mathcal{P}$, and in particular $T : \mathcal{P} \to \mathcal{P}$. Also, the standard arguments yield that T is completely continuous.

We now turn to the cases of the theorem.

(i) Assume $f_0 = 0$ and $f_\infty = \infty$. First, dealing with $f_0 = 0$, there exist $\eta > 0$ and $H_1 > 0$ such that $f(x) \le \eta x$, for $0 < x \le H_1$, and

$$\eta \int_0^1 K(s,s)a(s)\,ds \le 1.$$

So, if we choose $u \in \mathcal{P}$ with $\|u\| = H_1$, and if we recall from Remark 1 that $|u|_\infty \le \|u\|$, we have from (8),

$$\begin{aligned}(Tu)^{(n-2)}(t) &= \int_0^1 K(t,s)a(s)f(u(s))\,ds\\ &\le \int_0^1 K(s,s)a(s)f(u(s))\,ds\\ &\le \int_0^1 K(s,s)a(s)\eta u(s)\,ds\\ &\le \eta\int_0^1 K(s,s)a(s)\,ds\|u\|\\ &\le \|u\|,\quad 0\le t\le 1.\end{aligned}$$

As a consequence $\|Tu\| = |(Tu)^{(n-2)}(t)|_\infty \le \|u\|$. Thus, if we set

$$\Omega_1 = \{x\in\mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u\in\mathcal{P}\cap\partial\Omega_1. \tag{13}$$

Next, dealing with $f_\infty = \infty$, there exist $\lambda > 0$ and $\bar{H}_2 > 0$ such that $f(x) \ge \lambda x$, for $x \ge \bar{H}_2$, and

$$\frac{\lambda}{(n-2)!4^{n-1}}\int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2},s)a(s)\,ds \ge 1.$$

Now, let $H_2 = \max\{2H_1, (n-2)!4^{n-1}\bar{H}_2\}$ and set

$$\Omega_2 = \{x\in\mathcal{B} \mid \|x\| < H_2\}.$$

So, if $u\in\mathcal{P}$ with $\|u\| = H_2$, and if we recall from Remark 2 that $u(t) \ge \frac{1}{(n-2)!4^{n-1}}\|u\| \ge \bar{H}_2$ on $[\frac{1}{2},\frac{3}{4}]$, we have

$$\begin{aligned}(Tu)^{(n-2)}(\frac{1}{2}) &= \int_0^1 K(\frac{1}{2},s)a(s)f(u(s))\,ds\\ &\ge \int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2},s)a(s)\lambda u(s)\,ds\\ &\ge \lambda\int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2},s)a(s)\frac{1}{(n-2)!4^{n-1}}\|u\|\,ds\\ &= \frac{\lambda}{(n-2)!4^{n-1}}\int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2},s)a(s)\,ds\|u\|\\ &\ge \|u\|,\end{aligned}$$

so that $\|Tu\| \ge \|u\|$. Consequently,

$$\|Tu\| \ge \|u\|, \text{ for } u\in\mathcal{P}\cap\partial\Omega_2. \tag{14}$$

Therefore, by part (i) of Theorem 1 applied to (13) and (14), T has a fixed point $u(t)\in\mathcal{P}\cap(\bar{\Omega}_2\backslash\Omega_1)$ such that $H_1 \le \|u\| \le H_2$, and as such, $u(t)$ is a desired solution of (1), (2). (We remark that the arguments carry through, if we had set $H_2 = \max\{H_1, (n-2)!4^{n-1}\bar{H}_2\}$ and if $H_2 = H_1$, then there is a solution $u \in \mathcal{P}$ with $\|u\| = H_1$.) This completes the case when f is superlinear.

(ii) Now, assume $f_0 = \infty$ and $f_\infty = 0$. Dealing with $f_0 = \infty$, there exist $\bar{\eta} > 0$ and $J_1 > 0$ such that $f(x) \geq \bar{\eta} x$, for $0 < x \leq J_1$, and

$$\frac{\bar{\eta}}{(n-2)!4^{n-1}} \int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2}, s)a(s)\, ds \geq 1.$$

This time, we choose $u \in \mathcal{P}$ with $\|u\| = J_1$. Since $|u|_\infty \leq \|u\| = J_1$, we have $f(u(s)) \geq \bar{\eta} u(s)$, $0 \leq s \leq 1$. Also, we know $u(s) \geq \frac{1}{(n-2!)4^{n-1}} \|u\|$, $\frac{1}{2} \leq s \leq \frac{3}{4}$. Thus,

$$\begin{aligned}
(Tu)^{(n-2)}(\frac{1}{2}) &= \int_0^1 K(\frac{1}{2}, s)a(s)f(u(s))\, ds \\
&\geq \int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2}, s)a(s)\bar{\eta} u(s)\, ds \\
&\geq \frac{\bar{\eta}}{(n-2)!4^{n-1}} \int_{\frac{1}{2}}^{\frac{3}{4}} K(\frac{1}{2}, s)a(s)\, ds \|u\| \\
&\geq \|u\|,
\end{aligned}$$

and in particular, $\|Tu\| \geq \|u\|$. Setting

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < J_1\},$$

we conclude

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{15}$$

For the final part of this case, we deal with $f_\infty = 0$. There exist $\bar{\lambda} > 0$ and $\bar{J}_2 > 0$ such that, $f(x) \leq \bar{\lambda} x$, for $x \geq \bar{J}_2$, and

$$\bar{\lambda} \int_0^1 K(s, s)a(s)\, ds \leq 1.$$

There are two further sub-cases to be considered:

(I) We suppose first that f is bounded. Then, there exists $N > 0$ such that $f(x) \leq N$, for all $0 < x < \infty$. Let $J_2 = \max\{2J_1, N \int_0^1 K(s, s)a(s)\, ds\}$. Then, for $u \in \mathcal{P}$ with $\|u\| = J_2$, since $|u|_\infty \leq \|u\|$ and $K(t, s) \leq K(s, s)$, $0 \leq s, t \leq 1$, we have

$$\begin{aligned}
(Tu)^{(n-2)}(t) &= \int_0^1 K(t, s)a(s)f(u(s))\, ds \\
&\leq N \int_0^1 K(s, s)a(s)\, ds \\
&\leq J_2 \\
&= \|u\|, \quad 0 \leq t \leq 1.
\end{aligned}$$

Consequently, $\|Tu\| \leq \|u\|$.

(II) For the second sub-case, suppose that f is unbounded. Then, there exists $J_2 > \max\{2J_1, \bar{J}_2\}$ such that $f(x) \leq f(J_2)$, for $0 < x \leq J_2$. We now choose $u \in \mathcal{P}$ with $\|u\| = J_2$. Again, recalling $|u|_\infty \leq \|u\|$ and $K(t, s) \leq K(s, s)$ leads to

$$\begin{aligned}(Tu)^{(n-2)}(t) &= \int_0^1 K(t,s)a(s)f(u(s))\,ds \\ &\le \int_0^1 K(s,s)a(s)f(J_2)\,ds \\ &\le \bar{\lambda}\int_0^1 K(s,s)a(s)\,ds J_2 \\ &\le \|u\|, \quad 0 \le t \le 1.\end{aligned}$$

Thus, $\|Tu\| \le \|u\|$.

We conclude from each sub-case, (I) and (II), if we set

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < J_2\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{16}$$

Therefore, by part (ii) of Theorem 1 applied to (15) and (16), T has a fixed point $u(t) \in \mathcal{P} \cap (\Omega_2 \backslash \Omega_1)$ such that $J_1 \le \|u\| \le J_2$, and $u(t)$ is a sought solution of (1), (2). This completes the argument for the case of f sublinear.

The proof is complete. □

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