Electronic Journal of Differential Equations
Vol. **1995**(1995), No. 04, pp. 1-5. Published April 3, 1995.
ISSN 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

THE HARNACK INEQUALITY FOR ∞-HARMONIC FUNCTIONS

Peter Lindqvist
and
Juan J. Manfredi

Abstract. The Harnack inequality for nonnegative viscosity solutions of the equation $\Delta_\infty u = 0$ is proved, extending a previous result of L.C. Evans for smooth solutions. The method of proof consists in considering $\Delta_\infty u = 0$ as the limit as $p \to \infty$ of the more familiar p-harmonic equation $\Delta_p u = 0$.

The purpose of this note is to present a proof of the Harnack inequality for nonnegative viscosity solutions of the $\infty-$harmonic equation

$$\sum_{i=1,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \tag{1}$$

where $u = u(x_1, \cdots, x_n)$. For classical C^2-solutions this has recently been obtained by Evans, see [E]. While Evans works directly with equation (1), we approximate it by the $p-$harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{2}$$

and let $p \to \infty$. (See [A], [K], and [BDMB] for background and information about the ∞-Laplacian.)

The Harnack inequality for nonnegative $p-$harmonic functions can be proved by the now standard iteration methods of DeGiorgi and Moser, see [S] and [DB-T]. Unfortunately, in both of these methods the Harnack constants blow up as $p \to \infty$. Another approach to the Harnack inequality, valid only when $p > n$, follows from energy bounds for $\nabla(\log u)$, see [M] and [KMV]. We begin with a well known estimate:

1991 Mathematics Subject Classifications: 35J70, 26A16.
Key words and phrases: Harnack inequality, p-harmonic equations.

Submitted: February 22, 1995.

Lemma. *Suppose that u_p is a nonnegative weak solution of (2) in a domain $\Omega \subset \mathbb{R}^n$. Then, we have*

$$\int_\Omega |\zeta\, \nabla \log u_p|^p dx \leqq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla\zeta|^p dx \tag{3}$$

whenever $\zeta \in C_0^\infty(\Omega)$.

Proof. We may assume that $u_p > 0$. (Consider $u_p(x) + \varepsilon$ and let $\varepsilon \to 0^+$.) Use the test function $|\zeta|^p u_p^{1-p}$ in the weak formulation of (2). This simple calculation is given in [L, Corollary 3.8]. □

Our main result states that one can take the limit as $p \to \infty$ in (3).

Theorem. *Suppose that u is a nonnegative viscosity solution of (1) in a domain $\Omega \subset \mathbb{R}^n$. Then we have*

$$\|\zeta\, \nabla \log u\|_{\infty,\Omega} \;\leqq\; \|\nabla\zeta\|_{\infty,\Omega} \tag{4}$$

whenever $\zeta \in C_0^\infty(\Omega)$.

Proof. Select a bounded smooth domain D such that

$$\operatorname{supp}\zeta \subset D \subset \overline{D} \subset \Omega.$$

By a fundamental result of Jensen $u \in W^{1,\infty}(D)$ and it is the unique viscosity solution of (1) with boundary values $u|_{\partial D}$. For these results and the definition of viscosity solutions we refer to [J].

For $p > n$ let u_p be the solution to the problem

$$\begin{cases} \operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0 & \text{in } D \\ u_p - u \in W_0^{1,p}(D). \end{cases}$$

By the results of [BDBM, Section I], there exists a sequence $p_j \to \infty$ such that u_{p_j} tends to a viscosity solution v of (1) in $C^\alpha(\overline{D})$ for any $\alpha \in [0,1)$ and weakly in $W^{1,m}(D)$ for any finite m. Since u and v have the same boundary values, the uniqueness theorem of Jensen [J] implies that $u \equiv v$. Note, in addition, that any other subsequence of u_p has a subsequence converging to a viscosity solution of (1) and that this limit is u. We conclude that

$$u_p \to u \qquad \text{in} \qquad C^\alpha(\overline{D}) \qquad \text{for any} \quad \alpha \in [0,1) \tag{5}$$

and

$$u_p \rightharpoonup u \qquad \text{in} \qquad W^{1,m}(D) \qquad \text{for any finite} \quad m \tag{6}$$

as $p \to \infty$.

Fix $m \geq n$ and consider $p > m$. We have

$$\begin{aligned} \int_D |\zeta\, \nabla \log u_p|^m dx &\leqq \Big(\int_D |\ \zeta\, \nabla \log u_p\ |^p dx\Big)^{m/p} |D|^{(p-m)/p} \\ &\leqq \left(\frac{p}{p-1}\right)^m \Big(\int_D |\nabla\zeta|^p\, dx\Big)^{m/p} |D|^{(p-m)/p}, \end{aligned}$$

where we have used the Lemma in the second inequality. Therefore, we get

$$\Big(\int_D |\zeta\,\nabla\log u_p|^m dx\Big)^{1/m} \leqq \frac{p}{p-1}\Big(\int_D |\nabla\zeta|^p dx\Big)^{1/p}|D|^{(p-m)/pm}. \tag{7}$$

Assume momentarily that $\zeta\,\nabla\log u_p$ converges weakly to $\zeta\,\nabla\log u$ in $L^m(D)$. By the weak lower semi-continuity of the norm we obtain

$$\Big(\int_D |\zeta\,\nabla\log u\ |^m dx\Big)^{1/m} \leqq\ \|\nabla\zeta\|_{\infty,D}|D|^{1/m}. \tag{8}$$

Observe that (7) holds for the translated functions $u_p(x)+\varepsilon$, where $\varepsilon > 0$ is fixed, in place of u_p. Since these functions are bounded away from zero, it is elementary to check that $\zeta\,\nabla\log(u_p+\varepsilon)$ converges weakly to $\zeta\,\nabla\log(u+\varepsilon)$ in $L^m(D)$. It now follows from (5) and (6) that estimate (8) holds for $u(x)+\varepsilon$.

We now let $\varepsilon\to 0$. By the Monotone Convergence theorem, we obtain estimate (8) for u.

Finally, letting $m\to\infty$ we finish the proof of (4). □

If B_r and R_R are two concentric balls in Ω with radius r and R, the usual choice of a radial test function ζ $(0\leqq\zeta\leqq 1,\quad \zeta=1$ in $B_r,\quad \zeta=0$ outside $B_R)$ in (4) yields the estimate

$$\|\nabla\log u\|_{\infty,B_r}\leqq\frac{1}{R-r} \tag{11}$$

provided that $B_R\subset\Omega$. In particular, we obtain the following result.

Corollary 1. *(a) If u is a nonnegative viscosity solution of (1) in a domain $\Omega\subset\mathbb{R}^n$, then for a. e. $x\in\Omega$*

$$|\nabla u(x)|\leqq\frac{u(x)}{d(x,\partial\Omega)}. \tag{12}$$

(b) If u is a bounded viscosity solution of (1) in a domain $\Omega\subset\mathbb{R}^n$, then for a. e. $x\in\Omega$ we have

$$|\nabla u(x)|\leqq\frac{2\|u\|_\infty}{d(x,\partial\Omega)}. \tag{13}$$

Proof. It remains to consider only the second case, which follows from the first by considering $v=u+\|u\|_\infty$. □

Next, we state the Harnack inequality, which follows from (11).

Corollary 2. *Suppose that u is a nonnegative viscosity solution of (1) in $B_R(x_0)$. Then if $x,y\in B_r(x_0), 0\leqq r<R$, we have*

$$u(x)\leqq e^{|x-y|/(R-r)}u(y). \tag{14}$$

Proof. By integrating (11) on a line segment from x to y we obtain

$$|\log u(x)-\log u(y)|\leq\frac{|x-y|}{R-r},$$

from which (14) follows by exponentiating. □

Remarks.

§1. The Lemma holds for nonnegative super-solutions of the $p-$Laplacian by exactly the same proof. Thus for $p > n$ we get an estimate like (10) with m replaced by p, from which a Harnack inequality follows easily. This suggests the possibility that corollary 2 holds, indeed, for nonnegative viscosity super-solutions of (1).

§2. If one uses the estimate in [L, (4.10)]

$$\int_\Omega |\nabla u_p|^p u_p^{-1-\varepsilon}\zeta^p dx \leqq \left(\frac{p}{\varepsilon}\right)^p \int_\Omega u_p^{p-1-\varepsilon}|\nabla\zeta|^p dx$$

where $0 < \varepsilon < p-1$ instead of (3), we obtain the estimate

$$\|\zeta u^{-\alpha}\nabla u\|_{\infty,\Omega} \leqq \frac{1}{\alpha}\|u^{1-\alpha}\nabla\zeta\|_{\infty,\Omega}$$

for any $\alpha > 0$ and for any nonnegative viscosity solution u of (1) in Ω. Roughly speaking, estimates for the $p-$Laplacian that are independent of p, always yield estimates for $\infty-$harmonic functions.

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Department of Mathematics
Norwegian Institute of Technology
N-7034 Trondheim
Norway
E-mail address: lqvist@imf.unit.no

Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260
USA
E-mail address: juanjo@na0b.math.pitt.edu