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PICONE'S IDENTITY AND THE MOVING PLANE PROCEDURE

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Abstract. Positive solutions of a class of nonlinear elliptic partial differential equations are shown to be symmetric by means of the moving plane argument coupled with Spectral Theory results and Picone's Identity. The method adapts easily to situations where the moving plane procedure gives rise to variational problems with positive eigenfunctions.

0. Introduction

Consider the problem:

$$\begin{aligned} -\Delta u &= \lambda p(x) g(u) \qquad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\quad\ \ \text{on} \quad \partial\Omega \end{aligned} \tag{1}$$

where Ω is a cylinder in R^n : $\Omega = (-1,1) \times \Omega'$ with Ω' a domain (= bounded, open, connected set) of R^{n-1}. Are all $C^2(\overline{\Omega})$ positive solutions symmetric in x_1? If the boundary of Ω' is reasonably smooth, then under suitable conditions on p, g the classic moving plane argument of Serrin, [21], as extended by Gidas, Ni, Nirenberg in [14], [15], Berestycki and Nirenberg, [5], Amick and Fraenkel, [3], and elsewhere, [8], [18], [25], applies and the answer is positive. More recently, [6], Berestycki and Nirenberg showed that the result is still true for general nonlinear equations if, for example, $u \in W^{2,n}_{\ell oc}(\Omega) \cap C(\overline{\Omega})$ with no regularity assumed on $\partial\Omega$, while Dancer, [11], dealt with the case of $u \in H^{1,2}_0(\Omega) \cap L^\infty(\Omega)$.

It is the purpose of this paper to discuss the symmetry of positive solutions under somewhat weaker conditions than those that to the best of our knowledge have been applied earlier. We avoid direct use of pointwise considerations near $\partial\Omega$ by employing related arguments from Spectral Theory and Picone's Identity. In this way, we are able in particular to bypass various "corner Lemma" and Maximum Principle procedures. Most of the paper is devoted to the problem

$$\begin{aligned} -\Delta u &= f(u) \qquad \text{in} \quad \Omega \\ u &= 0 \qquad\quad\ \ \text{on} \quad \partial\Omega \end{aligned} \tag{2}$$

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with Ω bounded, $\Omega \subset R^n$ (we consider explictly the case $n \geq 3$), or $\Omega = R^n$ itself since these cases illustrate all the ideas. While our approaches are different, our results are closest to those obtained by Dancer, [11]. We feel that one possible advantage of our approach is that it adapts easily to variational problems with positive eigenfunctions. This is illustrated explicitly for the case of $\Omega = R^n$, and at the end we indicate some extensions to more general problems. We were unable to extend our approach to the more general nonlinear equations considered in [6]. Our results in particular show:

Theorem 0. *Let $0 < p(x) \in L^\alpha(\Omega)$, $\alpha > n/2$, symmetric in x_1. Assume p is nonincreasing in x_1 for $x_1 \geq 0$ and: $g \in C^{1+\theta}_{\ell oc}$, $g(\xi) > 0$ for $\xi \geq 0$.*

(a) *For λ small enough, Problem 1 has a bounded (i.e. L^∞) positive solution.*

(b) *All bounded positive solutions to (1) are symmetric with respect to x_1, and if moreover $p \in L^{n+\varepsilon}(\Omega)$ then $\frac{\partial u}{\partial x_1} < 0$ for $0 < x_1 < 1$.*

By a solution in Theorem 0 – and throughout the paper – we mean at least a function $u \in H^{1,2}_0(\Omega) \cap C^\theta(K)$ for any $K \Subset \Omega$, $\theta = \theta(K)$, which satisfies the problem in the weak sense, i.e. $B(u, v) = \big(f(u), v\big)$ for all $v \in H^{1,2}_0(\Omega)$ where B denotes the form associated with $-\Delta$. We do not usually ask that $u \in C(\overline{\Omega})$. Our approach instead requires a variational linear structure and some higher integrability properties of u. The latter, in at least some cases, may be obtained either from the equation u satisfies or from a-priori estimates used in finding u. Heuristically, what we do is related to the sufficient condition (ii) of page 4 of [6] and to [7] but without the need for $u \in C[\overline{\Omega}]$ or other specified behaviour at $\partial\Omega$, [7], thanks to the variational structure of the linearized problem. The regularity of $\partial\Omega$ is also irrelevant for most of our procedures, although we assume in most of the arguments that the domains obtained by reflecting about the moving planes are contained in Ω. In particular, this means that Ω cannot shrink as we move in from the boundary by moving the planes, nor can Ω have any "holes" in the regions involved in the reflection process. If f is not smooth, some assumptions on $\partial\Omega$ and/or u are added to ensure that now, $u \in C(\overline{\Omega})$, while for some f we show by the same methods, the symmetry and uniqueness of positive solutions without reference to the moving plane procedure.

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1. Preliminary Considerations

As mentioned in the Introduction, our procedure involves Spectral Theory and Picone's Identity. We begin therefore by considering a family of linear eigenvalue problems. Observe that we deal with functions in $H^{1,2}_0(\Omega)$ in what follows. Without loss of generality we may assume they are defined in the whole of R^n by means of the trivial extension. Given any function g, we set: $g^+ = \max\,(g, 0)$; $g^- = \max\,(-g, 0)$.

Let $\{\Omega_\lambda\}_{\lambda \in [a,b]}$ be a family of bounded open sets in R^n, χ_λ their characteristic functions, and assume $p_\lambda : [a, b] \to L^\alpha(\Omega_\lambda)$. That is: $\chi_\lambda p_\lambda \in L^\alpha(R^n)$, for $\lambda \in [a, b]$ and some fixed $\alpha > n/2$. We then have:

Theorem 1.

(a) *For each* $\lambda \in [a,b]$, *the map* $\ell_\lambda : H_0^{1,2}(\Omega_\lambda) \to L^2(\Omega_\lambda)$ *formally given by* $\ell_\lambda = -\Delta - p_\lambda I$ *has a least eigenvalue* $\mu_0(\lambda)$ *to which corresponds a nonnegative eigenfunction* $u_0(\lambda)$, *positive in at least one of the components of* Ω_λ.

(b) *For any* $K \Subset \Omega_\lambda$ *there exists* $\delta > 0$ *such that* $u_0(\lambda) \in C^\delta(K)$.

(c) *There exists a function* $\omega \in \big(H^{1,2}(\widetilde{\Omega}_\lambda) - H_0^{1,2}(\widetilde{\Omega}_\lambda)\big) \cap C(\widetilde{\Omega}_\lambda)$ *with* $\omega \geq 0$ *and* $\ell_\lambda(\omega) \geq 0$ a.e. $\widetilde{\Omega}_\lambda$ *for any component* $\widetilde{\Omega}_\lambda$ *of* Ω_λ *iff* $\mu_0(\lambda) > 0$.

(d) $\mu_0(\lambda) = 0$ *iff there exists at least one* $\omega \in H^{1,2}(\Omega_\lambda) \cap C(\Omega_\lambda)$, *with* $\omega \geq 0$ *and* $\ell_\lambda(\omega) \geq 0$ a.e., *nontrivial in each component of* Ω_λ, *and, for any such* ω *there exists in each component of* Ω_λ *a constant* $c \geq 0$ *with* $c\omega = u_0(\lambda)$.

(e) *If* $\|p_\lambda\|_{L^\alpha}$ *is bounded, there exists a constant* C_0 *such that if* meas $(\Omega_\lambda) < C_0$ *then* $\mu_0(\lambda) > 0$.

Proof. (a) If need be we add a positive constant to p_λ and observe that ℓ_λ^{-1} defines a compact self-adjoint map $L^2(\Omega_\lambda) \to L^2(\Omega_\lambda)$ by Sobolev's Theorem since $\alpha > n/2$, [16]. The existence of the least eigenvalue $\mu_0(\lambda)$ then follows. That the associated eigenfunction $u_0(\lambda)$ has the desired positivity properties is immediate, since by the Courant min.-max. characterization of $\mu_0(\lambda)$ we conclude that $u_0(\lambda) \geq 0$, [16], and indeed it follows that $u_0(\lambda) > 0$ in at least one component by the weak Harnack Inequality, [16].

(b) This is given in [16].

(c) It is here that we employ Picone's Identity. Once again by the weak Harnack Inequality, $\omega > 0$ in Ω_λ since ω is assumed nontrivial in each component. Let $\varphi \in C_0^\infty(\Omega_\lambda)$, $\varepsilon > 0$. We recall Picone's Identity, see eg. [2]:

$$\begin{aligned}\int_{\Omega_\lambda} (\omega+\varepsilon)^2 \big[\nabla[\frac{\varphi}{\omega+\varepsilon}]\big]^2 &= \int_{\Omega_\lambda} \big[|\nabla\varphi|^2 - p_\lambda \varphi^2\big] \\ &\quad - \int_{\Omega_\lambda} \big[\nabla(\frac{\varphi^2}{\omega+\varepsilon})\nabla(\omega) - \frac{p_\lambda \varphi^2(\omega+\varepsilon)}{\omega+\varepsilon}\big] \\ &\leq \int_{\Omega_\lambda} \big[|\nabla\varphi|^2 - p_\lambda\varphi^2\big] + \int_{\Omega_\lambda} \frac{p_\lambda \varepsilon \varphi^2}{\omega+\varepsilon}\,.\end{aligned}$$

Since $|\varepsilon/(\omega+\varepsilon)| \leq 1$, and $\varepsilon/(\omega+\varepsilon) \to 0$ pointwise, we let $\varepsilon \to 0$, apply Lebesgue's Dominated Convergence Theorem, followed by letting $\varphi \to u_0(\lambda)$ in $H_0^{1,2}$ to conclude that either in each component $\widetilde{\Omega}_\lambda$ we have $\widetilde{c}\omega = u_0(\lambda)$ for some constant $\widetilde{c} \geq 0$ or else $\mu_0(\lambda) > 0$. The first case is impossible since $\omega \notin H_0^{1,2}(\widetilde{\Omega}_\lambda)$ for any $\widetilde{\Omega}_\lambda$, and we conclude $\mu_o(\lambda) > 0$. On the other hand, if $\mu_0(\lambda) > 0$ then $\ell_\lambda \eta = p_\lambda$ has a solution $\eta \in H_0^{1,2}(\Omega_\lambda)$. Set $\omega = \eta + 1$ then $\ell_\lambda(\omega) = 0$ and Courant's min.-max. principle shows that if ω is nontrivial in a component then $\omega^- = 0$ a.e., whence $\omega > 0$ (again by Harnack's inequality). Finally $\omega \in H^{1,2}(\widetilde{\Omega}_\lambda) - H_0^{1,2}(\widetilde{\Omega}_\lambda)$ for otherwise $1 = \omega - \eta \in H_0^{1,2}(\widetilde{\Omega}_\lambda)$. Since an equivalent norm on $H_0^{1,2}(\widetilde{\Omega}_\lambda)$ is $(\int_{\widetilde{\Omega}_\lambda} |\nabla u|^2)^{1/2}$ then C meas $(\widetilde{\Omega}_\lambda) \leq \|1\|^2_{H^{1,2}(\widetilde{\Omega}_\lambda)} = 0$ and the result follows.

(d) If $\mu_0(\lambda) = 0$ then for any such ω we conclude $c\omega = u_0(\lambda)$ by part (c). Observe that we may always construct at least one such ω by using the eigenfunction itself in some components and solving $\ell_\lambda(\omega) = 1$ in others. Conversely, since such a ω exists then $\mu_0(\lambda) \geq 0$ by part (c). On the other hand, if $\mu_0(\lambda) > 0$ then part (c) shows there exists a ω, as desired, with $\omega \in H^{1,2}(\widetilde{\Omega}_\lambda) - H_0^{1,2}(\widetilde{\Omega}_\lambda)$ whence $c\omega \neq u_0(\lambda)$ in at least one component.

(e) We need only apply Sobolev's Estimate, [16] to obtain:

$$\int_{\Omega_\lambda} |\nabla\varphi|^2 - p_\lambda(x)\varphi^2 \geq \int_{\Omega_\lambda} |\nabla\varphi|^2[1 - K\|p_\lambda\|_{L^\alpha} \text{ (meas } (\Omega_\lambda)^\beta)]$$

where $\beta = \frac{2\alpha-n}{\alpha n}$, and $\varphi \in C_0^\infty(\Omega_\lambda)$, and observe that $\|p_\lambda\|_{L^\alpha}$ is bounded.

As a consequence we have:

Corollary 2. *If $\mu_0 \in C(a,b)$ with $\mu_0(\lambda) > 0$ for all $b - \lambda$ small enough, and for each λ there exists a $\omega \geq 0$, dependent on λ, in each $H^{1,2}(\widetilde{\Omega}_\lambda) - H_0^{1,2}(\widetilde{\Omega}_\lambda)$ such that $\ell_\lambda(\omega) \geq 0$, then $\mu_0(\lambda) > 0$ for all $\lambda \in (a,b)$.*

Otherwise there would exist λ_0 with $\mu_0(\lambda_0) = 0$. The existence of such a ω then contradicts Theorem 1(d).

The continuity of μ_0 – indeed of the entire spectrum – is a classical problem, discussed by Courant and Hilbert, [10], and studied more recently in a variety of papers: [4,13,20,26]. Based on these results we have:

Theorem 3. *Let $\Omega^*_\lambda = \Omega \cap \{x_1 > \lambda\} \neq \emptyset$ for $\lambda \in [a,b]$ and let $\ell^*_\lambda = -\Delta u - P_\lambda u$ defined on $H_0^{1,2}(\Omega^*_\lambda)$ where $P \in C([a,b];\ L^\alpha(\Omega^*_\lambda))$ for some $\alpha > n/2$, and Ω is a fixed bounded domain. Then $\mu^*_0(\lambda)$, the least eigenvalue of ℓ^*_λ in Ω^*_λ, is continuous. If* meas (Ω^*_λ) *is small enough, $\mu^*_0(\lambda) > 0$.*

By $P \in C\big([a,b];\ L^\alpha(\Omega^*_\lambda)\big)$ we mean $P_\lambda \in L^\alpha(R^n)$ and $\chi_{\lambda^*}P_\lambda \in C\big([a,b];\ L^\alpha(R^n)\big)$ where χ_{λ^*} denotes the characteristic function of Ω_{λ^*}.

Proof. Observe that these are nested domains. Choose λ_0 in the interval of interest, and without loss of generality pass to a subsequence and suppose first $\lambda_m \downarrow \lambda_0$, with $\delta = \lim\ \big(\mu^*_0(\lambda_m)\big)$. Note that $\mu^*_0(\lambda_m)$ are bounded since $\alpha > n/2$. Let $\{\omega_m\}$ be associated, normalized in L^2, eigenfunctions and observe that $\omega_m \geq 0$ $\quad(\omega_m > 0$ if Ω^*_λ is connected). By the trivial extension, $\|\omega_m\|_{L^2(\Omega^*_{\lambda_0})} = 1$ and $\int_{\Omega^*_{\lambda_0}} P_{\lambda_m}\omega_m^2$ can be estimated. To see this, note that:

$$|\int_{\Omega^*_{\lambda_0}} P_{\lambda_m}\omega_m^2| = |\int_{\Omega^*_{\lambda_0}} P_{\lambda_m}\omega_m^\varepsilon\omega_m^{2-\varepsilon}| \leq K\|P_{\lambda_m}\|_{L^\alpha}\|\omega_m\|^\varepsilon_{L^2}\|\nabla\omega_m\|^{2-\varepsilon}_{L^2}$$

where $\varepsilon = 2 - (n/\alpha)$. Whence:

$$\|\nabla\omega_m\|^2_{L^2} \leq K_1 + K_2\|\nabla\omega_m\|^{2-\varepsilon}_{L^2}.$$

We conclude that $\|\omega_m\|_{H_0^{1,2}(\Omega^*_{\lambda_0})} \sim \|\nabla\omega_m\|_{L^2(\Omega^*_{\lambda_0})}$ is bounded. Passing to a subsequence, also denoted by $\{\omega_m\}$, we may conclude convergence (weakly) in $H_0^{1,2}(\Omega^*_{\lambda_0})$ and (strongly) in $L^q(\Omega^*_{\lambda_0})$, for $q < (2n)/(n-2)$, to some $\omega \in H_0^{1,2}(\Omega^*_{\lambda_0})$. Obviously, $\omega \geq 0$, nontrivial, and $\ell_{\lambda_0}(\omega) = \delta\omega$ in $H_0^{1,2}(\Omega^*_{\lambda_0})$. We claim that $\delta = \mu^*_0(\lambda_0)$. If $\Omega^*_{\lambda_0}$ is connected, this is immediate by the positivity of ω – and again employing Picone's Identity. If $\delta \neq \mu_0^*(\lambda_0)$, then $\mu_0^*(\lambda_0) < \delta$ by the min.-max. principle. We conclude that there exists some $\varphi \in C_0^\infty(\Omega_{\lambda_0}^*)$ such that $(\ell_{\lambda_0}^*\varphi, \varphi) < (\delta - 2\varepsilon)(\varphi, \varphi)$. But since φ has compact support, then $(\ell_{\lambda_m}^*\varphi, \varphi) < (\delta - \varepsilon)(\varphi, \varphi)$ for all large m and some $\varepsilon > 0$, i.e. $\mu_0^*(\lambda_m) < \delta - \varepsilon$ contradicting the definition of δ. Suppose next that $\lambda_m \uparrow \lambda_0$ and again set $\delta = \lim\ \big(\mu_0^*(\lambda_m)\big)$. The same procedure as above shows the existence of a subsequence ω_m and function ω. We need to show $\omega \in H_0^{1,2}(\Omega_{\lambda_0}^*)$. To see this, let g be a cut off function: $g \in C^\infty(R, R)$, with $g(\xi) = 0$ if $\xi < 2$, $\quad g(\xi) = 1$ if $\xi > 3$ and set $z_m(x) = g\big((x_1 - \lambda_0)/(\lambda_0 - \lambda_m)\big)$. We then have:

$$\|z_m\omega_m\|^2_{H^{1,2}(\Omega^*_{\lambda_0})} \leq C\big[\|\omega_m\|^2_{H^{1,2}(\Omega^*_{\lambda_0})} + \|\,|\nabla z_m|\omega_m\|^2_{L^2(\Omega^*_{\lambda_0})}\big].$$

The first term is clearly bounded. For the second, note:

$$\|\,|\nabla z_m|\omega_m\|^2_{L^2(\Omega^*_{\lambda_0})} \leq C\,\frac{1}{(\lambda_0 - \lambda_m)^2}\int_{\Omega^*_{\lambda_m}\cap\{x_1<\lambda_0+3(\lambda_0-\lambda_m)\}} \omega_m^2$$

Poincare's Inequality, [16], then shows this term is bounded as well. Observe that $z_m\omega_m$ is obviously in $H_0^{1,2}(\Omega_{\lambda_0}^*)$ and thus, without loss of generality, weakly convergent in this space. Since $z_m \to 1$ pointwise in $\Omega_{\lambda_0}^*$ we conclude that $\omega \in H_0^{1,2}(\Omega_{\lambda_0}^*)$. Clearly δ is an eigenvalue as ω is nontrivial. By the min.-max. principle, it is the least. Finally the positivity of $\mu_0^*(\lambda)$ for meas (Ω_λ^*) small, follows from Theorem 1(e) with p replaced by P.

Theorem 4. *Let $T_\lambda^{-1} : R^n \to R^n$ by $y = T_\lambda^{-1}(x)$ where $y = (2\lambda - x_1, \overline{x})$, with $\overline{x} = (x_2, \dots, x_n)$. Let $\Omega_\lambda = T_\lambda^{-1}(\Omega_\lambda^*)$, $\quad p_\lambda(x) = P_\lambda\big(T_\lambda(x)\big)$ where $P_\lambda, \Omega_\lambda^*$ are as before. Then $\mu_0(\lambda)$, the least eigenvalue of $\ell_\lambda = -\Delta - p_\lambda$ in Ω_λ, is also continuous.*

We merely map the quadratic form associated with ℓ_λ in Ω_λ to that for ℓ_λ^* in Ω_{λ^*}. Notice that this leaves $-\Delta$ unchanged, and that the Jacobian is -1.

2. The Nonlinear Problem

Consider now the nonlinear problem (2). We recall $\Omega_\lambda^* = \Omega \cap \{x_1 > \lambda\}$ and $\Omega_\lambda = \{x | x^\lambda \in \Omega_\lambda^*\}$ where $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$, and $0 < u \in H_0^{1,2}(\Omega) \cap C^\theta(K)$. Standard regularity theory shows that if f is smooth then u is classical in Ω but we shall not need the extra regularity. Our assumptions then are

(I) $\Omega_\lambda \subset \Omega \cap \{x_1 < \lambda\}$ for $\lambda_0 \leq \lambda < \lambda_1$, $\quad \Omega_\lambda = \emptyset$ if $\lambda \geq \lambda_1$;

(II) Let $\ell_\lambda \equiv -\Delta - p_\lambda(x)I$ be defined on $H_0^{1,2}(\Omega_\lambda)$ where $p_\lambda(x) = [f(u) - f(v_\lambda)]/(u - v_\lambda)$, $\quad v_\lambda(x) = u(x^\lambda)$. Assume $P \in C\big[[\lambda_0, \lambda_1);\ L^{\frac{n+\varepsilon}{2}}(\Omega_\lambda^*)\big]$ for some $\varepsilon > 0$, with $\|P_\lambda\|_{L^{\frac{n+\varepsilon}{2}}(\Omega_\lambda^*)}$ bounded for $\lambda \in [\lambda_0, \lambda_1)$.

Note that $\mu_0(\lambda)$ – the least eigenvalue of ℓ_λ – is then continuous in λ, and $p_\lambda(y) \equiv P_\lambda(y)$. The results in Section 1 then yield:

Theorem 5. *Let* (I), (II) *hold. Then:*

(a) *$\mu_0(\lambda) > 0$ for $\lambda_0 < \lambda < \lambda_1$, and $u > v_\lambda$ in Ω_λ.*

(b) *If $\Omega_{\lambda_0} = \Omega \cap \{x_1 < \lambda_0\}$ then $u \equiv v_{\lambda_0}$ on Ω_{λ_0}.*

(c) *If* $x_0 \in \Omega \cap \{x_1 = \lambda\}$, $\frac{\partial u}{\partial x_1}(x_0)$ *exists and* $P_\lambda \in L^{n+\varepsilon}(\Omega^*_\lambda)$, *then* $\frac{\partial u}{\partial x_1}(x_0) < 0$.

Proof. We apply the earlier results using $\omega = u - v_\lambda$ in Theorem 1. We first show that Theorem 1-d can be used to conclude that $\mu_0(\lambda) > 0$. Specifically, assume otherwise i.e. $\mu_0(\lambda') = 0$, $\mu_0(\lambda) > 0$ for $\lambda > \lambda'$, for some $\lambda' \in (\lambda_0, \lambda_1)$, and note that $\omega^- \in H_0^{1,2}(\Omega_{\lambda'})$ and hence the min.-max. principle shows $\omega^- = 0$, since, by continuity, if $\omega^- \neq 0$ then $(u - v_\lambda)^- \neq 0$ for some $\lambda > \lambda'$ and thus $\mu_0(\lambda) \leq 0$.

Assume now that $\omega \in H_0^{1,2}(\widetilde{\Omega}_{\lambda'})$. Choose a small ball $B \subset \widetilde{\Omega}_{\lambda'}$ and look at the cylinder $Z = (-a, \widetilde{x}_1) \times S$ where: $(\widetilde{x}_1, \overline{y^*})$ is the center of B and $S = \{\overline{y} | (\widetilde{x}_1, \overline{y}) \in B\}$.

For notational convenience denote λ' by λ henceforth. Since v_λ can be approximated in $H^{1,2}$ by functions which vanish near $Z \cap \partial\widetilde{\Omega}_\lambda$ and $\omega \in H_0^{1,2}$, then this is also true of u. We may assume u admits in $Z - \widetilde{\Omega}_\lambda$ a trivial extension (also denoted by u) and thus if a is large enough $u \equiv 0$ on $(-a) \times S$. Now by a fundamental result employing Fubini's Theorem, [24], there exists a $\overline{y}_1 \in S$ such that $u(x_1, \overline{y}_1) = \int_{-a}^{x_1} \frac{\partial u(\xi, \overline{y}_1)}{\partial x_1} \, d\xi$ for almost all x_1.

Since u is continuous in $Z \cap \Omega_\lambda$ and in $Z - Z \cap \overline{\Omega}_\lambda$ and clearly so is the integral, we conclude that equality must actually hold in these regions. We thus have that $\int_{-a}^{x_1} \frac{\partial u}{\partial x_1}(\xi, \overline{y}_1) d\xi = 0$ if $(x_1, \overline{y}_1) \in Z - Z \cap \overline{\Omega}_\lambda$ and $\int_{-a}^{x_1} \frac{\partial u}{\partial x_1}(\xi, \overline{y}_1) d\xi = u(x_1, \overline{y}_1) \geq \delta > 0$ if $(x_1, \overline{y}_1) \in Z \cap K$, if $K \Subset \Omega$, by the positivity of the solution u in the compacta of Ω.

Now let α be the least number such that we have $C = \{(x_1, \overline{y}_1) | \alpha < x_1 \leq \widetilde{x}_1\} \subset \widetilde{\Omega}_\lambda$. It follows that $(\alpha, \overline{y}_1) \in \partial\widetilde{\Omega}_\lambda$ and $C_\lambda \equiv \{x | x^\lambda \in C\} \subset \Omega$.

By definition and assumption, $\Omega_{\lambda-\varepsilon} \subset \Omega$ for $\varepsilon > 0$ small enough, and we conclude that $(\alpha, \overline{y}_1) \in \Omega$. I.e. $\overline{C} \subset \Omega$ and thus $u > \delta > 0$ in $\overline{C}$. This contradicts the absolute continuity of the integral and it follows that $\omega \notin H_0^{1,2}(\widetilde{\Omega}_{\lambda'})$ and thus $\mu_0(\lambda') > 0$ by Theorem 1-d.

Since $\mu_0(\lambda) > 0$ and $(u - v_\lambda)^- \in H_0^{1,2}(\Omega_\lambda)$, then the min.-max. principle again implies $(u - v_\lambda)^- = 0$, whence $u \geq v_\lambda$. The earlier arguments show that $u \not\equiv v_\lambda$ in Ω_λ, and then $u > v_\lambda$. This shows part (a).

As for part (b), we have $u(x) \geq u(x^{\lambda_0})$ in Ω_{λ_0} by continuity, since $u > v_\lambda$ on Ω_λ, $\lambda > \lambda_0$. However Ω is symmetric in this case about $x_1 = \lambda_0$. If we first perform a reflection about $x_1 = \lambda_0$, and then repeat the above procedure we would conclude $u(x^{\lambda_0}) \geq u\big((x^{\lambda_0})^{\lambda_0}\big) = u(x)$ and the result.

Finally, for part (c), note that $x_0 \in \Omega$ and $P_\lambda \in L^{n+\varepsilon}$, imply that ω is, without loss of generality, in $C^{1+\theta}(\overline{B})$ for some ball $B \subset \Omega_\lambda$ with $x_0 \in \partial B$, [16]. Choose a function $z \in C^{1+\theta}(\overline{B})$ such that $-\Delta z - P_\lambda z = 0$ in B, $z > 0$ in $\overline{B}$. Then by considering the equation ω/z satisfies we conclude, again by [16], that $\omega/z \in C^2(B) \cap C^{1+\theta}(\overline{B})$ and $\frac{\partial}{\partial x_1}(\omega/z)(x_0) < 0$, i.e. $\frac{\partial \omega}{\partial x_1}(x_0) = 2\frac{\partial u}{\partial x_1}(x_0) < 0$.

Next assume that f depends on x as well, i.e. $f \equiv f(x, u)$. As was the case in the previous references, this situation can also be dealt with if we assume f is monotone in x_1 : $f(x, \xi) \geq f(x^\lambda, \xi)$ for $\lambda > 0$. There is no significant change in the proofs. Note that we could thus deal with some cases where f had singularities with respect to x on $\{x_1 = \lambda_0\} \cap \Omega$, or with singularities along planes $\{x_m = c\} \cap \Omega$, $m \neq 1$, as examples with $f(x, u) = p(x)g(u)$ easily show, as long as the resulting P_λ was in $L^\alpha(\Omega_\lambda)$.

One limitation in the applicability of Theorem 5 is given by condition (II). Observe that since $\|P_\lambda\|_{L^{\frac{n+\varepsilon}{2}}}$ is assumed bounded, then it suffices that $\chi_\lambda P_\lambda(x)$ be pointwise continuous, a.e. in λ, which will be immediately the case if f is smooth in u. This follows from the observation that if $\{g_n\}$ is a sequence of functions bounded in L^α and $g_n \to g \in L^\alpha$ pointwise then $g_n \to g$ in $L^{\alpha-\varepsilon}$ for any $\varepsilon > 0$, and this is because we can find a constant c –independent of n– such that $g_n = \bar{g}_n^c + g_n'$ with $\bar{g}_n^c = g_n$ if $|g_n| \le c$, $|\bar{g}_n^c| = c$ otherwise, and g_n' of small $L^{\alpha-\varepsilon}$ norm. We thus only need to check the boundedness of $\chi_\lambda P_\lambda$. Recall that we may also express P_λ as $P_\lambda = \int_0^1 f'\big(tu + (1-t)v_\lambda\big)dt$ and thus if, for example, $u \in L^\infty(\Omega)$ and $f \in C^{1+\theta}_{\ell oc}$, then Theorem 5 applies. Finally, observe that the results apply if f is Lipschitz (locally Lipschitz if u is in L^∞). This follows by setting $p_\lambda = [f(u) - f(v_\lambda)]/[u - v_\lambda]$ if $u(x) \ne v_\lambda(x)$; $p_\lambda = 0$ otherwise, and this example also shows that the continuity of μ_0 is really only needed from the left: If we let λ' be given by $\mu_0(\lambda) > 0$ if $\lambda > \lambda'$, as before, then we repeat the above arguments, in particular Theorem 1-c, and conclude that both $\mu_0(\lambda') > 0$ and $u > v_{\lambda'}$. We next observe that $\chi_\mu P_\mu \to \chi_{\lambda'} P_{\lambda'}$ pointwise, and that $|P_\mu| < C$ for some C, by the Lipschitz assumption on f, as $\mu \to (\lambda')^-$. We have then the contradiction $\lambda_0(\mu) > 0$ for $|\mu - \lambda'|$ small.

3. $\Omega = R^n$

The above approach also works for the case of $\Omega = R^n$ and we now consider the modifications needed to deal with this case. Detailed references of other results for this case may be found in [17]. Specifically, assume $-\Delta u = f(x,u)$ weakly in R^n, with $0 < u \in E(R^n)$, $\quad u \to 0$ at ∞. Here $E(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_E^2 = \int_\Omega |\nabla u|^2$.

Our conditions on f are as follows:

(II') Let p_λ be defined as in condition (II) and assume $p:\ (0,\infty] \to L^\alpha(\Omega_\lambda)$ continuously, with $\alpha = n/2$ and $p_\infty \equiv f(x,u)/u$;

(III) $f \in C^{1+\theta}_{\ell oc}$ and $f(x,0) = 0$;

(IV) $f(x,\xi) \ge f(x^\lambda,\xi)$ for: $x \in \Omega_\lambda$, $\quad \lambda > 0$ and $\xi > 0$, furthermore given any $\lambda > 0$, $\varepsilon > 0$ there exists $x \in \Omega_\lambda$ such that $f(x,\xi) > f(x^\lambda,\xi)$ for some $0 < \xi < \varepsilon$.

Observe that in the special case: $f(x,\xi) = p(x)\xi^\delta$, then condition (IV) holds if, for example, $\frac{\partial p}{\partial x_1} \le 0$, $\quad \frac{\partial p}{\partial x_1}\ (\varepsilon,\bar{x}) < 0$ for $0 < \varepsilon$ small enough. If $u \in C^1$, say, decays fast enough at ∞ then $u \in E(R^n)$. We give explicit conditions for this to be the case, as well as other convenient results in Theorem 6.

Theorem 6.
(a) *If* $u \in C^1$ *and:* $u \in L^{\frac{2n}{n-2}}$, $\quad uf(x,u) \in L^1$ *then* $u \in E(R^n)$.
(b) *Let* $\Omega_\lambda = \{x | x_1 < \lambda\}$, *then* $\omega = u - v_\lambda \in E(\Omega_\lambda)$ *if* $u \in E(R^n)$.

We recall that it is often possible to show that a solution $0 < u \in E(R_n)$ must, as a consequence, tend to zero at infinity, [1].

Proof of Theorem 6. (a) We observe by direct calculation:

$$\int_{R^n} |\nabla(\varphi u)|^2 = \int_{R^n} u^2 |\nabla\varphi|^2 + \int_{R^n} \varphi^2 u f(x,u)$$
$$\le \|u\|^2_{L^{\frac{2n}{n-2}}(\mathrm{supp}\ |\nabla\varphi|)} \cdot \|\ |\nabla\varphi|\ \|^2_{L^n} + \int_{R_n} |uf(x,u)|$$

where φ denotes a cut-off function: $\varphi(x) = g(\frac{|x|}{m})$, $0 \leq g \leq 1$, smooth, and $g(\xi) = 1$ if $\xi \leq 1$, $g(\xi) = 0$ if $\xi \geq 2$. As $m \to \infty$ we have $\{\varphi u\}$ bounded in $E(R^n)$ and thus without loss of generality weakly convergent (to u) in $E(R^n)$.

(b) Since $u \in E(R^n)$, then $v_\lambda \in E(R^n)$ and thus ω by definition. Furthermore if $\varphi_m \in C_0^\infty(R^n)$, $\varphi_m \to u$ in $E(R^n)$ then $\varphi_m(x) - \varphi_m(x^\lambda) \in E(\Omega_\lambda)$ since $\varphi_m(x) = \varphi_m(x^\lambda)$ on $x_1 \equiv \lambda$, and $\{\varphi_m(x) - \varphi_m(x^\lambda)\}$ is Cauchy in $E(\Omega_\lambda)$, converging to $u - v_\lambda$.

Sufficient explicit conditions for (II′) to hold are:

Lemma 7. *Assume* $|f_u(x,\xi)| \leq k(x) + h(x)\xi^\gamma$ *for* $\xi \geq 0$ *with* $k(x) \in L^{\frac{n}{2}} \cap L^\infty(R^n)$, $h(x) \in L^s \cap L^\infty(R^n)$ *with* $s \leq (2n\alpha)/\big(2n - \alpha\gamma(n-2)\big)$, *then* $p : (0,\infty] \to L^\alpha(\Omega_\lambda)$ *continuously.*

Note that we require that $0 \leq \gamma < 4/(n-2)$ in Lemma 7. This is because we do not postulate any specific decay conditions at ∞ on u, apart from $u \to 0$ and those implicitly associated with $u \in E$. Note that the given bounds on s, γ reduce exactly to the sufficient condition for existence as stated in [1], if e.g. $f(x,\xi) = h(x)\xi^{\gamma+1}$.

Proof of Lemma 7. We clearly have:

$$\begin{aligned} |p_\lambda(x)| &= \Big|\int_0^1 f_u\big(x, tu + (1-t)v_\lambda\big)dt\Big| \\ &\leq C[k(x) + h(x)(|u|^\gamma + |v_\lambda|^\gamma)]. \end{aligned}$$

The continuity of u and $f \in C^{1+\theta}_{\ell oc}$ imply $p_\lambda(x) \to p_\mu(x)$ pointwise in R^n as $\lambda \to \mu$ for $\mu \in (0,\infty]$. Since p_λ is also uniformly bounded in L^∞ for $\lambda \in (0,\infty]$, then we immediately have $p_\lambda \to p_\mu$ in $L^\alpha(B)$ for any fixed ball $B \subset R^n$. That $\|p_\lambda - p_\mu\|_{L^\alpha(R^n - B)}$ is small for B large is immediate by the integrability assumption on k, h and the embedding $E(R^n) \hookrightarrow L^{\frac{2n}{n-2}}(R_n)$.

We consider the eigenvalue problem: $-\Delta z = \xi_\lambda p_\lambda z$ for $z \in E(\Omega_\lambda)$. Observe that $(-\Delta)^{-1}(p_\lambda u)$ – for fixed λ– can be viewed as a compact map $E(\Omega_\lambda) \to E(\Omega_\lambda)$, [1], and thus there exist for each λ an eigenvalue ξ_λ and associated positive eigenfunction η_λ given by:

$$\frac{1}{\xi_\lambda} = \sup_{\substack{\varphi \in E(\Omega_\lambda) \\ \varphi \not\equiv 0}} \frac{(p_\lambda\varphi, \varphi)}{\|\varphi\|^2_{E(\Omega_\lambda)}}\,.$$

That $\xi_\lambda > 0$ follows from the next Lemma. In any case observe that $f(x,u)$ must be positive somewhere, since $-\Delta u = f(x,u)$ and $u > 0$. Since we also have $p_\lambda \to p_\infty = f(x,u)/u$ pointwise, then p_λ cannot be nonpositive for λ large. I.e. for such λ at least, $\xi_\lambda > 0$.

Lemma 8.

(a) *Let* $\omega = u - v_\lambda$ *and* ξ_λ, η_λ *exist then* $(\xi_\lambda - 1)(p_\lambda\eta_\lambda, \omega)_{\Omega_\lambda} \geq 0$.

(b) ξ_λ *is continuous in* λ, $\xi_\lambda \geq 1$ *and* $w > 0$ *for all* $\lambda > 0$.

Proof. (a) Observe that $\ell_\lambda\omega = -\Delta\omega - p_\lambda\omega = f(x,v_\lambda) - f(x^\lambda, v_\lambda) = r(x) \geq 0$ whence $\xi_\lambda(p_\lambda\eta_\lambda, \omega)_{\Omega_\lambda} = (\eta_\lambda, \ell_\lambda\omega)_{\Omega_\lambda} + (p_\lambda\eta_\lambda, \omega)_{\Omega_\lambda}$ and the result since $\eta_\lambda > 0$.

(b) We show first that $\xi_\lambda \geq 1$. We claim that this is true for λ large enough since otherwise there exists a sequence $\lambda_m \to \infty$ for which this is not the case.

But as $\lambda_m \to \infty$, $\quad p_{\lambda_m} \to p_\infty$ in L^α, and thus $\xi_{\lambda_m} > 0$ exists. Furthermore: as $\lambda_m \to \infty$, $\quad \omega \to u$ pointwise and thus $\omega \to u$ in $L^\tau(B)$ for any large τ and fixed ball $B \subset R^n$ by the uniform boundedness of ω, u. Similarily, we note that ξ_{λ_m} is bounded by the min.-max. principle and setting ξ_∞ equal to the limit of ξ_{λ_m}, we may assume that $\|\eta_{\lambda_m}\|_E = 1$ and thus $\eta_{\lambda_m} \to \eta$ weakly in $E(R^n)$ and strongly in $L^{\frac{2n-\varepsilon}{n-2}}(B)$ where η denotes a (positive) eigenfunction of $-\Delta\eta = \xi_\infty p_\infty \eta$ in $E(R^n)$, since $\eta \not\equiv 0$, as a consequence of $\xi_\infty(p_\infty \eta, \eta) = 1$. We thus have the existence of two positive eigenfunctions: η, u corresponding to the eigenvalues $\xi_\infty, 1$ respectively. However, Picone's Identity also shows the simplicity of the eigenvalue associated with a positive eigenfunction η such that $(p_\infty \eta, \eta) > 0$, and thus $\xi_\infty = 1$ and $\eta \equiv u$. We conclude by part (a) that $(p_\infty u, u) = \|u\|^2_{E(R^n)} \leq 0$ if $\xi_{\lambda_m} < 1$, which is a contradiction. Let λ_0 denote the least λ such that $\xi_\lambda \geq 1$ and ω is nontrivial (and thus positive) for $\lambda > \lambda_0$. The next arguments also show that if $\lambda_0 = 0$ we are done, hence suppose $\lambda_0 > 0$. Assume first ω is nontrivial in Ω_{λ_0}. By the presumed continuity, $\xi_{\lambda_0} = 1$ since ω is also nontrivial for some $\lambda < \lambda_0$, and $-\Delta\omega - p_{\lambda_0}\omega \geq 0$. We conclude $(p_{\lambda_0}\eta_{\lambda_0}, \omega)_{\Omega_{\lambda_0}} = (p_{\lambda_0}\eta_{\lambda_0}, \omega)_{\Omega_{\lambda_0}} + (r, \eta_{\lambda_0})_{\Omega_{\lambda_0}}$, i.e. $r \equiv 0$ and since $\omega \in E(\Omega_{\lambda_0})$, it must be an eigenfunction corresponding to ξ_{λ_0}. Indeed, again by Picone's Identity, if $\omega \not\equiv 0$ then $\omega = c_{\lambda_0}\eta_{\lambda_0}$ in Ω_{λ_0} for some constant c_{λ_0}. Note that this result applies to all λ for which $\xi_\lambda = 1$ and $\omega \not\equiv 0$ in Ω_λ, and, identically, if $\xi_\lambda > 1$ then $(-\Delta\omega, \omega^-) \geq (p_\lambda \omega, \omega^-)$ whence $(-\Delta\omega^-, \omega^-) \leq (p_\lambda \omega^-, \omega^-)$, i.e.

$$1 \leq \frac{(p_\lambda \omega^-, \omega^-)}{\|\omega^-\|^2_{E(\Omega_\lambda)}}$$

and we have a contradiction unless $\omega^- = 0$, i.e. once again, if $\omega \not\equiv 0$ then $\omega \geq 0$ and $\omega > 0$ by the weak Harnack Inequality. It follows that since for any $\lambda > \lambda_0$ we have $\xi_\lambda \geq 1$ and ω is nontrivial then $\omega > 0$ in Ω_λ, $\quad \omega = 0$ on $x_1 = \lambda$ and thus $\frac{\partial u}{\partial x_1} < 0$ if $x_1 > \lambda_0$. Now suppose ω is trivial in Ω_{λ_0} and then observe $f\big(x, u(x)\big) \equiv f\big(x^{\lambda_0}, u(x)\big)$ for $x \in \Omega_{\lambda_0}$, contradicting assumption (IV) on f. Hence if $\lambda_0 > 0$, then $\xi_{\lambda_0} = 1$ and $\omega > 0$ is its associated eigenfunction. But this is impossible since then $r \equiv 0$ in Ω_{λ_0} and again this violates (IV). Finally, the continuity of ξ_λ follows from the properties of p_λ and the min.-max. definition of ξ_λ. Specifically, observe first that if $\xi_\mu > 0$ then p_μ is somewhere positive and thus so is p_λ for $|\lambda - \mu|$ small (λ sufficiently large if $\mu = \infty$). We conclude ξ_λ is bounded above and below, and set ξ^* equal to any limit point of ξ_λ. The earlier arguments in this proof show that a subsequence of the normalized (in E) positive eigenfunctions η_λ converges to η weakly in $E(R^n)$ and strongly in $L^{\frac{2n-\varepsilon}{n-2}}(B)$. We again note that $\xi^*(p_\mu \eta, \eta)_{\Omega_\mu} = 1$, $-\Delta\eta = \xi^* p_\mu \eta$ in Ω_μ, and $\eta \geq 0$, nontrivial. If $\lambda \uparrow \mu$ then $\eta \in E(\Omega_\mu)$ and again by Picone's Identity, $\xi^* = \xi_\mu$ and $\eta = \eta_\mu$. If $\lambda \downarrow \mu$ then we need only show $\eta \in E(\Omega_\mu)$, since the rest is the same. But this is immediate here by the smoothness of the (plane) boundary. The uniqueness of ξ_μ, η_μ then shows the continuity.

We then have under the above assumptions (II$'$), (III), (IV) on f :

Theorem 9. *If $f(x,\xi)$ is symmetric (in x_1) about $x_1 = 0$, then u is symmetric in x_1, and $x_1 \frac{\partial u}{\partial x_1} < 0$ if $x_1 \neq 0$.*

Proof. We have by Lemma 8 that $\xi_\lambda \geq 1$ and $w > 0$, i.e. $u > v$ and $\frac{\partial u}{\partial x_1} < 0$ for all $\lambda > 0$ and thus $u \geq v_\lambda$ for $\lambda = 0$. Repeating the procedure for λ negative we obtain the result.

4. Extensions

We now briefly and heuristically comment on some extensions where the eigenvalue arguments and Picone's Identity still work.

Observe that the same procedure can yield some modest results even for problems not involving purely Dirichlet conditions. Consider for example the cylinder $(-1,1) \times \Omega'$ with Dirichlet conditions on $\{-1\} \times \Omega'$, $\{+1\} \times \Omega'$ but Neumann elsewhere. The procedure in such a case is identical. Note that a key step involves the fact that the part of the boundary with Neumann conditions does not reflect inside the regions Ω_λ.

Assume now that f is not smooth. Suppose first as in [3], [14] that $f = f_1 + f_2$ with f_1 smooth and f_2 monotone increasing and continuous. Let $f_2(\xi) \equiv 0$ if $\xi < c$ for some $c > 0$, and suppose Ω is bounded. We now also require that u be in $C(\overline{\Omega}) \cap C^1(\Omega)$. This is similar to the requirements of [6]. If $f_1(u), f_2(u)$ are in L^p for some $p > n$ then it suffices to assume that Ω satisfies an exterior cone condition at every point of $\partial\Omega$, [16]. A more detailed study of the requirements on $\partial\Omega$ can be found in [23]. We now set $p_\lambda(x) = [f_1(u) - f_1(v_\lambda)]/(u - v_\lambda)$ in (II) and repeat the earlier procedures. Observe that $\big((u-v_\lambda)^-, \ell_\lambda((u-v_\lambda))\big) = \big(f_2(u) - f_2(v_\lambda), (u-v_\lambda)^-\big)$. If $u \not\geq v_\lambda$ we have an immediate contradiction to $\mu_0(\lambda) \geq 0$ unless there is a point x^0_λ such that $\big(f_2(u) - f_2(v_\lambda)\big)(u-v_\lambda)^-|(x^0_\lambda) < 0$. On the other hand, $f_2(u) \equiv 0$ in a neighborhood of $\partial\Omega$ by the continuity of u and thus we may keep our arguments away from $\partial\Omega$. We conclude in such eventuality that dist $(x^0_\lambda, (\partial\Omega_\lambda - \{x_1 = \lambda\})) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. To apply these observations, suppose that for some value of $\lambda = \lambda^*$ we have $u \geq v_{\lambda^*}$ in Ω_{λ^*} and $\mu_0(\lambda^*) > 0$, then by the assumed continuity, $\mu_0(\lambda) > 0$ for λ near λ^*. If $u \not\geq v_\lambda$ for such λ then we construct a sequence of points $\{x^0_\lambda\}$ with: $x^0_\lambda \to x_0$, dist $(x_0, \partial\Omega) \geq \varepsilon_0$, $u(x^0_\lambda) < v_\lambda(x^0_\lambda)$, $x_0 \in \partial\Omega_{\lambda^*}$. Consequently, $x_0 \in \{x_1 = \lambda^*\}$ and $\frac{\partial u}{\partial x_1}(x_0) \geq 0$, contradicting Theorem 5(c). Observe that the same procedures also work if f_2 has a simple jump discontinuity at c, or if $f = f_1(x,\xi) + f_2(x,\xi)$, with obvious changes.

Suppose next that we consider $-\Delta u = f(u)$ in R^n and assume the existence of $0 < u \to 0$ at ∞, with maximum at the origin, and that $f(\xi) \in C^{1+\theta}_{\ell oc}$, with $f(0) = f'(0) = 0$. Theorem 6 gives conditions on u and f which suffice for $u \in E(R^n)$. To apply the above spectral arguments and obtain monotonicity results, we then only further require the continuity of $\chi_\lambda p_\lambda$ in $L^{n/2}(R^n)$, and a detailed calculation shows that for this to be the case it suffices that $u \in L^{\frac{n\theta}{2}}(R^n)$. Note that by following the arguments in [14] we conclude that $u - v_\lambda \not\equiv 0$ in Ω_λ, for $\lambda > 0$, since the maximum of u is at $x = 0$.

We note that we can begin our eigenvalue procedures for any Ω_{λ_0} for which we can conclude that $\mu(\lambda_0) > 0$. If some information is known about the norm of u, then we need not start by considering a very thin domain Ω_λ to ensure $\mu(\lambda_0) > 0$. We can bypass in this way the requirement that $\Omega_\lambda \subset \Omega$ for all relevant λ, and still obtain the monotonicity result $u > v_\lambda$, for $\lambda > \lambda_0$ say. We note that, as a consequence, we immediately have the observation that if f is Lipschitz, with small constant, and Ω is symmetric in x_1 then all positive solutions must be symmetric, regardless of whether Ω is convex or not in x_1. This result is essentially known, [12]. This approach can also be applied if some extra conditions are imposed on f, as the following arguments in particular indicate.

Consider now the situation where $f(x,\xi)$ is not smooth at $\xi = 0$. In general we could not obtain results for this case. In special situations, however, we could actually obtain better results than those obtained earlier. Specifically, assume now: $f \in \operatorname{Lip}_{\ell oc}\big(\overline{\Omega} \times (0,\infty)\big)$ and $0 \le f(x,\xi)/\xi$ is nonincreasing in ξ for $\xi > 0$. The prototype f we have in mind is $f(x,\xi) = p(x)\xi^\theta + q(x)$ with $p, q \ge 0$, smooth and $\theta \le 1$. Our result for these f is as follows (see also [9]):

Theorem 10. *Suppose Ω is a bounded domain and that u is a positive solution of $-\Delta u = f(x,u)$. Then u is unique up to a constant multiple. If $f(x,c\xi) \not\equiv cf(x,\xi)$ for any $c > 0$, $\xi > 0$ $\quad (c \neq 1)$, and $x \in \Omega$ then u is unique. If both Ω and $f(x,\cdot)$ are symmetric about $x_1 = 0$ then so is u.*

We observe that we do not require that Ω be convex in x_1 nor assume conditions on $\vec{\nabla}_x f$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$. We apply once again Picone's Identity and conclude:

$$\int_\Omega |\nabla\varphi|^2 = \int_\Omega \frac{f(x,u)}{u}\,\varphi^2 + \int_\Omega u^2|\nabla\,(\frac{\varphi}{u})|^2.$$

Now let v denote another positive solution and without loss of generality, $(u-v)^- \not\equiv 0$. By our assumptions on $f(x,\xi)/\xi$ we have:

$$\begin{aligned} J(\varphi) &\equiv \int_\Omega |\nabla\varphi|^2 - [\frac{f(x,u)-f(x,v)}{u-v}]\varphi^2 \\ &= \int_\Omega u^2|\nabla\,(\frac{\varphi}{u})|^2 + \int_\Omega [\frac{f(x,u)}{u} - \frac{f(x,u)-f(x,v)}{u-v}]\varphi^2 \ge 0 \end{aligned} \tag{3}$$

since $\frac{f(x,u)-f(x,v)}{u-v} \le \frac{f(x,u)}{u}$. Note that here we set $[\frac{f(x,u)-f(x,v)}{u-v}] = 0$ if $u = v$.

Now let $w = (u-v)^-$ and observe that we may construct a sequence of functions $0 \le \varphi_m \le (u-v)^-$ with compact support such that $\varphi_m \to (u-v)^-$ in $H^{1,2}$. We conclude that

$$\begin{aligned} \big|(\frac{f(x,u)-f(x,v)}{u-v})\big|\varphi_m^2 &\le \big|[\frac{f(x,u)-f(x,v)}{u-v}]\big|[(u-v)^-]^2 \\ &= |f(x,u)-f(x,v)|(u-v)^-, \end{aligned}$$

and by integrating and recalling the equation $(u-v)$ satisfies, that $J(w) = 0$ since:

$$\int_\Omega |\nabla\varphi_m|^2 \to \int_\Omega |\nabla w|^2 = -\int_\Omega \nabla(u-v)\cdot\nabla w.$$

In the same way, we obtain $0 \le J(w+\varepsilon\varphi)$ for given ε and any $\varphi \in C_0^\infty$. We conclude that $-\Delta w - [\frac{f(x,u)-f(x,v)}{u-v}]w = 0$ in Ω. But $w \ge 0$ in Ω and since $\frac{f(x,u)-f(x,v)}{u-v}$ is locally in $L^\infty(\Omega)$ by the fact that $u, v \in C^\delta(K)$, Harnack's Inequality, [16], implies that $w > 0$ or $w \equiv 0$. By assumption, we have $w \not\equiv 0$. But then (3) yields $w = cu$ for some $c > 0$, i.e. $v = (1+c)u$. It also follows that $f\big(x, (1+c)u\big) = (1+c)f(x,u)$ for $x \in \Omega$ from the equations that u, v satisfy, and if this is impossible, we conclude $u \equiv v$. Finally, suppose both Ω and $f(x,\cdot)$ are symmetric in x_1, and now choose v by $v(x,\bar{x}) = u(-x_1, \bar{x})$. Clearly $0 < v$ satisfies the same equation in Ω by the assumed symmetry, and we again have $w > 0$ or $w \equiv 0$. Now $w > 0$ is impossible since $w = 0$ on $x_1 = 0$, and it follows that $w \equiv 0$, i.e. $u \geq v$. In the same way we obtain $v \geq u$ and the result.

The proof of Theorem 10 also yields monotonicity results: suppose, for example, $f = f(u)$ and Ω_λ is properly contained in Ω for some $\lambda > 0$ then $u > v_\lambda$ follows by choosing $v = v_\lambda$ in the proof, and we thus have $\frac{\partial u}{\partial x_1} < 0$ on $\Omega \cap \{x_1 = \lambda\}$.

As a final remark, we recall that the classic moving plane argument can be extended to systems: $-\Delta \vec{u} = \vec{f}(\vec{u})$, [22], if we merely assume $\frac{\partial f_i}{\partial u_j} \geq 0$. We were not able to obtain a similar extension under our conditions on $\partial\Omega$, $\vec{u}$, $\vec{f}$.

5. Examples

We conclude with some simple examples. We begin with:

Proof of Theorem 0. (a) Since $p \in L^\alpha$, the linear problem: $-\Delta\widetilde{u} = p \geq 0$, $\widetilde{u} \in H_0^{1,2}(\Omega)$ has a positive solution $\widetilde{u}$ in $L^\infty(\Omega) \cap C(\Omega)$, [16]. Observe that we thus have $-\Delta\widetilde{u} \geq \lambda p(x) g(\widetilde{u})$ for λ such that $\lambda g(\widetilde{u}) \leq 1$. Since $\underset{\sim}{u} = 0$ is a subsolution we have the existence of a solution $0 < u \in L^\infty(\Omega) \cap C(\Omega)$ by Schauder's Fixed Point Theorem. (b) In this case $p_\lambda = p(x)\int_0^1 g'\big(tu + (1-t)v_\lambda\big)$ and $\|P_\lambda\|_{L^\alpha}$ is clearly bounded and pointwise continuous in λ by the continuity of u, g' and the continuity in $L^{\frac{n+\varepsilon}{2}}$ follows. The symmetry and differentiability of the positive solutions follow from the comments after Theorem 5. We mention that the uniqueness questions for some of these problems is discussed in [22].

As another example, consider the problem: $-\Delta u = q(x)u^\gamma$ in R^n, with: $1 < \gamma < (n+2)/(n-2)$ and q smooth such that $0 < q \in L^s \cap L^\infty(R^n)$ for $s = 2n/\big((n+2) - \gamma(n-2)\big)$. The Mountain Pass Lemma, [1], [19] then yields the existence of a decaying positive solution $u \in E(R^n)$. If we further assume that q is symmetric with respect to $x_1 = 0$ and $x_1 \frac{\partial q}{\partial x_1} < 0$ for $x_1 \neq 0$ then any such solutions is symmetric in x_1 and $x_1 \frac{\partial u}{\partial x_1} < 0$ for $x_1 \neq 0$.

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