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Asymptotic instability of nonlinear differential equations *

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Abstract

This article shows that the zero solution to the system

$$x' = A(t)x + f(t,x), \quad f(t,0) = 0$$

is unstable. To show instability, we impose conditions on the nonlinear part $f(t,x)$ and on the fundamental matrix of the linear system $y' = A(t)y$. Our results generalize the instability results obtained by J. M. Bownds, Hatvani-Pintér, and K. L. Chiou.

1 Introduction

Bownds [1] studied stability properties of the second order differential equations

$$y'' + a(t)y = 0, \qquad t \geq 0 \tag{1}$$

$$x'' + a(t)x = f(t,x,x'), \quad t \geq 0\,, \tag{2}$$

where $a(t)$ is a continuous real-valued function. It is proved in [1] that, if (1) has a stable zero solution and has another solution with the property

$$\limsup_{t\to\infty}(|y(t)| + |y'(t)|) > 0\,, \tag{3}$$

then, under suitable conditions on f, there exists a solution x to (2) which satisfies (3). Bownds [1] conjectured that this result is true without the stability assumption for (1), conjecture that was later proven in [6]. This result and some other ideas from [6] have opened interesting possibilities in the study of asymptotic instability, as shown in [7].

This article concerns the generalization of the results given in [2] for the systems

$$\begin{aligned} y'(t) &= A(t)y(t), \quad t \geq 0 && (4)\\ x'(t) &= A(t)x(t) + f(t,x(t)), \quad f(t,0) = 0,\ t \geq 0 && (5)\\ x'(t) &= A(t)x(t) + b(t), \quad t \geq 0\,, && (6) \end{aligned}$$

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where $A(t)$, $f(t,x)$ and $b(t)$ are continuous functions, and $f(t,x)$ satisfies

$$|f(t,x)| \leq \gamma(t)|x|^m , \tag{7}$$

where m is a positive constant and γ is an integrable function. The following two theorems are proven in [2].

Theorem A [2] *Assume that the fundamental matrix, Φ, of System (4) satisfies*

$$|\Phi(t)\Phi^{-1}(s)| \leq K\frac{h(t)}{h(s)} , \quad s \geq t \geq t_0 , \tag{8}$$

for some constant K. If (7) is fulfilled with $\gamma \in L^1[0,\infty)$, and there exists a solution $y(t)$ of (4) such that

$$\limsup_{t\to\infty} |y(t)|0 , \tag{9}$$

then there exists a nontrivial solution $x(t)$ to (5) satisfying (9).

Theorem B [2] *If the linear system (4) has a solution y such that,*

$$0 < \limsup_{t\to\infty} |y(t)| \leq \infty , \tag{10}$$

then there exists a solution $x(t)$ of (6) satisfying (10).

Our goal is to extend Theorems A and B for functions $f(t,x)$ for which (7) holds more general functions γ. This generalization is obtained by using the notion of h-asymptotic instability.

2 Preliminaries

Let V^n denote one of the spaces $\mathbf{R}^n$ or $\mathbf{C}^n$. In this space $|x|$ denotes a fixed norm of a vector x, and $|A|$ denotes the corresponding matrix-norm of matrix A. Throughout this article, the function h is assumed to be positive and continuous, the interval $[0,+\infty)$ is denoted by J, and we use the following notation:

- $|x|_h = \sup_{t\geq 0} \left|\frac{x(t)}{h(t)}\right|$
- $C_h = \{x : J \to V^n : \ x \text{ is continuous and } |x|_h < \infty\}$,
- $B_h[0,1] := \{x \in C_h : \ |x|_h \leq 1 \ \}$,
- $L^1_h = \{x : J \to V^n : \ \int_0^\infty \frac{|x(t)|}{h(t)} ds < \infty\}$.

The following definitions are taken from [8].

Definition 1 *We say that the null solution to (5) is:*

h-Unstable *on J iff there exist an $\varepsilon > 0$ and $t_0 \in J$, such that for each $\delta > 0$, there exist an initial condition ξ_δ and a $t_\delta > 0$, such that*

$$|h(t_0)^{-1}\xi_\delta| < \delta, \quad \text{and } |h(t_\delta)^{-1}x(t_\delta, t_0, \xi_\delta)| \geq \varepsilon.$$

Asymptotically h-unstable *on J iff $x = 0$ is h-unstable or there exists a $t_0 \in J$, such that for any $\delta > 0$, there exists $\xi \in V^n$ such that*

$$|h(t_0)^{-1}\xi_\delta| < \delta, \quad \text{and } \limsup_{t\to\infty} |h(t)^{-1}x(t, t_0, \xi_\delta)| > 0.$$

3 h-Asymptotic instability

Theorem 1 *Assume that the fundamental matrix of system (4) satisfies (8), and the function $f(t,x)$ in (5) satisfies (7) with $\gamma \in L^1_{h^{1-m}}$. If there exists a solution of (4), such that*

$$0 < \limsup_{t\to\infty} |\frac{y(t)}{h(t)}| < \infty, \tag{11}$$

then there exists a nontrivial solution x of (5) with Property (11).

Proof. From (11), we may assume that for a fixed ε, with $0 < \varepsilon < 1$,

$$|h(t)^{-1}y(t)| \leq 1 - \varepsilon, \quad \forall t \geq 0\,. \tag{12}$$

Since $\gamma \in L^1_{h^{1-m}}$, there exists a positive t_0, such that

$$K \int_t^\infty h(s)^{m-1}\gamma(s)\, ds < \varepsilon\,, \quad \forall t \geq t_0, \tag{13}$$

where K is the same constant that appears in (8). We find a solution to (5) by finding a solution to the integral equation

$$x(t) = y(t) - \Phi(t)\int_t^\infty \Phi^{-1}(s) f(s, x(s))\, ds, \quad t \geq t_0\,,$$

on the set $B_h[0,1]$. For $x \in B_h[0,1]$, define

$$\mathcal{U}(x)(t) = y(t) - \int_t^\infty \Phi(t)\Phi^{-1}(s) f(s, x(s))\, ds. \tag{14}$$

Using (7), (8), and (12), we obtain

$$|h(t)^{-1}\mathcal{U}(x)(t)| \leq 1 - \varepsilon + \int_t^\infty |h(t)^{-1}\Phi(t)\Phi^{-1}(s)|\gamma(s)|x(s)|^m\, ds\,.$$

For $t \geq t_0$ we obtain

$$\begin{aligned} |h(t)^{-1}\mathcal{U}(x)(t)| &\leq 1-\varepsilon+\int_t^\infty |h(t)^{-1}\Phi(t)\Phi^{-1}(s)|\gamma(s)|h(s)h^{-1}(s)x(s)|^m\,ds \\ &\leq 1-\varepsilon+K\int_t^\infty h(s)^{m-1}\gamma(s)\,ds \\ &\leq 1-\varepsilon+\varepsilon=1\,. \end{aligned}$$

Hence $\mathcal{U}: B_h[0,1] \to B_h[0,1]$.

Now, we prove that $\mathcal{U}$ is continuous in the following sense: Suppose that a sequence $\{x_n\}$ in C_h converges uniformly to x on each compact subinterval of J, then $\mathcal{U}(x_n)$ converges uniformly to $\mathcal{U}(x)$ on each compact subinterval of J.

For a fixed $T > t_0$, we will show the uniform convergence of $\{\mathcal{U}(x_n)\}$ on $[t_0, T]$. Choose $t_1 > T$, such that $t > t_1$ implies

$$K\int_t^\infty h(s)^{-1}\gamma(s)\,ds \leq \frac{\varepsilon}{4}\,. \tag{15}$$

By the uniform convergence of $\{x_n\}$ on the interval $[t_0, t_1]$, there exists a positive integer $N = N(\varepsilon, t_1)$, such that $n \geq N$ implies

$$|f(s,x_n(s))-f(s,x(s))| \leq \varepsilon\left[2Kt_1\sup_{[t_0,t_1]}|h(t)^{-1}|\right]^{-1},\ \forall s\in[t_0,t_1]. \tag{16}$$

For $t \in [t_0, T]$ we write

$$|h(t)^{-1}\left[\mathcal{U}(x_n)(t)-\mathcal{U}(x)(t)\right]| \leq I_1+I_2+I_3, \tag{17}$$

where

$$\begin{aligned} I_1 &= \int_t^{t_1} |h(t)^{-1}\Phi(t)\Phi^{-1}(s)||f(s,x_n(s))-f(s,x(s))|\,ds \\ I_2 &= \int_{t_1}^\infty |h(t)^{-1}\Phi(t)\Phi^{-1}(s)||f(s,x_n(s))|\,ds \\ I_3 &= \int_{t_1}^\infty |h(t)^{-1}\Phi(t)\Phi^{-1}(s)||f(s,x(s))|\,ds\,. \end{aligned}$$

From (7) and (13) we obtain $I_2 \leq \frac{\varepsilon}{4}$ and $I_3 \leq \frac{\varepsilon}{4}$. From (15) we have $I_1 \leq \frac{\varepsilon}{2}$. These estimates and (17) yield

$$|h(t)^{-1}[\mathcal{U}(x_n)(t)-\mathcal{U}(x)(t)]| \leq \varepsilon,\ \forall t\in[t_0,T],$$

which proves the uniform convergence of $\mathcal{U}(x_n)$ to $\mathcal{U}(x)$ on $[t_0, T]$.

Now, we prove that the set of functions $\mathcal{U}(B_h[0,1])$ is equicontinuous at each point $t \in [t_0, \infty)$.

For each $x \in B_h[0,1]$, the function $z(t) = \mathcal{U}(x)(t)$ is a solution of the non-homogeneous linear system

$$z'(t) = A(t)z(t) + f(t, x(t)).$$

Since

$$|h(t)^{-1}z(t)| = |h(t)^{-1}\mathcal{U}(x)(t)| \le 1$$

and $|f(t, x(t))|$ is uniformly bounded on any finite t-interval, the set of all functions $z(t) = \mathcal{U}(x)(t)$, with $x \in B_h[0,1]$, is equicontinuous at each point of $[t_0, \infty)$. In this manner all the hypotheses of the Schauder-Tychonoff theorem [5] are satisfied. Consequently, there exists $x \in B_h[0,1]$ such that $x(t) = \mathcal{U}(x)(t)$, i.e. x satisfies the integral equation

$$x(t) = y(t) - \Phi(t)\int_t^\infty \Phi^{-1}(s)f(s, x(s))\,ds\,.$$

¿From (11) and

$$\lim_{t\to\infty}\int_t^\infty \Phi(t)\Phi^{-1}(s)f(s, x(s))\,ds = 0\,,$$

we obtain

$$\lim_{t\to\infty}|h(t)^{-1}[x(t) - y(t)]| = 0\,. \tag{18}$$

¿From (11) and (18) we conclude that (11) is satisfied with y replaced by x, and this proof is complete. ♢

Remarks Note that Theorem A follows from Theorem 1, by putting $h(t) = 1$. Also note that under the conditions of Theorem 1, if we assume that

$$\limsup_{t\to\infty} h(t) = \infty\,,$$

then the trivial solution of (5) is unstable in the sense of Liapounov.

Let us consider Equation (5) with

$$A(t) = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{t} \end{pmatrix}.$$

In this case the fundamental matrix Φ for system (4) satisfies

$$|\Phi(t)\Phi^{-1}(s)| \le t/s, \quad s \ge t.$$

Assume that $f(t, x)$ satisfies (7) with $t^{m-1}\gamma \in L^1$. Then, according to Theorem 1, Equation (5) yields a solution x satisfying

$$\limsup_{t\to\infty}|t^{-1}x(t)| > 0.$$

This property implies instability in the sense of Liapounov. Note that this result cannot be obtained from Perron's theorem [3], from Coppel's instability theorem [4], or from Theorem A.

Our next goal is to generalize Theorem B.

Theorem 2 *If there exists a solution y of (4) satisfying*

$$0 < \limsup_{t\to\infty} |h(t)^{-1} y(t)| \le \infty\,, \tag{19}$$

then there exists a solution x of (6) with the same property.

Proof. Note that every solution $x(t)$ of (6) has the form

$$x(t) = \Phi(t)c + \Phi(t)\int_0^t \Phi^{-1}(t)b(s)\,ds\,. \tag{20}$$

Let $y(t) = \Phi(t)c$ be a solution that satisfies (19). If

$$\limsup_{t\to\infty} |h(t)^{-1}\Phi(t)\int_0^t h(s)^{-1}\Phi^{-1}(s)b(s)ds| = 0, \tag{21}$$

we multiply (20) by $h(t)^{-1}$ to obtain

$$\begin{aligned}&\limsup_{t\to\infty} |h(t)^{-1}x(t)| \\ &\quad > \limsup_{t\to\infty} |h(t)^{-1}y(t)| - \limsup_{t\to\infty} |h(t)^{-1}\Phi(t)\int_0^t h(s)^{-1}\Phi^{-1}(s)b(s)\,ds|\,.\end{aligned}$$

¿From (19) and (21) it follows that $\limsup_{t\to\infty} |h(t)^{-1}x(t)|$ belongs to $(0,\infty]$. Therefore, (19) is satisfied with y replaced by x.

On the other hand, if

$$0 < \limsup_{t\to\infty} |h(t)^{-1}\Phi(t)\int_0^t h(s)^{-1}\Phi^{-1}(s)b(s)ds| \le \infty\,, \tag{22}$$

the assertion of this theorem follows independently of (19). ♢

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