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COMPLEX DYNAMICAL SYSTEMS ON BOUNDED SYMMETRIC DOMAINS

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Abstract. We characterize those holomorphic mappings which are the infinitesimal generators of semi-flows on bounded symmetric domains in complex Banach spaces.

1. Introduction

Let D be a bounded domain in a complex Banach space X. By $\mathrm{Hol}(D,X)$ we denote the set of holomorphic mappings from D into X. Let $\mathrm{Hol}(D)$ be the semigroup (with respect to composition) of all holomorphic self-mappings of D, and let $\mathrm{Aut}(D) \subset \mathrm{Hol}(D)$ be the subgroup consisting of all holomorphic automorphisms of D.

A family $S = \{F_t\} \subset \mathrm{Hol}(D)$, $t \geq 0$ $(-\infty < t < \infty)$, is called a continuous one-parameter semigroup (group) if

$$F_{s+t} = F_s \circ F_t, \quad t \geq 0 \quad (-\infty < t < \infty), \tag{1}$$

and

$$\lim_{\substack{t \to 0^+ \\ (t \to 0)}} F_t(x) = x, \quad x \in D. \tag{2}$$

A mapping $f \in \mathrm{Hol}(D,X)$ is said to be an infinitesimal generator of a semi-flow (complete flow) if there exists a one-parameter semigroup (group) $S_f = \{F_t\}$ such that for each $x \in D$,

$$f(x) = \lim_{\substack{t \to 0^+ \\ (t \to 0)}} \frac{x - F_t(x)}{t}, \tag{3}$$

where once again the limit is taken with respect to the norm of X. We denote by $\mathrm{hol}(D)$ the family of all (infinitesimal) holomorphic generators on D.

Note that if $f \in \mathrm{hol}(D)$ generates a complete flow $S_f = \{F_t\}_{t \in \mathbb{R}}$, then $F_t \in \mathrm{Aut}(D)$ and $F_t^{-1} = F_{-t}$ for all $t \in \mathbb{R}$. In this case one writes that $f \in \mathrm{aut}(D)$.

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It can be shown (see, for example, [10] and [11]) that since $f \in \mathrm{hol}(D)$ is locally bounded on D, the Cauchy problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + f(u(t,x)) = 0 \\ u(0,x) = x, \quad x \in D, \end{cases} \tag{4}$$

can be solved on $\mathbb{R}^+ = [0,\infty)$ for each $x \in D$ and $u(t,x) = F_t(x)$. Thus (4) defines an analytic dynamical system and $S_f = \{F_t\}_{t\geq 0}$ is a uniquely defined semi-flow on D.

Moreover, the convergence in (2) is uniform on each ball strictly inside D. If, in addition, $f \in \mathrm{aut}(D)$, then the Cauchy problem (4) can be solved for all $t \in \mathbb{R} = (-\infty,\infty)$.

Note also that if $g \in \mathrm{Hol}(D,X)$, then by allowing f to operate on g by means of the formula $(fg)(x) = g'(x) \circ f(x)$ we can interpret f as a derivation of $\mathrm{Hol}(D,X)$, i.e., as a holomorphic vector field. Using this terminology, $f \in \mathrm{hol}(D)$ will be called a semi-complete vector field, and $f \in \mathrm{aut}(D)$ a complete vector field (see, for example, [7], [6], [13] and [10]). It is known that $\mathrm{aut}(D)$ is a real Banach Lie algebra, while $\mathrm{hol}(D)$ is only a real cone (see [1], [10] and [11]).

Our purpose in this paper is to describe the class of semi-complete vector fields on a bounded symmetric domain. To motivate our approach we briefly review some previous results.

For the one-dimensional case, namely, $D = \Delta$, the open unit disk in the complex plane $\mathbb{C}$, an implicit condition which characterizes $\mathrm{hol}(\Delta)$ was obtained by E. Berkson and H. Porta [4].

It was shown by M. Abate [1] that their condition can be rewritten explicitly in the form

$$\mathrm{Re}\, f(x)\bar{x} \geq -\frac{1}{2}\mathrm{Re}\, f'(x)(1-|x|^2). \tag{5}$$

As a matter of fact, this condition is the special case $n = 1$ of a more general (and more complicated) condition, which is valid for the open Euclidean unit ball in $\mathbb{C}^n$ (see [1]).

On the other hand, it follows directly from the definition, that if $f \in \mathrm{hol}(D)$ has a continuous extension to $\bar{\Delta}$, then

$$\mathrm{Re}\, f(x)\bar{x} \geq 0 \quad \text{ for all } x \in \partial\Delta. \tag{6}$$

Unfortunately, it is not clear how to derive (6) from (5) in such a situation. At the same time, by rewriting (6) in the form

$$\mathrm{Re}\,[f(x) - f(0)]\bar{x} \geq -\mathrm{Re}\, f(0)\bar{x},$$

and dividing the left-hand side by $|x|^2 = 1$, we get

$$\mathrm{Re}\left(\frac{f(x)-f(0)}{x}\right) \geq -\mathrm{Re}\,\overline{f(0)}x, \quad x \in \partial\Delta.$$

Now it follows by the maximum principle for harmonic functions that the last inequality holds also for $x \in \Delta$. Multiplying it by $|x|^2$, $x \in \Delta$, $x \neq 0$, we obtain

$$\mathrm{Re}\, f(x)\bar{x} \geq \mathrm{Re}\, f(0)\bar{x}(1-|x|^2),\ x \in \Delta. \tag{7}$$

We claim that even if $f \in \mathrm{Hol}(\Delta,\mathbb{C})$ does not extend continuously to $\bar{\Delta}$, condition (7) is necessary and sufficient for f to be an infinitesimal generator of a semi-flow.

Indeed, for the case of the open unit ball B in a Hilbert space H, it was shown in [11], by using its hyperbolic metric, that the condition

$$\operatorname{Re}\langle f(x),x\rangle \geq \operatorname{Re}\langle f(0),x\rangle(1-\|x\|^2),\ \ x\in B, \tag{8}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in H, characterizes the class $\mathrm{hol}(B)$.

Note that a crucial point of the approach in [11] was the smoothness of the boundary of B. It is clear that such a property is no longer valid for the finite product B^n equipped with the max norm, and all the more so for the open unit ball in $\mathcal{L}(H,H)$, the space of bounded linear operators from H into H.

Another technical way to extend (8) to B^n, by using a special curve defined by a family of Möbius transformations, was employed in [12].

Therefore a natural idea which arises is that this be done for each Banach space X the open unit ball D of which is a homogeneous domain (i.e., for each pair $x,y\in D$ there is $F\in\mathrm{Aut}(D)$ such that $F(x)=y$).

Indeed, since every such ball is a bounded symmetric domain (see the definition below), one can propose using the more general and well-developed theory of such domains to derive an analog of condition (8) which will characterize $\mathrm{hol}(D)$.

It will become clear that such an approach does not require difficult calculations, and moreover, it establishes new facts concerning the description of semi-complete vector fields.

A domain D is called symmetric if for all $a\in D$ there exists $F_a\in\mathrm{Aut}(D)$ such that $F_a^2=I_D$ and a is an isolated fixed point of F_a.

For the case when D is a bounded symetric domain, the class $\mathrm{aut}(D)$ of all complete vector fields on D has been well-described with the help of an algebraic approach (see, for example, [7], [13], [3] and [6]). Namely, it is known that $\mathrm{aut}(D)$ is a real Banach Lie algebra and each $f\in\mathrm{aut}(D)$ is a polynomial of degree at most 2. Moreover, if

$$p=\{f\in\mathrm{aut}(D):\ f'(0)=0\} \tag{9}$$

and

$$k=\{f\in\mathrm{aut}(D):\ f(0)=0\}, \tag{10}$$

then $\mathrm{aut}(D)$ is the direct sum decomposition

$$\mathrm{aut}(D)=p\oplus k,$$

and each element of X can be realized as the constant term of a unique element of p, i.e., for each $y\in X$ there is a unique two-homogeneous polynomial P_y such that the mapping $g_y\in\mathrm{Hol}(X,X)$ defined by the formula

$$g_y(x)=y+P_y(x) \tag{11}$$

belongs to $p\subset\mathrm{aut}(D)$.

Furthermore, by Kaup's theorem [8], every bounded symmetric domain D can be realized as the open unit ball of a JB^*-triple system, and moreover, it is a homogeneous domain, i.e., for each pair $x,y\in D$ there is $F\in\mathrm{Aut}(D)$ such that $F(x)=y$.

Note also that an automorphism which moves the origin to $y\in D$ can be generated by $g\in p\subset\mathrm{aut}(D)$, i.e., g has the form (11) (see, for example, [13] and [6]).

So, in the sequel we will always assume that a bounded symmetric domain is realized as a convex balanced domain. At the same time, in this case the gauge of D (the Minkowski functional) can be defined as $c_D(0,\cdot)$, where $c_D(\cdot,\cdot)$ is the infinitesimal Carathéodory metric on D, and D is the indicatrix of this gauge, i.e.,

$$D = \{x \in X : c_D(0,x) < 1\}.$$

Thus, since D is bounded, $c_D(0,\cdot)$ is a norm which is equivalent to the norm of X, and D can be considered the open unit ball of X when it is equipped with this norm. So, our problem may be formulated as follows.

Let X be a complex Banach space such that the open unit ball D of X is a homogeneous domain. What are the geometric conditions which characterize semi-complete vector fields on D?

Let X' be the dual space of X. As usual, we use the pairing $\langle x, x'\rangle$ to denote the action of a linear functional $x' \in X'$ on an element $x \in X$. In particular, for $X = H$, a Hilbert space, $\langle\cdot,\cdot\rangle$ means the inner product in H. Recall also that the normalized duality mapping $J : X \to 2^{X'}$ is defined by

$$J(x) = \{x' \in X' : \langle x, x'\rangle = \|x\|^2 = \|x'\|^2\}.$$

2. Main result

Theorem 1. *Let X be a complex Banach space such that the open unit ball D of X is a homogeneous domain. Then the following assertions hold:*

1. *If $f \in \operatorname{hol}(D)$, then for each $x \in D$ and for each $x' \in J(x)$,*
$$Re\,\langle f(x), x'\rangle \geq Re\,\langle f(0), x'\rangle(1 - \|x\|^2). \tag{12}$$
2. *If $f \in \operatorname{Hol}(D,X)$ is bounded on each subset strictly inside D and for each $x \in D$ there exists $x' \in J(x)$ such that (12) holds, then $f \in \operatorname{hol}(D)$.*
3. *If $f \in \operatorname{hol}(D)$ and $S_f = \{F_t\}_{t\geq 0}$ is the semi-flow generated by f, then $F_t \in \operatorname{Hol}(D)$ satisfies the following estimate:*
$$\|F_t(x)\| \leq \frac{\|x\| + 1 - e^{-2\|f(0)\|t}(1 - \|x\|)}{\|x\| + 1 + e^{-2\|f(0)\|t}(1 - \|x\|)}. \tag{13}$$

To prove our theorem we need several preliminary assertions.

Proposition 1. [10], [11]. *Let D be a bounded convex domain in X. Then $f \in \operatorname{Hol}(D,X)$ is semi-complete (i.e., belongs to $\operatorname{hol}(D)$) if and only if for each $\lambda > 0$ the nonlinear resolvent $R(\lambda, f) = (I + \lambda f)^{-1}$ is a well-defined holomorphic self-mapping of D.*

In addition, if $S_f = \{F_t\}_{t\geq 0}$ is the semi-flow generated by f, then it can be given by the exponential formula

$$F_t = \lim_{n\to\infty} R^n(\frac{1}{n}t, f), \quad t \geq 0, \tag{14}$$

where the limit in (14) is taken with respect to the norm of X uniformly on each subset strictly inside D.

Proposition 2. [10], [11]. *Let D be as in Proposition 1. Then $\operatorname{hol}(D)$ is a real cone, i.e., for each pair f and g from $\operatorname{hol}(D)$ and all $\alpha, \beta > 0$, the mapping $\alpha f + \beta g$ also belongs to $\operatorname{hol}(D)$.*

Since $\operatorname{aut}(D) = \operatorname{hol}(D) \cap (-\operatorname{hol}(D))$ is a linear space, Proposition 2 immediately implies the following assertion.

Proposition 3. *Let D be a bounded balanced convex symmetric domain in X. Then each element $f \in \mathrm{hol}(D)$ can be represented as*

$$f \ = \ h + g, \tag{15}$$

where $h \in \mathrm{hol}(D)$ with $h(0) = 0$ and $g = g_y \in p \subset aut(D)$ is defined by (11) with $y = f(0)$. This representation is unique.

Proposition 4. *Let $f \in \mathrm{hol}(D)$ be as above, and let $g_{f(0)} \in p \subset aut(D)$ be defined by (11). Then for each $x \in D$ and for each $x' \in J(x)$ the following inequality holds:*

$$Re\,\langle f(x), x'\rangle \geq Re\,\langle g_{f(0)}(x), x'\rangle. \tag{16}$$

Proof. Indeed, it follows by (15) that $h = f - g_{f(0)}$ belongs to $\mathrm{hol}(D)$ and

$$h(0) = 0. \tag{17}$$

Let $S_h = \{\mathcal{H}_t\}_{t\geq 0} \subset \mathrm{Hol}(D)$ be the semi-flow generated by h, i.e., for each $x \in D$,

$$\lim_{t\to 0^+} \frac{x - \mathcal{H}_t(x)}{t} = h(x).$$

It follows by the uniqueness of the solution to the Cauchy problem (4) and by (17) that the origin is a common fixed point of $S_h = \{\mathcal{H}_t\}_{t\geq 0}$ for all $t \geq 0$. Since $\|\mathcal{H}_t(x)\| \leq 1$, it follows by the Schwarz Lemma that $\|\mathcal{H}_t(x)\| \leq \|x\|$ for all $x \in D$. Now using (17), we get

$$\mathrm{Re}\,\langle h(x), x'\rangle \geq 0 \tag{18}$$

for all $x' \in J(x)$. By the definition of h, (18) is exactly (16), and we are done.

Now it is very easy to prove the necessity of (12) for f to be a semi-complete vector field. In fact, for each $u \in \partial D$ and each $g \in \mathrm{aut}(D)$ we have

$$\mathrm{Re}\,\langle g(u), u'\rangle = 0 \tag{19}$$

whenever $u' \in J(u)$ (note that g is holomorphically extensible to ∂D). In particular, this holds for $g_y = y + P_y(x) \in p$ where P_y is a homogeneous polynomial of degree 2. Therefore, if for $x \in D, x \neq 0$, we set $u = \frac{1}{\|x\|}x$, we obtain

$$\begin{aligned}
\mathrm{Re}\,\langle g_y(x), x'\rangle \ &= \ \mathrm{Re}\,\langle y + P_y(x), x'\rangle \ = \ \mathrm{Re}\,\langle y, x'\rangle + \mathrm{Re}\,\langle P_y(x), x'\rangle \\
&= \ \mathrm{Re}\,\langle y, x'\rangle + \|x\|^3 \mathrm{Re}\,\langle P_y(u), u'\rangle \\
&= \ \mathrm{Re}\,\langle y, x'\rangle + \|x\|^3 (\mathrm{Re}\,\langle P_y(u), u'\rangle + \langle y, u'\rangle) \\
&\quad\ -\|x\|^3 \mathrm{Re}\,\langle y, u'\rangle \\
&= \ \mathrm{Re}\,\langle y, x'\rangle - \|x\|^2 \mathrm{Re}\,\langle y, \|x\| u'\rangle \\
&= \ \mathrm{Re}\,\langle y, x'\rangle (1 - \|x\|^2).
\end{aligned}$$

Using this equality with $y = f(0)$ and (16) we obtain (12). Assertion 1 of our theorem is proved. To prove assertions 2 and 3 we first establish a somewhat more general proposition.

Proposition 5. *Let X be an arbitrary complex Banach space, and let D be the open unit ball in X. Suppose that $f \in \mathrm{Hol}(D, X)$ is bounded on each subset strictly inside D and satisfies the following condition: For each $x \in D$ and some $x' \in J(x)$,*

$$Re\,\langle f(x), x'\rangle \geq \alpha(\|x\|) \cdot \|x\|, \tag{20}$$

where $\alpha : [0, 1] \to \mathbb{R}$ is an increasing continuous function on $[0, 1]$ such that

$$\alpha(0) \cdot \alpha(1) \leq 0. \tag{21}$$

Then

1. f *is a semi-complete vector field on* D.
2. *If* $S_f = \{F_t\}$ *is the semi-flow generated by* f*, then for all* $t \geq 0$ *and* $x \in D$*,*
$$\|F_t(x)\| \leq \beta_t(\|x\|), \tag{22}$$
where β_t *is the solution of the Cauchy problem*
$$\begin{cases} \frac{d\beta_t(s)}{dt} + \alpha(\beta_t(s)) = 0, \\ \beta_0(s) = s, \quad s \in [0,1]. \end{cases} \tag{23}$$

Proof. Fix $r \in (0,1)$ and consider the equations
$$x + \lambda f(x) = z \tag{24}$$
$$s + \lambda\alpha(s) = \|z\|, \tag{25}$$
where $z \in \bar{D}_r = \{x \in X : \|x\| \leq r < 1\}, s \in [0,1]$, and $\lambda > 0$. It follows from (21) that for a fixed $z \in \bar{D}_r$, the function $\gamma(s) = s + \lambda\alpha(s) - \|z\|$ satisfies the conditions $\gamma(0) \leq 0,\ \gamma(1) > 0$. Hence equation (25) has a unique solution $s_0 = s_0(z) \in [0,1)$. So, for an arbitrary $\delta > 0$ we can find $\epsilon > 0$ such that $\gamma(s_0 + \delta) \geq \epsilon$. Now taking $x \in D$ such that $\|x\| = s = s_0 + \delta$, we have by (20) for such x and any $x' \in J(x)$,
$$\begin{aligned} \mathrm{Re}\,\langle x + \lambda f(x) - z, x'\rangle &= \mathrm{Re}\,(\langle x, x'\rangle + \lambda\langle f(x), x'\rangle - \langle z, x'\rangle) \\ &\geq s^2 + \lambda\alpha(s)\cdot s - \|z\|\cdot s \\ &= s\gamma(s) \ \geq\ s\cdot\epsilon. \end{aligned}$$
It follows by the same considerations as in Theorem 3 in [2] that equation (24) has a unique solution $x = x(z)$ such that $\|x(z)\| \leq s_0 + \delta$. Since $\delta > 0$ is arbitrary, we must have
$$\|x(z)\| \ \leq\ s_0.$$
In terms of nonlinear resolvents the last inequality can be rewritten as
$$\begin{aligned} \|R(\lambda, f)(z)\| &= \|(I_X + \lambda f)^{-1}(z)\| \ \leq\ R(\lambda,\alpha)(\|z\|) \\ &= (I_{\mathbb{R}} + \lambda\alpha)^{-1}(\|z\|). \end{aligned}$$
Now using Proposition 1 and the exponential formula (14) we deduce our assertion.

To prove our theorem we need only observe that the function
$$\alpha(s) \ =\ -\|f(0)\|(1 - s^2) \tag{26}$$
satisfies all the conditions of Proposition 5, and that the solution $\beta_t(s)$ of the Cauchy problem (23) with α defined by (26) has the same form as the right-hand side of (13). The theorem is proved.

Remark 1. *If* X *is a* J^**-algebra, then condition (16) can be rewritten in the form*
$$Re\,\langle f(x), x'\rangle \geq Re\,\langle f(0) - x[f(0)]^* x, x'\rangle, \tag{27}$$
which also characterizes those mappings $f \in \mathrm{Hol}(D,X)$ *which are semi-complete vector fields on the open unit ball of X.*

For example, consider the case of the algebra $X = \mathcal{L}_c(H_1, H_2)$ of all linear compact operators $\mathcal{A} : H_1 \to H_2$ ($\mathcal{A}$ is defined on the whole of H_1 and maps it compactly into H_2), when H_1 and H_2 are Hilbert spaces.

Let $\mathcal{D}$ be the open unit operator ball of $\mathcal{L}_c(H_1, H_2)$, that is, $\mathcal{D} = \{\mathcal{A} \in \mathcal{L}_c(H_1, H_2) : \|\mathcal{A}\| < 1\}$. Suppose that the mapping f belongs to $\mathrm{Hol}(D, X)$. It is easy to see that for any $\mathcal{A} \in \mathcal{L}_c(H_1, H_2)$ there exists $x_{\mathcal{A}} \in H_1$ such that $\|\mathcal{A}\| = \|\mathcal{A}x_{\mathcal{A}}\|$ and

$\|x_{\mathcal{A}}\| = 1$. Indeed, $\|\mathcal{A}\| = \sup\limits_{\substack{\|x\|=1 \\ x\in H_1}} \|\mathcal{A}x\|$, so there exists $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ and $\|\mathcal{A}x_n\| \to \|\mathcal{A}\|$, as $n \to \infty$. Since H_1 is a Hilbert space, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to some $x_A \in H_1$. Since $\mathcal{A}$ is compact, $\mathcal{A}x_{n_k} \to \mathcal{A}x_A$ as $k \to \infty$. Hence $\|\mathcal{A}x_{\mathcal{A}}\| = \|\mathcal{A}\|$ and $\|x_{\mathcal{A}}\| = 1$.

For any $\mathcal{A} \in \mathcal{L}_c(H_1, H_2)$ we construct the support functional $g_{\mathcal{A}} \in (\mathcal{L}_c(H_1, H_2))^*$ in the following way:

$$g_{\mathcal{A}}(T) := (Tx_{\mathcal{A}}, \|\mathcal{A}\|^{-1}\mathcal{A}x_A), \; T \in \mathcal{L}_c(H_1, H_2).$$

((x, y) is the scalar product in H_2).

We have $|g_{\mathcal{A}}(T)| \le \|Tx_{\mathcal{A}}\|\|x_{\mathcal{A}}\| \le \|T\|, \quad g_{\mathcal{A}}(\mathcal{A}) = \|\mathcal{A}\|$, hence $\|g_{\mathcal{A}}\| = 1$. Thus $g_{\mathcal{A}}$ belongs to $J(\mathcal{A})$.

The following condition is a natural analog of (7) for this algebra:

$$\operatorname{Re} \mathcal{A}^* f(A) \ge \operatorname{Re} \mathcal{A}^* f(0)(\mathcal{I} - |\mathcal{A}|^2) \tag{28}$$

(here $|\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A}$).

We claim that this simple condition implies (27). Indeed, (28) is equivalent to

$$\begin{aligned}
\operatorname{Re}(\mathcal{A}^* f(\mathcal{A})x, x) &\ge \operatorname{Re}(\mathcal{A}^* f(0)(\mathcal{I} - |\mathcal{A}|^2)x, x) \\
&= \operatorname{Re}((\mathcal{A}^* f(0)x, x) - A^* f(0)\mathcal{A}^*\mathcal{A}x, x)) \\
&= \operatorname{Re}((\mathcal{A}^* f(0)x, x) - (\mathcal{A}^*\mathcal{A}[f(0)]^*\mathcal{A}x, x)).
\end{aligned}$$

Hence for $x = x_{\mathcal{A}}$ we obtain:

$$\operatorname{Re}(f(\mathcal{A})x_{\mathcal{A}}, \mathcal{A}x_{\mathcal{A}}) \ge \operatorname{Re}((f(0)x_{\mathcal{A}}, \mathcal{A}x_{\mathcal{A}}) - (\mathcal{A}[f(0)]^*\mathcal{A}x_{\mathcal{A}}, \mathcal{A}x_{\mathcal{A}}),$$

or, setting $\mathcal{A}'$ to be $g_{\mathcal{A}}$,

$$\operatorname{Re}\langle f(\mathcal{A}), \mathcal{A}'\rangle \ge \operatorname{Re}\langle f(0) - \mathcal{A}[f(0)]^*\mathcal{A}, \mathcal{A}'\rangle,$$

which is precisely (27).

Note that in the particular case when $\min(\dim H_1, \dim H_2) < \infty$, $\mathcal{L}_c(H_1, H_2) = \mathcal{L}(H_1, H_2)$, the space of all bounded linear operators $\mathcal{A} : H_1 \to H_2$. So in this case all of the above is also true for the open unit ball $\mathcal{D}$ of $\mathcal{L}(H_1, H_2)$.

Remark 2. *If $f \in \operatorname{hol}(D)$, then it follows from the representation (15) (see Proposition 3) that the linear operator $A = f'(0)$ is accretive.*

Indeed, if $h = f - g_{f(0)}$, then $h'(0) = f'(0) = A$. But $h(0) = 0$ and the origin is a common fixed point of the semi-flow $S_h = \{\mathcal{H}_t\}_{t\ge0}$. Using the Cauchy inequalities, it is easy to check that the family $\{B_t = (\mathcal{H}_t)'(0)\}_{t\ge0}$ is a semigroup of linear contractions generated by A. Therefore A is accretive by the Lumer-Phillips Theorem.

Thus, if in the J^*-algebra X we consider the Riccati flow equation

$$\begin{cases} \dot{x}_t = a + bx_t - x_t a^* x_t, \\ x_0 = x \in D, \end{cases}$$

then this equation has a solution on $D \times \mathbb{R}^+$ if and only if the element $b \in X$ defines an accretive linear operator by $x \mapsto bx$.

Remark 3. *As a matter of fact, if under the conditions of our Theorem, the operator $B = iA$, where $A = f'(0)$, is Hermitian, i.e., Re $\langle Ax, x'\rangle = 0$ for all $x \in X$ and $x' \in J(x)$, then $f \in \operatorname{hol}(D)$ actually belongs to aut(D).*

Indeed, it is enough to prove that h in the representation (15) has the form

$$h(x) \;=\; f'(0)x. \tag{29}$$

To see this, let us represent $h(x)$ by the Taylor formula

$$h(x) = h'(0)x + k(x),$$

where $k(x)$ contains the terms of order greater or equal to 2. Then, by (18), we have

$$\mathrm{Re}\,\langle h(x), x'\rangle = \mathrm{Re}\,\langle h'(0)x, x'\rangle + \mathrm{Re}\,\langle k(x), x'\rangle \geq 0.$$

Since $h'(0) = f'(0)$ we see that

$$\mathrm{Re}\,\langle k(x), x'\rangle \geq 0.$$

Since $k(0) = 0$, we get by the theorem that $k \in \mathrm{hol}(D)$. But $k'(0) = 0$ and it follows by the infinitesimal version of the Cartan Uniqueness Theorem (see [10]) that $k = 0$ and we are done.

Following S. G. Krein [9] (see also E. Vesentini [14]), a linear operator $A : X \to X$ such that $\mathrm{Re}\,\langle Ax, x'\rangle = 0$ for all $x \in X$ and $x' \in J(x)$ is called a conservative operator. So we have the following result.

Corollary 1. *Let $f \in \mathrm{hol}(D)$. Then f is a complete vector field ($f \in aut(D)$ if and only if the operator $f'(0)$ is conservative.*

The following proposition is a direct consequence of assertion 3 of the Theorem. It is motivated by Proposition 7 in [5].

Corollary 2. *Let $S = \{F_t\}_{t\geq 0}$ be a one-parameter semigroup of holomorphic self-mappings of D such that F_t converges to I, as $t \to 0^+$, locally uniformly on D. Then for each $\rho \in (0,1)$, $M \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$, there exists a positive number $A = A(\rho, M, \alpha) < 1$ such that*

$$\sup\{\|F_t(x)\| : \|\xi\| \leq M, \quad \|x\| \leq \rho,\ 0 \leq t \leq \alpha\} \leq A,$$

where $\xi = \frac{d^+F_t(0)}{dt}$.

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