ELECTRONIC JOURNAL OF DIFFERENTIAL EQUATIONS, Vol. **1998**(1998), No. 07, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

Exponential stability of a Von Karman model with thermal effects *

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Abstract

A one-dimensional Von Karman model with thermal effects is studied. We derive the equations that constitute the mathematical model, and prove existence and uniqueness of a global solution. Then using Lyapunov functions, we show that solutions decay exponentially.

1 Introduction

In the last few years, the asymptotic behaviour of the coupling between elastic and heat phenomena has been studied by several authors. Most of their results concern the linear case, see for example [5, 17, 6, 1, 13] and references therein. Analysis of these articles shows that the linear thermoelastic plate models (coupling of plate and heat) and the standard linear thermoelastic system (coupling between the wave and heat equations) have different properties. The first model is always exponentially stable (namely the energy approaches zero exponentially when time approaches infinity), while the second model has this property only in certain domains. The second model consists of the system

$$
\begin{gathered}
\partial_{tt} u - \Delta u - \beta \operatorname{grad}(\operatorname{div} u) + m \operatorname{grad} \theta = 0 \quad \text{in } \Omega \\
\partial_t \theta - k \Delta \theta + m \operatorname{div} \partial_t u = 0 \quad \text{in } \Omega \\
u = \theta = 0 \quad \text{on } \partial\Omega \,,
\end{gathered}
$$

where β, m, k are positive constants, u is the displacement and θ the temperature. For this model, D. B. Henry, A. Perissinitto and O. Lopes [7] proved that the exponential stability is equivalent to that of the decoupled system

$$
\begin{gathered}
\partial_{tt} u - \Delta u - \beta \operatorname{grad}(\operatorname{div} u) + (m^2/k) \operatorname{grad} \Delta^{-1} \operatorname{div} \partial_t u = 0 \quad \text{in } \Omega \\
\partial_t \theta - k \Delta \theta + m \operatorname{div} \partial_t u = 0 \quad \text{in } \Omega \\
u = \theta = 0 \quad \text{on } \partial\Omega \,.
\end{gathered}
$$

* *1991 Mathematics Subject Classifications:* 35M10, 73B30.
Key words and phrases: Thermoelastic systems, Von Karman, asymptotic behaviour, exponential stability, semigroup, Lyapunov function.

Submitted December 29, 1997. Published February 27, 1998.
Research partially supported by the French-Algerian agreement No. 96 MDU 378.

Here the operator grad Δ^{-1} div is a projection whose range is the irrotational part of the velocity field. The question is whether the control of this part of the velocity field is sufficient to ensure uniform stability. In the one-dimensional model, in higher dimensions in the presence of symmetry properties, and in very special domains (excluding convex domains) the answer is positive. See [7, 14] for the one-dimensional case, [3, 17] for the presence of symmetry, and [8, 11] for special domains.

Results for various nonlinear models have been obtained in [2, 15, 17] and their references. In particular, [2, 15] concern Von Karman models with thermal effects. In [1], the authors construct simple Lyapunov functions for a different thermoelastic plate model. In this paper, we use these functions to prove stability results for a Von Karman model with thermal effects.

The plan of this paper is to derive the equations, then prove existence and uniqueness of a global weak solution, and finally demonstrate exponential stability of the model.

Our proof of existence and uniqueness of weak solutions is directly inspired by the techniques used in [9], where uniform stabilization of a nonlinear beam by a nonlinear boundary feedback is obtained.

We restrict our work to the one-dimensional problem, for the following two reasons. The first one is the difficulty in obtaining uniqueness for the multi-dimensional Von Karman models in the energy space we consider. To our knowledge, there exist only partial results in this case, [16, 18]. In [16], existence and uniqueness of a global strong solution in two dimensional bounded domains is proven, but without uniqueness for finite energy solutions. In [18], the authors prove existence and uniqueness for finite energy solutions in $\mathbb{R}^2$, in rectangular domains, and outside a convex obstacle. These difficulties also appear in the thermal case. In fact, for (1) with $\gamma > 0$, it is known that the linear part has no regularization property. The second reason is the presence of planar strain in the coupling (see the first and third equation in (1)). Recall that exponential stability for the thermoelastic system has been proved in the one-dimensional case, and only for special domains in higher dimensions.

2 Derivation of the model

Consider the planar motion of a beam that occupies, in the reference position, the region

$$U = \{(x, y, z); \quad 0 \leq x \leq L,\ -1 \leq y \leq 1,\ \frac{-h}{2} \leq z \leq \frac{h}{2}\}\,.$$

In this setting, L is the length of the beam, and the segment $\{0 \leq x \leq L,\ y = z = 0\}$ is called the medium line of the beam.

The fact that the beam is stretchable implies the existence of nonlinear terms in the equations describing the motion. In addition to the mechanical load, we assume that the body is subjected to an unknown heat distribution, τ, that vanishes at the boundary of the beam.

Let the displacement be denoted by $(u,w)=((u_1,u_2),w)$, and the domain by

$$\Omega=\{(x,y,0),\,0<x<L,\,-1<y<1\}\,.$$

It is known [9, 10] that, up to a normalization of both the physical constants and h, the mechanical energy of the system is given by

$$\begin{aligned} K(t) &= \frac{1}{2}\{\int_\Omega |\partial_t u|^2\,dxdy+\int_\Omega |\partial_t w|^2\,dx\,dy+\gamma^2\int_\Omega |\partial_t\nabla w|^2\,dx\,dy \\ &\quad +(C(\varepsilon(u(t)+f(\nabla w(t)),\varepsilon(u(t)+f(\nabla w(t)))_0 \\ &\quad +\int_\Omega |\Delta w|^2\,dx-\int_\Omega \alpha(\widetilde{\theta}\,\mathrm{div}\,u+\theta\Delta w)\,dx\,dy\,\}\,, \end{aligned}$$

where $C(\varepsilon(u(t)+f(\nabla w(t))$ is the strain tensor in the plane (x,y), ε is the tensor of deformations, α a positive constant, and $\widetilde{\theta}$ and θ are thermal strain resultants with

$$f(\nabla w)=\frac{1}{2}\nabla w\otimes\nabla w\,,\qquad \widetilde{\theta}=\frac{1}{h}\iint_{-\frac{h}{2}}^{\frac{h}{2}}\tau\,dz\,,\qquad \theta=\frac{12}{h^3}\iint_{-\frac{h}{2}}^{\frac{h}{2}}z\tau\,dz\,.$$

We also assume that the motion occurs in the xz-plane, in which case the energy becomes

$$\begin{aligned} K(t) &= \int_0^L\Big(|\partial_t u_1(t)|^2+|\partial_t w(t)|^2+\gamma^2\,|\partial_t\partial_x w(t)|^2\Big)\,dx \\ &\quad +\int_0^L[|\partial_{xx}w|^2+|\partial_x u_1+\frac{1}{2}(\partial_x w)^2|^2]\,dx-\int_0^L\alpha(\varphi\partial_x u_1+\psi\partial_{xx}w)\,dx\,, \end{aligned}$$

where

$$\varphi=\int_{-1}^1\widetilde{\theta}\,dy\,,\qquad \psi=\int_{-1}^1\theta\,dy\,.$$

Finally, we suppose that on the boundary, the displacement is only horizontal, which implies

$$w(x,.)=\partial_x w(x,.)=0\,,\quad \text{for } x=0,\ x=L\,.$$

Then the dynamical variation δ satisfies

$$\delta K=0\,,$$

and we deduce the following equations (where u_1 is denoted by u).

$$\begin{cases} \partial_{tt}u-\partial_x(\partial_x u+\frac{1}{2}(\partial_x w)^2)=\partial_x\varphi, \quad (x,t)\in]0,L[\times\mathbb{R}^+ \\ \partial_{tt}(I-\gamma^2\partial_{xx})w+\partial_{xxxx}w \\ -\partial_x[(\partial_x u+\frac{1}{2}(\partial_x w)^2)\partial_x w]=-\alpha\partial_{xx}\psi,\ (x,t)\in]0,L[\times\mathbb{R}^+ \\ \partial_x u(0,t)=\partial_x u(L,t)=w(0,t)=w(L,t)=\partial_x w(0,t)=\alpha\partial_x w(L,t)=0\,. \end{cases}$$

The two heat equations have the following form. (see [10])

$$\begin{cases} \partial_t\varphi - \partial_{xx}\varphi = \alpha\partial_x\partial_t u, \ (x,t) \in]0,L[\times\mathbb{R}^+ \\ \partial_t\psi - \partial_{xx}\psi = \alpha\partial_{xx}\partial_t w, \ \ (x,t) \in]0,L[\times\mathbb{R}^+ \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \ \ t \in \mathbb{R}^+ \end{cases}$$

So that for $\alpha = 1$, we obtain

$$\begin{cases} \partial_{tt}u - \partial_x(\partial_x u + \frac{1}{2}(\partial_x w)^2) = \partial_x\varphi, \ \ (x,t) \in]0,L[\times\mathbb{R}^+ \\ \partial_{tt}(I - \gamma^2\partial_{xx})w + \partial_{xxxx}w \\ -\partial_x[(\partial_x u + \frac{1}{2}(\partial_x w)^2)\partial_x w] = -\partial_{xx}\psi, \ (x,t) \in]0,L[\times\mathbb{R}^+ \\ \partial_t\varphi - \partial_{xx}\varphi = \partial_x\partial_t u, \ (x,t) \in]0,L[\times\mathbb{R}^+ \\ \partial_t\psi - \partial_{xx}\psi = \partial_{xx}\partial_t w, \ \ (x,t) \in]0,L[\times\mathbb{R}^+ \\ \partial_x u(0,t) = \partial_x u(L,t) = 0 \\ w(0,t) = w(L,t) = \partial_x w(0,t) = \partial_x w(L,t) = 0, \ t \in \mathbb{R}^+ \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \ \ t \in \mathbb{R}^+ \\ u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x), x \in]0,L[\\ w(x,0) = w(x), \ \partial_t w(x,0) = w_1(x), \ \ x \in]0,L[\\ \varphi(x,0) = \varphi(x), \ \psi(x,0) = \psi_0(x), \ \ \ x \ \in]0,L[\end{cases} \tag{1}$$

3 Existence and uniqueness of a solution

Existence follows from the argument in the paper by J. Lagnese and G. Leugering [9]. Nevertheless, we provide all the details for the coupled equation. Let $\Omega = (0,L)$, and rewrite the system above in the form

$$\begin{aligned} CY' &= AY + F(Y) \\ Y(0) &= Y_0\,, \end{aligned} \tag{2}$$

where

$$Y = \begin{bmatrix} u \\ v \\ \varphi \\ w \\ z \\ \psi \end{bmatrix}, \quad C = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & (I-\gamma^2\partial_{xx}) & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \partial_{xx} & 0 & \partial_x & 0 & 0 & 0 \\ 0 & \partial_x & \partial_{xx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\partial_{xxxx} & 0 & -\partial_{xx} \\ 0 & 0 & 0 & 0 & \partial_{xx} & \partial_{xx} \end{pmatrix},$$

and

$$F(Y)=\begin{pmatrix} 0 \\ \frac{1}{2}\partial_x((\partial_x w)^2) \\ 0 \\ 0 \\ \partial_x[\ (\partial_x u+\frac{1}{2}\ (\partial_x w)^2)\partial_x w] \\ 0 \end{pmatrix}.$$

Let the energy space be

$$H=\widetilde{H}^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times H_0^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\,,$$

where $\widetilde{H}^1(\Omega)$ is the Sobolev space $H^1(\Omega)$ with null average,

$$\widetilde{H}^1(\Omega)=\{u\in H^1(\Omega);\ \int_\Omega u(x)\,dx=0\}\,.$$

Let $|.|$ denote the norm in $L^2(\Omega)$, and $\|.\|$ denote the norm in H,

$$\|Y\|^2=|\nabla u|^2+|v|^2+|\varphi|^2+|\Delta w|^2+\left|L_\gamma^{1/2}z\right|^2+|\psi|^2\ .$$

with $L_\gamma=(I-\gamma^2\Delta)$ and Δ being the Dirichlet-Laplace operator.

Theorem 1 *For all $Y_0\in H$ there exists a unique weak solution Y of (2) such that*

$$Y\in C(\mathbb{R}^+,H)\,.$$

We prove this theorem as follows: First it is shown that the linear part defines a semigroup of solutions, and the nonlinear part is Lipschitz. From these two facts, we conclude the existence of a local solution. The proof is then completed by establishing estimates, on the local solution, that avoid blowup in finite time; hence, ensuring global existence.

Lemma 2 *$C^{-1}A$ is a generator of a semigroup of contractions in H.*

Proof. One has

$$\begin{aligned} D(C^{-1}A) \;=\; &\big\{Y\in H\,:\,\Delta u\in L^2(\Omega),\, v\in\widetilde{H}^1(\Omega),\, \varphi\in H^2(\Omega)\cap H_0^1(\Omega),\\ &\quad w\in H^4(\Omega)\cap H_0^2(\Omega)\ z\in H_0^1(\Omega),\, \psi\in H^2(\Omega)\cap H_0^1(\Omega)\big\}\,. \end{aligned}$$

The operator $C^{-1}A$ is a generator of a semigroup of contractions, because it is the diagonal matrix of two operators that are generators of semigroups of contractions. Those two operators are: the thermoelasticity

$$A_1=\begin{pmatrix} 0 & I & 0 \\ \partial_{xx} & 0 & \partial_x \\ 0 & \partial_x & \partial_{xx} \end{pmatrix},$$

and the thermoplates

$$A_2=\begin{pmatrix} 0 & I & 0 \\ -L_\gamma^{-1}\partial_{xxxx} & 0 & -L_\gamma^{-1}\partial_{xx} \\ 0 & \partial_{xx} & \partial_{xx} \end{pmatrix}.$$

Existence and uniqueness of a local solution

The nonlinear part $C^{-1}F$ of (2) can be considered as a perturbation of the operator $C^{-1}A$. So, to prove local existence and uniqueness of a solution, we have to verify that $C^{-1}F$ is locally Lipschitz continuous in H. (See Theorem 4.3.4. p. 57 in [4])

Lemma 3 *The function F is locally Lipschitz continuous in H*

Proof. For $Y \in H$, define the energy

$$E(Y) = \frac{1}{2}\left\{\left|\partial_x u + \frac{1}{2}(\partial_x w)^2\right|^2 + |v|^2 + |\varphi|^2 + |\partial_{xx} w|^2 + \left|L_\gamma^{1/2} z\right|^2 + |\psi|^2\right\}.$$

For $Y_1, Y_2 \in B(0,R)$, one has

$$\begin{aligned}
&\|F(Y_1) - F(Y_2)\| \\
&= \frac{1}{2}\left|\partial_x((\partial_x w_1)^2) - \partial_x((\partial_x w_2)^2)\right| \qquad (3)\\
&\quad + \left|L_\gamma^{-1/2}\left(\partial_x[\,(\partial_x u_1 + \frac{1}{2}\,(\partial_x w_1)^2)\partial_x w_1] - \partial_x[\,(\partial_x u_2 + \frac{1}{2}\,(\partial_x w_2)^2)\partial_x w_2]\right)\right|.
\end{aligned}$$

The first term on the right is estimated as follows:

$$\begin{aligned}
&\left|\partial_x((\partial_x w_1)^2) - \partial_x((\partial_x w_2)^2)\right| \\
&= |\partial_x\ ((\partial_x w_1 - \partial_x w_2)(\partial_x w_1 + \partial_x w_2))| \\
&\le |\partial_{xx}(w_1 - w_2)|\left(\|\partial_x w_1\|_{L^\infty(\Omega)} + \|\partial_x w_2\|_{L^\infty(\Omega)}\right) \\
&\quad + \|\partial_x w_1 - \partial_x w_2\|_{L^\infty(\Omega)}\left(|\partial_{xx} w_1| + |\partial_{xx} w_2|\right).
\end{aligned}$$

As the space has dimension one, we have the embedding

$$H^1(\Omega) \subset L^\infty(\Omega).$$

Therefore, there exist positive constants denoted by C such that

$$\begin{gathered}
\left(\|\partial_x w_1\|_{L^\infty(\Omega)} + \|\partial_x w_2\|_{L^\infty(\Omega)}\right) \le C\left(\|Y_1\| + \|Y_2\|\right), \\
\|\partial_x w_1 - \partial_x w_2\ \|_{L^\infty(\Omega)} \le C\|Y_1 - Y_2\|.
\end{gathered}$$

Since $(\|Y_1\| + \|Y_2\|) \le 2R$, the first term on the right-hand side of (3) is bounded by

$$K(R)\|Y_1 - Y_2\|.$$

Let's estimate the second term in the right-hand side of (3).

$$\begin{aligned}
&\left|L_\gamma^{-1/2}\left(\partial_x[\,(\partial_x u_1 + \frac{1}{2}(\partial_x w_1)^2)\partial_x w_1] - \partial_x[(\partial_x u_2 + \frac{1}{2}(\partial_x w_2)^2)\partial_x w_2]\right)\right| \\
&\le \left|(\partial_x u_1 + \frac{1}{2}(\partial_x w_1)^2) - (\partial_x u_2 + \frac{1}{2}(\partial_x w_2)^2)\right|\|\partial_x w_1\|_{L^\infty(\Omega)} \\
&\quad + \left|(\partial_x u_2 + \frac{1}{2}(\partial_x w_2)^2)\right|\|\partial_x w_1 - \partial_x w_2\|_{L^\infty(\Omega)}.
\end{aligned}$$

So that

$$\begin{aligned}&\left|L_\gamma^{-1/2}\left(\partial_x[(\partial_x u_1+\tfrac{1}{2}\ (\partial_x w_1)^2)\partial_x w_1]-\partial_x[(\partial_x u_2+\tfrac{1}{2}\ (\partial_x w_2)^2)\partial_x w_2]\right)\right|\\&\qquad\le E(Y_1-Y_2)\left(\|\partial_x w_1\|_{L^\infty(\Omega)}+\left|(\partial_x u_2+\tfrac{1}{2}\ (\partial_x w_2)^2)\right|\right)\,,\end{aligned}$$

where once again we have used the embedding of $H^1(\Omega)$ into $L^\infty(\Omega)$.

Furthermore, we have

$$\begin{aligned}&\|Y\|^2\\&=\quad\left|\partial_x u+\frac{1}{2}(\partial_x w)^2-\frac{1}{2}(\partial_x w)^2\right|^2+|v|^2+|\varphi|^2+|\partial_{xx}w|^2+\left|L_\gamma^{1/2}z\right|^2+|\psi|^2 \qquad (4)\\&\le\quad 2\left|\partial_x u+\frac{1}{2}(\partial_x w)^2\right|^2+|v|^2+|\varphi|^2+|\partial_{xx}w|^2+\left|L_\gamma^{1/2}z\right|^2+|\psi|^2+2\left|\frac{1}{2}(\partial_x w)^2\right|^2\\&\le\quad 2\sqrt{E(Y(t))}\,E(Y(t))\,,\end{aligned}$$

and

$$E(Y(t))\le 2\|Y(t)\|^2\|Y(t)\|\,. \qquad (5)$$

Since for all $Y\in B(0,R)$, there exist constants $C_1(R), C_2(R)$ such that

$$C_1\|Y\|^2\le E(Y)\le C_2\|Y\|^2\,.$$

From the previous estimates, we deduce

$$\|F(Y_1)-F(Y_2)\|\le C_3\|Y_1-Y_2\|\,,$$

where C_3 is a constant depending on R. This proves the existence of a local solution to (2).

Existence of a global solution

Existence of a global solution follows from the decay of the energy $E(Y)$. First, we notice that for initial data in the domain of $C^{-1}A$, the local solution of (2) remains in the same domain. To see this, we have to verify only that

$$C^{-1}F(D(C^{-1}A)\cap B(O,R))\subset D(C^{-1}A)\,,$$

which is obtained from calculations similar to the ones above, and by the embedding of $H^1(\Omega)$ into $L^\infty(\Omega)$.

For $Y_0\in D(C^{-1}A)$, the corresponding solution of (2) satisfies

$$\frac{d}{dt}E(Y)=-\left|\partial_x\varphi\right|^2-\left|\partial_x\psi\right|^2\ . \qquad (6)$$

So that

$$E(Y(t))\le E(Y(0))\,,$$

and using (5) and (6), one gets

$$\|Y(t)\|^2\le 2E(Y(0))^{3/2}\,.$$

Which proves boundedness of Y in the H-norm, and therefore, global existence is proven. (see Theorem 4.3.4 page 57 in [4])

4 Exponential decay

Theorem 4 *For all $R > 0$ and all $Y_0 \in B(0,R)$ there exist positive constants $M(R)$ and $\omega(R)$ such that solutions to (2) satisfy*

$$E(Y(t)) \leq M(R)e^{-\omega(R)\,t}E(Y_0)\,.$$

Proof. Our argument is based on the choice of a suitable Lyapunov function,

$$\begin{aligned}\sigma_\varepsilon(t) \quad = \quad & E(Y(t)) + \varepsilon\left(\int_\Omega \psi(-\partial_{xx})^{-1}L_\gamma\, z dx + \frac{1}{2}\left(\int_\Omega v\, u\, dx + \frac{1}{2}\int_\Omega L_\gamma z\, w\, dx\right)\right) \\ & -\varepsilon\ \left(\alpha\int_\Omega L_\gamma z\ \ (h(x)\partial_x w)\, dx - \ \frac{1}{2}\int_\Omega \varphi q\, dx\right)\,,\end{aligned}$$

where

$$(-\partial_{xx})^{-1}: L^2(\Omega)\to H^2(\Omega)\cap H^1_0(\Omega)\,,\quad h(x) = \frac{2}{L}x - 1\,,\quad q(x) = \int_0^x v(y,t)\,dy\,,$$

and ε and α are positive constants which will be chosen later.

This Lyapunov function consists of two parts: One concerns the thermoelastic equations and the other the thermoplates. For the thermoelasticity, J.S. Gibson, G.Rosen and Tao [6] have constructed the same multiplier, but it does not work for the thermoplates equations. For this system, we use the multiplier introduced by F.Ammar Khodja and A.Benabdallah [1] and prove that it works for the nonlinear term.

Our purpose is to show that

$$\frac{d}{dt}\sigma_\varepsilon(t) \leq -c\sigma_\varepsilon(t)\,,\quad c>0\,,$$

from which we will deduce that

$$\sigma_\varepsilon(t) \leq \sigma_\varepsilon(0)e^{-ct}\,. \tag{7}$$

Then, noticing that there exist two positive constants a_1, a_2 such that

$$a_1E(Y(t)) \leq \sigma_\varepsilon(t) \leq a_2E(Y(t))$$

we conclude the theorem. Inequality (7) is obtained in the following 5 steps.

1.) Estimate for $\frac{d}{dt}\int_\Omega \psi(-\partial_{xx})^{-1}L_\gamma z\, dx$:

$$\frac{d}{dt}\int_\Omega \psi(-\partial_{xx})^{-1}L_\gamma z dx = \int_\Omega \psi_t(-\partial_{xx})^{-1}L_\gamma z dx + \int_\Omega \psi(-\partial_{xx})^{-1}L_\gamma z_t\, dx\,.$$

But

$$\psi_t = \partial_{xx}\psi - \partial_{xx}z\,.$$

So

$$\begin{aligned}\int_\Omega \psi_t(-\partial_{xx})^{-1}L_\gamma\, z dx &= \int_\Omega \psi L_\gamma z\, dx - \int_\Omega zL_\gamma z\, dx \\ &\le -|L_\gamma^{1/2}z|^2 + |L_\gamma^{1/2}\psi||L_\gamma^{1/2}z| \\ &\le -(1-\delta_1)|L_\gamma^{1/2}z|^2 + \frac{1}{4\delta_1}|L_\gamma^{1/2}\psi|^2 \\ &\le -(1-\delta_1)|L_\gamma^{1/2}z|^2 + \frac{c_1}{\delta_1}|\partial_x\psi|^2\,,\end{aligned}$$

where δ_1 is an arbitrary positive constant which will be chosen later.

On the other hand

$$\int_\Omega \psi(-\partial_{xx})^{-1}L_\gamma z_t\, dx = \int_\Omega (-\partial_{xx})^{-1}\psi\ L_\gamma z_t\, dx\,,$$

but

$$L_\gamma z_t = -\partial_{xxxx}w + \partial_x[(\partial_x u + \frac{1}{2}(\partial_x w)^2)\partial_x w] - \partial_{xx}\psi \tag{8}$$

and

$$\begin{aligned}-\int_\Omega (-\partial_{xx})^{-1}\psi\ \partial_{xxxx}w dx &= \int_\Omega \psi\, \partial_{xx}w\, dx - \partial_x(-\partial_{xx})^{-1}\psi(L)(\partial_{xx}w)(L) \\ &\quad +\partial_x(-\partial_{xx})^{-1}\psi(0)(\partial_{xx}w)(0)\,.\end{aligned}$$

So

$$\begin{aligned}-\int_\Omega (-\partial_{xx})^{-1}\psi\ \partial_{xxxx}w\, dx &\le |\psi||\partial_{xx}w| + |\partial_x(-\partial_{xx})^{-1}\psi(L)||\partial_{xx}w(L)| \\ &\quad +|\partial_x(-\partial_{xx})^{-1}\psi(0)|\,|\partial_{xx}w(0)|\,.\end{aligned}$$

But

$$\begin{aligned}&\left|\int_\Omega \partial_x(-\partial_{xx})^{-1}\psi[(\partial_x u + \frac{1}{2}(\partial_x w)^2)\partial_x w]\, dx\right| \\ &\le \|\partial_x w\|_{L^\infty(\Omega)}|\partial_x u + \frac{1}{2}(\partial_x w)^2||\partial_x(-\partial_{xx})^{-1}\psi| \\ &\le \delta_2 R^2|\partial_x u + \frac{1}{2}(\partial_x w)^2|^2 + \frac{1}{4\delta_2}|\partial_x(-\partial_{xx})^{-1}\psi|^2\,.\end{aligned}$$

So, it follows

$$\begin{aligned}\frac{d}{dt}\int_\Omega \psi(-\partial_{xx})^{-1}L_\gamma z\, dx &\le -(1-\delta_1)|L_\gamma^{1/2}z|^2 + (\frac{c_1}{\delta_1} + \frac{c_2}{\delta_2})|\partial_x\psi|^2 \\ &\quad +\delta_2 R^2|\partial_x u + \frac{1}{2}(\partial_x w)^2|^2\ + |\psi||\partial_{xx}w| \\ &\quad +|\psi||\partial_{xx}w| + |\partial_x(-\partial_{xx})^{-1}\psi(L)||\partial_{xx}w(L)| \\ &\quad +|\partial_x(-\partial_{xx})^{-1}\psi(0)||\partial_{xx}w(0)| + |\psi|^2\,.\end{aligned}$$

2.) Estimate for $\frac{d}{dt}\int_\Omega vu\,dx$:

$$\frac{d}{dt}\int_\Omega v\,u\,dx \;= |v|^2 + \int_\Omega v_t u\,dx$$

and

$$\begin{aligned}\int_\Omega v_t u\,dx &= \int_\Omega \partial_x(\partial_x u + \frac{1}{2}(\partial_x w)^2)u\,dx - \int_\Omega \partial_x\varphi\,u\,,dx\\ &\leq -\int_\Omega (\partial_x u + \frac{1}{2}(\partial_x w)^2)\partial_x u\,dx + |\partial_x\varphi|\,|u|\;.\end{aligned}$$

Here we have used the boundary condition on u, $\partial_x u(L) = \partial_x u(0) = 0$. So

$$\frac{d}{dt}\int_\Omega vu\,dx \leq |v|^2 - \int_\Omega (\partial_x u + \frac{1}{2}(\partial_x w)^2)\partial_x u\,dx + |\partial_x\varphi|\,|u|\;.$$

3.) Estimate for $\frac{d}{dt}\int_\Omega L_\gamma w\,dx$: One has

$$\frac{d}{dt}\int_\Omega L_\gamma z\,w\,dx = \int_\Omega L_\gamma z\,z\,dx + \int_\Omega L_\gamma z_t w = \left|L_\gamma^{1/2}z\right|^2 + \int_\Omega L_\gamma z_t w\,dx\,.$$

Using (8) we obtain

$$\int_\Omega L_\gamma z_t w dx = -\left|\partial_{xx}w\right|^2 - \int_\Omega(\partial_x u + \frac{1}{2}(\partial_x w)^2)(\partial_x w)^2\,dx + \int_\Omega \partial_x\psi\partial_x w\,dx\,.$$

So

$$\begin{aligned}\frac{d}{dt}\int_\Omega L_\gamma z\,w\,dx- &= -\left|\partial_{xx}w\right|^2 - \int_\Omega(\partial_x u + \frac{1}{2}(\partial_x w)^2)(\partial_x w)^2\,dx\\ &\quad + \int_\Omega \partial_x\psi\partial_x w\,dx + \left|L_\gamma^{1/2}z\right|^2\;.\end{aligned}$$

4.) Estimate for $\frac{d}{dt}\int_\Omega L_\gamma z h(x)\partial_x w\,dx$:

$$-\frac{d}{dt}\int_\Omega L_\gamma z\,(h(x)\partial_x w)\,dx = -\int_\Omega L_\gamma z_t h(x)\partial_x w\,dx - \int_\Omega L_\gamma z h(x)\partial_x z\,dx\,.$$

An integration by parts of the second term of the right member of the previous equality gives

$$\int_\Omega L_\gamma z\,(h(x)\partial_x z)\,dx \leq c\left|L_\gamma^{1/2}z\right|^2\;.$$

Furthermore, (8) implies

$$\begin{aligned}\int_\Omega L_\gamma z_t h(x)\partial_x w\,dx &\leq c\left|(\partial_x u + \frac{1}{2}(\partial_x w)^2)\right|\left|\partial_{xx}w\;\right|\|\partial_x w\|_{\;L^\infty(\Omega)}\\ &\quad +(\delta_3 - \frac{3}{L})\left|\partial_{xx}w\;\right|^2 + \;c(\delta_3)\left|\partial_x\psi\right|^{\;2}\\ &\quad +\frac{1}{2}\left(\left|\partial_{xx}w\;\;(0)\right|^2 + \left|\partial_{xx}w\;\;(L)\right|^2\right)\;.\end{aligned}$$

5.) Estimate for $\frac{d}{dt}\int_\Omega \varphi\, q\, dx$:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \varphi q\, dx &= |v|^2 - \int_\Omega \partial_x\varphi v\, dx + \int_\Omega \varphi q_t\, dx\\ &\leq |v|^2 + |\partial_x\varphi|\,|v| + \int_\Omega \varphi q_t\, dx\,.\end{aligned}$$

To simplify notation, let

$$k(x,t) = \int_0^x \varphi(y,t)\, dy\,.$$

So that

$$\int_\Omega \varphi q_t\, dx = -\int_\Omega k\partial_x q_t\, dx = -\int_\Omega k v_t\, dx\,.$$

But

$$\begin{aligned}\int_\Omega k v_t dx &= \int_\Omega \partial_x(\partial_x u + \frac{1}{2}(\partial_x w)^2)k\, dx + \int_\Omega \partial_x\varphi k\, dx\\ &= -\int_\Omega (\partial_x u + \frac{1}{2}(\partial_x w)^2)\partial_x k\, dx - \int_\Omega \varphi\partial_x k\, dx\,.\end{aligned}$$

So

$$\begin{aligned}-\frac{d}{dt}\int_\Omega \varphi q\, dx &\leq -(1-2\delta_4)\,|v|^2 \ + \delta_5\left|(\partial_x u + \frac{1}{2}(\partial_x w)^2)\right|^2\\ &\quad +(\frac{1}{4\delta_4} + \frac{1}{4\delta_5} + c_0)\,|\partial_x\varphi|^2\ .\end{aligned}$$

Conclusion

Gathering all the above calculations and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\sigma_\varepsilon(t) &\leq -\ [1-\varepsilon(c(\delta_1)+c(\delta_2)+c_1+c(\delta_3))]\,|\partial_x\psi|^2\\ &\quad -[1-\varepsilon(c(\ \delta_4)+c(\delta_5)+c_2]\,|\partial_x\varphi|^2\\ &\quad -\varepsilon[(\frac{3}{4}-\delta_1)\left|L_\gamma^{\frac{1}{2}}z\right|^2 + (\frac{1}{2}-\delta_4)\,|v|^2]\\ &\quad -\varepsilon[(1-\delta_2 R^2 - \frac{4}{L^2}\alpha)\left|\partial_x u + \frac{1}{2}(\partial_x w)^2\right|^2]\\ &\quad -\varepsilon[(\frac{1}{4} - ((\frac{3}{L}+R^2)\alpha - \delta_3)\,|\partial_{xx}w|^2]\\ &\quad +\varepsilon\frac{\alpha}{2}[\ (|\partial_{xx}w(0)|^2 + |\partial_{xx}w(L)|^2)]\\ &\quad \frac{\varepsilon}{2\alpha}[\left|\partial_x(\partial_{xx})^{-1}\psi(0)\right|^2 + \left|\partial_x(\partial_{xx})^{-1}\psi(L)\right|^2]\\ &\quad +\varepsilon\delta_6\,|u|^2 + \frac{\varepsilon}{4\delta_6}\,|\partial_x\varphi|^2\\ &\quad -\frac{\varepsilon\alpha}{2}(|\partial_{xx}w(0)|^2 + |\partial_{xx}w(L)|^2)\,.\end{aligned}$$

It remains to choose, in the above steps, the constants δ_i, α, ε sufficiently small to make negative the constants before the energy. This is always possible, and then we obtain

$$\frac{d}{dt}\sigma_\varepsilon(t) \leq -cE(Y(t)).$$

This gives (7) and the theorem is proved. Notice that the previous constant c depends explicitly on R.

Acknowledgment The authors are indebted to the referee for all the comments, remarks, and bibliographical suggestions.

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