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On forced periodic solutions of superlinear quasi-parabolic problems *

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Abstract

We study the existence of periodic solutions for a class of quasi-parabolic equations involving the p-Laplacian (or any other nonlinear operators of similar class) perturbed by nonlinear terms and forced by rather irregular periodic in time excitations (including what we call abrupt changes). These equations may model problems for which, aside from the presence of the kind of nonlinear dissipation associated to the p-Laplacian, other nonlinear and not necessarily dissipative mechanisms occur. We look for boundedness conditions on these periodic excitations and nonlinear perturbations sufficient to guarantee the existence of periodic responses (solutions) of the same period.

1 Introduction

Monotone operators, in particular the ones that are subdifferentials of convex functions, like the p-Laplacian, appear frequently in equations modeling the behaviour of viscoelastic materials (see Le Tallec [10] for instance). These nonlinear operators may be accompanied by others nonlinear operators that may complicate the analysis of the equations and the prediction of the behaviour of the material. This is specially true when these accompanying operators may, under certain circumstance, impart energy into the system formed by the interaction between the material and external actions.

Mathematically speaking, this means that the overall operator associated to the problem is no longer always monotonic and not even always dissipative. Thus, due to this imbalance between the gain and the dissipation of energy, it is not clear whether when acted by periodic external forces, the internal dissipation will be enough for the system respond also in a periodic way. This raises the problem of knowing under what conditions such periodic responses exist.

Obviously, this sort of question also appears when the principal part of the operator modeling to the internal dissipation is linear. But, in this case, one can resort to a larger collection of mathematical techniques to study the problem, and, due to our better knowledge of the behavior of linear equations

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as compared to the present knowledge of the nonlinear ones, the answer to the previous question can be more easily obtained. In fact, there is plenty of articles considering several aspects (and not only the question of existence) of the problem. For instance, with the help of techniques of super and subsolution, Hess ([11], [12], [13], [14] [15]) studied the existence of periodic solutions and its related properties (like the study of principal eigenvalues and bifurcation), when the principal part of the involved operator is linear. Other aspects of this problem, always when the principal part of the involved operator is linear, can be found in papers by Dancer and Hess ([6], [7], [8]), Buonocore [4], Nkashama [20] and Esteban [10].

Here, on the other hand, we will be interested in giving a partial answer only to the the question of existence of forced periodic solutions, but for a class of nonlinear equations including the p-Laplacian as the principal part of the operator. The analysis of such cases is harder, as it is in general when the principal part of the operator is also nonlinear. It is also much less studied.

In fact, few papers consider the question in such situations. For instance, Mizoguchi [19] considered the problem of existence of positive periodic solutions for equations in which the principal part of the operator was related to the operator appearing in porous media equation.

Yamada [24] treated a problem involving the p-Laplacian, perturbed by a maximal monotonic operator, and proved the existence of periodic solutions when the boundary conditions changed periodically. We observe that the problem considered in [24] is always dissipative.

In [21], Ôtani proved rather general results concerning the existence of strong solutions of abstract equations of a certain class , which allows the p-Laplacian as the principal part of the operator. We will compare his results with ours later on in the paper.

For simplicity of exposition, we will start by considering in detail a rather simple model problem. We begin by studying cases where there are no changes in the basic type of internal dissipation mechanism during the periodic cycle (later we will consider a case where this change of type can occur). We will also assume that the periodic excitation is of a special type.

Let Ω be an open bounded and regular set in R^N and $0 < T < +\infty$. We will look for solutions $u : \Omega \times [0, T] \longrightarrow \mathbb{R}$ of quasi-parabolic problems of form

$$\begin{gathered} u_t - \Delta_p u = m(t)g(u) + h(x,t) \quad \text{in} \quad \Omega\,, \\ u|_{\partial\Omega} = 0\,, \\ u(.,0) = u(.,T)\,. \end{gathered} \tag{1}$$

Here, ∇u denotes the gradient of u, $\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, and u_t denotes the derivative of u with respect to time t. We remark that throughout this paper, we will assume that $m(t)g(0) + h(x,t)$ is not identically zero, otherwise $u \equiv 0$ would be a trivial solution.

Also, we will suppose that the above g is a continuous function such that for some constants a and s there holds

$$|g(v)| \leq a(|v|^s + 1) \quad \forall v \in \mathbb{R}.$$

Our goal here will be basically to find conditions on the growth of g that are enough to guarantee the existence of T-periodic solutions. We put ourselves in what we call the "worst-case situation from the energy point of view": we assume that the sole responsable for the dissipation in the problem is the principal monotone part of the operator, that is, there is no dissipation of energy coming from g. Thus, we do not pay attention to any particular behavior of g, except its growth rate and amplitude, and we explore exclusively the coercivity and the regularization coming from Δ_p.

For technical reasons, we will split the analysis in three cases: $0 \leq s < p-1$, $s = p-1$ and $s > p-1$. This will be done in Section 3

Next, in the following section, we will see that our method of proof can be easily extended to abstract problems involving operators similar to Δ_p. In fact, our results will be true for equations of form:

$$\begin{aligned} u_t + A(u) &= F(u)\,, \\ u(0) &= u(T)\,, \end{aligned} \tag{2}$$

with A a strongly monotone, coercive and hemicontinuous operator in a suitable reflexive separable Banach space and F a nonlinear mapping satisfying growth conditions similar to the ones for the previously described special case (1). In Section 4, we will also give examples of application of this abstract result to other equations.

Then, we will pay attention to situations where at certain specified times (and in a periodic way) there are sudden (abrupt) changes of the involved operator.

A problem of this sort is studied in Kawohl and Rühl [16], where it is proved the existence of periodic solutions for an equation involving the Laplacian, perturbed by another maximal monotonic operator, with a certain boundary condition changing abruptly and periodically in time. We remark again that, as in the case of [24], the system studied in [16] is always dissipative.

The results in Ôtani [21] can also be applied to certain situations of abrupt changes. They do not apply, however, to the sort of situation we will be interested, and we will comment more on this point later on.

Here, as before, we want to study situations of abrupt changes in cases where it is possible to impart energy into the system. Moreover, we would like to allow for abrupt changes in type of the principal part of the operator (which models the internal dissipation), effectively changing its degree of dissipation (which mathematically can be gauged, for instance, by its type of coerciveness).

For this, we will consider a model case given the following problem: being $p_i \geq 2$, $i = 1, 2$, Ω as before a bounded and regular region of $\mathbb{R}^N$, $\bar{t} \in (0, T)$ and $I_1 = (0, \bar{t})$, $I_2 = (\bar{t}, T)$, we want to find a suitable solution of

$$\begin{aligned} u_t(t,x) - \Delta_{p_1} u(t,x) &= m_1(t,x) g_1(u(t,x)) + h_1(t,x) \quad \text{for} \quad (t,x) \in I_1 \times \Omega\,, \\ u_t(t,x) - \Delta_{p_2} u(t,x) &= m_2(t,x) g_2(u(t,x)) + h_2(t,x) \quad \text{for} \quad (t,x) \in I_2 \times \Omega\,, \\ u(t,\cdot)|_{\partial\Omega} &= 0, \quad \text{for} \quad t \in (0,T), \\ u(0,x) &= u(T,x), \quad \text{for} \quad x \in \Omega\,. \end{aligned} \tag{3}$$

As in the case of our previous simpler model (1), our goal will be to find conditions so that there are periodic solutions when $|g_i(v)| \le a_i(|v|^{s_i} + 1) \quad \forall v \in \mathbb{R}$, $i = 1, 2$. We will establish a result of this sort in Section 5.

Section 6 finishes the paper with several remarks concerning the existence of strong solutions.

2 Preliminaries

For the sake of fixing notations and ease the reading, in this section we will recall results that we will be using in the paper.

Let V be a Banach space and V' be its topological dual. Let H be a Hilbert space such that $V \subset H \subset V'$, with continuous and dense inclusions. We will denote by $|\cdot|_V$, $|\cdot|_{V'}$, and $|\cdot|_H$ respectively the norms of V, V' and H; by $\langle\ ,\ \rangle_{V',V}$ we denote the duality pairing between V' and V, and by $\langle\ ,\ \rangle_H$, the inner product of H.

For $T > 0$, we consider the Banach space $L^p(0,T;V) = \{u : (0,T) \to V; \int_0^T |u(t)|_V^p\, dt < \infty\}$, with norm $|u|_{L^pV} = (\int_0^T |u(t)|_V^p\, dt)^{1/p}$. Let p' be defined by $1/p + 1/p' = 1$ and denote $u_t = \dfrac{du}{dt}$; we recall that the Banach space $\{u \in L^p(0,T;V);\ u_t \in L^{p'}(0,T;V')\}$, with norm given by $|u|_{L^pV} + |u_t|_{L^{p'}V'}$, is a subspace of $C([0,T];H)$, the class of continuous functions defined on $[0,T]$ with values in H. Moreover, for u and v in this space, there holds

$$\int_0^T \langle u_t(t), v(t)\rangle_{V'V} + \langle v_t(t), u(t)\rangle_{V'V}\, dt = \langle u(T), v(T)\rangle_H - \langle u(0), v(0)\rangle_H. \quad (1)$$

For a proof of these results, see Lions [18], pp. 156 and 321.

We will need the following compactness criterion given by the Aubin-Lions (see Lions [18], p. 58, Theorem 5.1 and Strauss [22], p. 34, Theorem 2):

Lemma 1 *Let X, Y and Z three Banach spaces, X and Z reflexive, such that $X \subset Y \subset Z$ with continuous inclusions. Moreover, assume that the inclusion $X \subset Y$ is compact and that $1 \le p \le \infty$ and $1 < q \le \infty$. Then, the space $\{u \in L^p(0,T;X);\ u_t \in L^q(0,T;Z)\}$ is compactly included in $L^p(0,T;Y)$.*

In the analysis of the cases with abrupt changes, we will have the opportunity to use the following result stated in Brezis [2], Section 1, Theorem 1.3.

Lemma 2 *Let E be an uniformly convex Banach space and let C be a closed convex subset of E. Let F be a family of (not necessarily strict) contractions from C to C with the property that for any $T, T' \in \mathcal{F}$ it is true that $TT' = T'T \in \mathcal{F}$. Suppose also that there is $\overline{x} \in C$ such that the set $\{T\overline{x},\ \forall T \in \mathcal{F}\}$ is bounded. Then, there exists $x_0 \in C$ such that $Tx_0 = x_0$ for any $T \in \mathcal{F}$. In particular, if $T : \mathcal{C} \to \mathcal{C}$ is a (not necessarily strict) contraction, T has a fixed point if and only if there is $\overline{x} \in C$ such that $\{T^n\overline{x}\}_{n\ge 0}$ is bounded.*

The following results will be useful to obtain strong solutions:

Lemma 3 (Tychonoff-Schauder Fixed Point Theorem *Let K be a compact convex subset of a locally convex topological vector space X. Let T be an upper semi-continuous (multivalued) mapping from K into X such that for each x in K, $T(x)$ is a closed convex subset of X whose intersection with K is nonempty. Then T has a fixed point in K, i.e., there exists an element x_0 in K such that $x_0 \in T(x_0)$.*

Lemma 4 *Let K and K_1 be two compact topological spaces and T be a multivalued mapping from K into K_1 with $T(x)$ closed for each $x \in K$. Then T is upper semicontinuous if and only if the graph of T, $G(T) := \{[u,w] \in K \times K_1\,;\, w \in T(u)\}$, is a closed subset of $K \times K_1$.*

The proof of the last two results can be found in Browder [3] (the first one is Corollary 2 of Theorem 6.3), while the second is Proposition 6.2.)

$L^q(\Omega)$ will denote the usual Banach space of real valued function defined on Ω, with norm $|u|_q = (\int_\Omega |u(x)|^q dx)^{1/q}$. When $q = 2$, $\langle \cdot,\cdot \rangle_2$ denotes the usual inner product in the Hilbert space $L^2(\Omega)$.

The norm of $u \in W_0^{1,p}(\Omega)$, the usual Sobolev space, will be denoted $|u|_{1,p} = (\int_\Omega |\nabla u(x)|^p \, dx)^{1/p}$. One can consult Adams [1] for general properties of Sobolev spaces. Here we only recall that if $p \geq 2$, then $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$, with continuous and dense inclusions.

To rigorously define the p-Laplacian, we first consider the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined for any $u \in W_0^{1,p}(\Omega)$ by $\Phi(u) = \frac{1}{p}|u|_{1,p}^p$.

Concerning this functional, it is known that it is of class C^1 and that, for any u and v belonging to $W_0^{1,p}(\Omega)$ one has

$$\langle \Phi'(u), v \rangle_{W^{-1,p'},W_0^{1,p}} = \int_\Omega |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla v(x) \rangle \, dx$$

When $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u) \in L^2(\Omega)$, we have $\langle \Phi'(u), v \rangle_{W^{-1,p'},W_0^{1,p}} = -\int_\Omega \Delta_p u(x) \cdot v(x) \, dx$.

Thus, from now on we are going to denote Φ' by $-\Delta_p$, and, for simplicity of notation, the duality map between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ just by $\langle \cdot,\cdot \rangle$.

It is also known that $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ has the following properties:

(i) *Strong Monotonicity:* There is $\alpha > 0$ such that for any $u, v \in W_0^{1,p}(\Omega)$ it holds

$$\langle (-\Delta_p)u - (-\Delta_p)v, u - v \rangle \geq \alpha |u - v|_{1,p}^p \,. \tag{2}$$

(ii) *Hemicontinuity:* For any $\lambda \in \mathbb{R}$ and any $u, v, w \in W_0^{1,p}(\Omega)$, the following function is continuous:

$$\lambda \to \langle -\Delta_p(u + \lambda w), v \rangle \,.. \tag{3}$$

(iii) *Coercivity:* For any $u \in W_0^{1,p}(\Omega)$, it holds that

$$\langle -\Delta_p u, u\rangle = |u|_{1,p}^p \,. \tag{4}$$

(iv) *Boundness:* For any $u \in W_0^{1,p}(\Omega)$, it is true

$$|-\Delta_p u|_{W^{-1,p'}} \le |u|_{1,p}^{p-1} \,. \tag{5}$$

Properties (4) and (5) easily follow from the definition of Δ_p; (3) follows from the fact that $\Phi \in C^1(W_0^{1,p}(\Omega))$. (2) follows from Tolksdorf [23], Section 2, Lemma 1.

We also need to clarify the meaning of weak and strong solutions:

Definitions:

(i) Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$. We say that u is a *weak solution* of

$$\begin{gathered} u_t - \Delta_p u = f\,, \\ u|_{\partial\Omega} = 0\,, \\ u(0) = u_0\,, \end{gathered} \tag{6}$$

when $u \in W = \{w \in L^p(0,T;W_0^{1,p}(\Omega)); w_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$, u satisfies (6) in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u(0) = u_0$ (which makes sense because $W \subset C([0,T];L^2(\Omega))$).

(ii) In the case where $f \in L^2(0,T;L^2(\Omega))$, u is called a *strong solution* of (6) if it is a weak solution , $u_t \in L^2(0,T;L^2(\Omega))$, and (6) is satisfied in $L^2(0,T;L^2(\Omega))$.

We will also use the T-periodic "almost linearized" problem associated to the previous initial value problem:

$$\begin{gathered} u_t - \Delta_p u = f(x,t) \qquad \text{in} \quad \Omega\,, \\ u|_{\partial\Omega} = 0\,, \\ u(.,0) = u(.,T)\,, \end{gathered} \tag{7}$$

Concerning this problem, the following result that can be found in Lions [18], Chapter 2, Section 7.4, p. 236:

Lemma 5 *Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $p \ge 2$ and*

$$W = \{w \in L^p(0,T;W_0^{1,p}(\Omega)); \quad w_t \in L^{p'}(0,T;W^{-1,p}(\Omega)\}. \tag{8}$$

Then, there is a unique weak solution $u \in W$ of (7). In particular, when f is T-periodic, u is also T-periodic.

This lemma allows the consideration of the Solution Operator associated to the T-periodic "almost linearized" Problem 7. We will need certain properties of this solution operator acting in suitable functional spaces. In the following we will state two of this sort of result; their proofs are quite classical, and so they will only be sketched.

Proposition 1 *Let $p \geq 2$ and $q > 1$ such that $1 - N/p > -N/q$. Then, the solution operator S defined as*

$$\begin{aligned} S : L^{p'}(0,T; W^{-1,p'}(\Omega)) &\to L^p(0,T; L^q(\Omega)) \\ f &\mapsto S(f) = u\,, \end{aligned} \tag{9}$$

where u is the solution of (7), is completely continuous. Moreover, there are positive constants C_1 and C_2 such that

$$|S(f)|_{L^p W_0^{1,p}} \leq |f|^{1/(p-1)}_{L^{p'} W^{-1,p'}} \quad , \tag{10}$$

$$|\frac{d}{dt} S(f)|_{L^{p'} W^{-1,p'}} \leq C_1 |f|_{L^{p'} W^{-1,p'}} \quad , \tag{11}$$

$$|S(f) - S(g)|_{L^p W_0^{1,p}} \leq C_2 |f - g|^{1/(p-1)}_{L^{p'} W^{-1,p'}} . \tag{12}$$

Proof: By taking $v = u$ in (1), working with classical energy estimates and using (5) we obtain (10) to (12). The continuous inclusion $L^p(0,T; W_0^{1,p}(\Omega)) \subset L^p(0,T; L^q(\Omega))$ and (12) now imply that S is continuous. Moreover, for any bounded subset B of $L^{p'}(0,T; W^{-1,p'}(\Omega))$, by (10) and (11), $S(B)$ is a bounded subset of W. But by Lemma 2.1 we have the compact inclusion $W \subset L^p(0,T; L^q(\Omega))$, then $S(B)$ is relatively compact in $L^p(0,T; L^q(\Omega))$, and, consequently, S is completely continuous. $\square$

The proof of the following result is similar to the previous one:

Proposition 2 *The Solution Operator S associated to (7) is also well defined when acting in the following spaces:*

$$\begin{aligned} S : L^{p'}(0,T; W^{-1,p'}(\Omega)) &\to L^\infty(0,T; L^2(\Omega)) \\ f &\mapsto S(f) = u\,. \end{aligned} \tag{13}$$

Moreover, it is a continuous and bounded operator satisfying:

$$|S(f) - S(g)|^2_{L^\infty L^2} \leq C |f - g|^{\frac{2}{p-1}}_{L^{p'} W^{-1/p'}} + C |f - g|^{p'}_{L^{p'} W^{-1,p'}} . \tag{14}$$

We remark that when $g = 0$ then $S(g) = 0$; consequently, this last inequality furnishes

$$|S(f)|^2_{L^\infty L^2} \leq C |f|^{\frac{2}{p-1}}_{L^{p'} W^{-1/p'}} + C |f|^{p'}_{L^{p'} W^{-1,p'}} . \tag{15}$$

with $C = C(\Omega, T, p, \alpha) > 0$.

Finally, we observe that in the derivations of the a priori estimates holding in this paper, we will follow the usual procedure and denote by C a generic constant depending only on the problem data. Sometimes we will display the dependence of these constant on the data, and, in those cases where we want to stress the role of the constants, we will use other symbols, like C_i, D_i, a_i, and so on.

3 Existence of Solutions in the Case of the First Model Equation

First of all we want to briefly describe the arguments we will be using to prove existence of periodic solutions. We will adapt rather classical techniques: we will consider the Solution Operator associated to the T-periodic "almost linearized" problem 7 acting in suitable functional spaces and the Nemytskii operator corresponding to the function in the right hand side of (1). Then, we will look for solutions of the problem as fixed points of the composition of these two operators. As it is usual in this setting, the main difficulty will be to obtain suitable a priori estimates to guarantee that we are in the conditions of some fixed point theorem (in our case, the Schauder Fixed Point Theorem.)

We start by investigating the existence of weak solutions of Problem 1 in the case $0 \leq s < p-1$. We have:

Theorem 1 *Suppose that $p \geq 2$, $m \in L^\infty(0,T)$, $h \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for some $a > 0$ and $s \in [0, p-1)$ there holds $|g(v)| \leq a(|v|^s + 1)$. Then, Problem (1) has a weak solution.*

Proof: The Nemytskii operator, N_H, associated to the function $H(u,x,t) = m(t)g(u)+h(x,t)$,which is defined by $N_H(u)(x,t) = H(u(x,t),x,t) = m(t)g(u(x,t))+h(x,t)$, is a continuous and bounded map from $L^p(0,T;L^p(\Omega)$ to $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

On the other hand, because $0 \leq s < p-1$, we have the continuous inclusion $L^{p/s}(0,T;L^{p/s}(\Omega)) \subset L^{p'}(0,T;W^{-1,p'}(\Omega))$, then we conclude that there is a positive constant $C = C(\Omega,T)$, depending only on Ω and T such that

$$|N_H(u)|_{L^{p'}W^{-1,p'}} \leq a_1 |u|^s_{L^pL^p} + a_2, \tag{1}$$

with

$$a_1 = Ca|m|_\infty, \quad a_2 = a_1(|\Omega|T)^{s/p} + |h|_{L^{p'}W^{-1,p'}}, \tag{2}$$

which gives the boundness of N_H.

It is easy to see that N_H is sequentially continuous in $L^{p/s}(0,T;L^{p/s}(\Omega))$ and consequently continuous in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. So

$$S \circ N_H : L^p(0,T;L^p(\Omega)) \to L^p(0,T;L^p(\Omega))$$

is completely continuous. Moreover, by using (10), (11), the fact that the inclusion $L^p(0,T;W_0^{1,p}(\Omega)) \subset L^p(0,T;L^p(\Omega))$ is continuous and also (1), we conclude that there are constants

$$a_3 = C(2a_1)^{1/(p-1)} \quad \text{and} \quad a_4 = C(2a_2)^{1/(p-1)} \tag{3}$$

such that

$$|S \circ N_H(u)|_{L^pL^p} \leq a_3 |u|_{L^pL^p}^{s/p-1} + a_4 \tag{4}$$

and

$$|\frac{d}{dt}(S \circ N_H(u))|_{L^{p'}W^{-1,p'}} \leq Ca_1|u|^s_{L^pL^p} + Ca_2. \tag{5}$$

Recalling that $0 \leq s < p-1$, from (4), we can conclude that for sufficiently large $R > 0$, when $|u|_{L^pL^p} \leq R$ then also $|S \circ N_H(u)|_{L^pL^p} \leq R$. Fixing such R, by restricting now to a closed ball of radius R in $L^p(0,T;L^p(\Omega))$, we can apply Schauder Fixed Point Theorem and conclude that there is $u \in W$ satisfying the abstract equations $u_t - \Delta_p u = N_H(u)$ and $u(0) = u(T)$, that is, there is a solution of (1.3). □

Now, we will consider equations (1) in the case $s = p-1$. Then, it holds:

Theorem 2 *Let $p \geq 2$, m, h and g as in Theorem 1, with $s = p-1$. Then, if $|m|_{L^\infty}$ is sufficiently small, (1) has a weak solution.*

Proof: Following the proof of Theorem 1, we observe that in this case we also have that $S \circ N_H : L^p(0,T;L^p(\Omega)) \to L^p(0,T;L^p(\Omega))$ is a completely continuous operator.

However, (4) reduces to $|S \circ N_H(u)|_{L^pL^p} \leq a_3|u|_{L^pL^p} + a_4$. If we had $0 < a_3 < 1$, we could take $R \geq \dfrac{a_4}{1-a_3}$ to obtain $a_3R + a_4 \leq R$, and the rest of the argument used in Theorem 1 could be applied, furnishing the existence of the required solution. But the condition $0 < a_3 < 1$ is attained when $\|m\|_{L^\infty}$ is small enough since, from (2) and (3), a_3 approaches zero as $|m|_{L^\infty}$ approaches zero. □

Now we will study the existence of solutions of 1 when $s > p-1$, $p \geq 2$.

This case will require a modified treatment. In fact, let us briefly describe what would be the trouble if one had proceeded with the argument exactly as before. Then, one would observe that when a_3 were small enough (which as previously could be attained when $|m|_{L^\infty}$ is sufficiently small), it would still be possible to find $R > 0$ such that $a_3R^{s/(p-1)} + a_4 \leq R$ (observe (4) in the case $s > p-1$.) However, in this case it would not be true that $L^{p/s}(0,T;L^{q/s}(\Omega)) \subset L^{p'}(0,T;W^{-1,p'}(\Omega))$ for $q > 1$. Consequently, the composition $S \circ N_H$ could not be defined in the previous functional spaces. Thus, to prove the existence of solutions when $s > p-1$, we will have to explore the additional properties of the Solution Operator S given by Proposition 2 in order to define $S \circ N_H$ in other suitable spaces. By doing this, we will be able to prove following result.

Theorem 3 *Let m, h and g as in Theorem 1. Let also $p \geq 2$ and $p-1 < s < p-1+\Gamma$ with either $\Gamma = 2p/N$ when $N > p$ or $\Gamma = 2$ when $N \leq p$. Then, if $|g(v)| \leq a(|v|^s + 1)$ and $|m|_{L^\infty}$ is sufficiently small, (1) has a weak solution.*

Proof: The argument is as follows: we will choose suitable $k, r > 1$ such that the operator $S \circ N_H : L^k(0,T;L^r(\Omega)) \to L^k(0,T;L^r(\Omega))$ is completly continuous. Then, with similar arguments to the ones used in Theorem 2, by choosing $|m|_\infty$ small enough, we will obtain a fixed point for $S \circ N_H$ and, thus, the desired solution.

We start by recalling that, by Propositions 1 and 2, the Solution Operator S associated to problem 7 is a continuous and bounded operator from $L^{p'}(0,T;W^{-1,p'}(\Omega))$ to either $L^\infty(0,T;L^2(\Omega))$ or $L^p(0,T,L^q(\Omega))$ (with q satisfying $1 - \dfrac{N}{p} > -\dfrac{N}{q}$ and to be chosen later on) as the image space.

By interpolation, $X = L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T,L^q(\Omega))$ is continuously imbedded in $L^{k(\theta)}(0,T;L^{r(\theta,q)}(\Omega))$, for any $\theta \in [0,1]$ and

$$k(\theta) = p/\theta \quad \text{and} \quad r(\theta,q) = q/[q(1-\theta)+2\theta]. \tag{6}$$

Moreover, for any $u \in X$ we have

$$|u|_{L^{k(\theta)}L^{r(\theta,q)}} \leq |u|^\theta_{L^pL^q} |u|^{1-\theta}_{L^\infty L^2}. \tag{7}$$

This implies that, for $\theta \in (0,1]$,

$$\begin{aligned} S : L^{p'}(0,T;W^{-1,p'}(\Omega)) &\to L^{k(\theta)}(0,T;L^{r(\theta,q)}(\Omega)) \\ f &\mapsto S(f) = u \end{aligned} \tag{8}$$

is a completely continuous operator. Moreover, by using (10), (15) and (7), we obtain:

$$|S(f)|_{L^{k(\theta)}L^{r(\theta,q)}} \leq |f|^{1/(p-1)}_{L^{p'}W^{-1,p'}} [c_5 + c_6 |f|^{p/2}_{L^{p'}W^{-1,p'}}]^{(1-\theta)/2}. \tag{9}$$

Now, we are going to choose a suitable $\theta_0 \in (0,1]$ such that when $p-1 < s < p-1+\Gamma$ we have $L^{k(\theta)/s}(0,T;L^{r(\theta,q)/s}(\Omega)) \subset L^{p'}(0,T;W^{-1,p'}(\Omega))$ continuously. For this, we observe that if we require

$$s \leq \min\{f_1(\theta) = (p-1)/\theta\ ,\ f_2(\theta) = 2(q-1)/[q(1-\theta)+2\theta]\} = s(\theta),$$

then, by (6), we have

$$\begin{aligned} k(\theta)/s = p/(\theta s) \geq p' = p/(p-1)\,, \\ r(\theta,q)/s = 2q/\{[q(1-\theta)+2\theta]s\} \geq q'\,, \end{aligned}$$

and, thus, the required continuous inclusion $L^{k/s}(0,T,L^{r/s}(\Omega) \subset L^{p'}(0,T;W^{-1,p'}(\Omega)$.

By analyzing the expressions of $f_1(\theta)$ and $f_2(\theta)$ it is easy to conclude that

$$\theta_0(q) = \frac{q(p+1)-2q}{q(p+1)-2p} \in (0,1), \tag{10}$$

furnishes the highest possible superior bound for $s(\theta)$ (under the required restrictions). That is, we must take $s \le s(\theta) = \dfrac{p-1}{\theta_0(q)} = p+1-2\,\dfrac{p}{q} = p+1+\Gamma$, with q satisfying $1-\dfrac{N}{p} > \dfrac{N}{q}$. This gives us the conditions on Γ described in the statement of the theorem. With $\theta_0(q)$ given by (10), we can now take the corresponding values of $k(\theta_0)$ and $r(\theta_0, q)$ given by the expressions in (6). Thus, S defined as in (8) will be continuous. The Nemytskii operator $N_H : L^k(0,T;L^r(\Omega)) \to L^{p'}(0,T;W^{-1,p'}(\Omega))$ will also be continuous and bounded in the sense that, with $C = C(\Omega, T, k, s, a) > 0$,

$$|N_H(u)|_{L^{p'}W^{-1,p'}} \le C|m|_{L^\infty}(|u|^s_{L^kL^r}+1) + |h|_{L^{p'}W^{-1,p'}}. \tag{11}$$

We conclude that $S \circ N_H : L^k(0,T;L^r(\Omega)) \to L^k(0,T;L^r(\Omega))$ is, as in the previous cases, completely continuous.

Now, from (9) and (11), with the help of Young inequality, we obtain that, when $|u|_{L^kL^r} \le R$ for any fixed $R > 0$, then

$$|S \circ N_H(u)|_{L^kL^r} \le D_1 R^{s(\zeta_1+\zeta_2)} + D_2, \tag{12}$$

where

$$\zeta_1 = 1/(p-1), \qquad \zeta_2 = (p(1-\theta_0)+2\theta_0)/2(p-1),$$

$$D_1 = C^{1-\theta_0}(C|m|_{L^\infty})^{\zeta_1}(\zeta_1/\zeta_2) + C^{1-\theta_0}(C|m|_{L^\infty})^{\zeta_2}$$

$$\begin{aligned} D_2 = C^{1-\theta_0}\{&(C|m|_{L^\infty})^{\zeta_1}[(\zeta_2-\zeta_1)/\zeta_2] + (C(|m|_{L^\infty} + |h|_{L^{p'}W^{-1,p'}})\}^{\zeta_1} \\ &+C^{(1-\theta_0)}C^{\zeta_2}(|m|_{L^\infty} + |h|_{L^{p'}W^{-1,p'}})^{\zeta_2}, \end{aligned}$$

again with $C = C(\Omega, p, q, T, a) > 0$. We observe that because $s > p-1$ then $s(\zeta_1+\zeta_2) > 1$.

So, proceeding as in the final part of the proof of Theorem 2, we start by fixing $R > 0$ large enough such that $D_2 \le R/2$; then, we observe that as $|m|_{L^\infty} \to 0$, we also have $D_1 \to 0$. Thus, by taking $|m|_{L^\infty}$ small enough, we have $D_1R^{s(\zeta_1+\zeta_2)} \le R/2$. Under these conditions, (12) implies that $S \circ N_H(\overline{B_R(0)}) \subset \overline{B_R(0)}$, where $\overline{B_R(0)}$ is the ball of radius R and center 0 in $L^k(0,T;L^r(\Omega))$. Thus using Schauder Fixed Point Theorem we conclude that $S \circ N_H$ has a fixed point, and consequently, (1) has a weak solution. $\square$

Remark: We must stress that the above results are concerned with the existence of *forced periodic solutions*. In particular, if h were identically zero (recall the observation just after equation (1,) from the above results we could not conclude that the constructed solution is nontrivial (i.e., $u \ne 0$.) This is the reason for the fact that, when $p = 2$, the above stated range of allowed s is larger than the corresponding range found when one is looking for nontrivial solutions in the case $g(u) = u^s$ and $h = 0$.

4 An Abstract Result

The results stated in the last section hold for more general dissipative operators sharing with Δ_p certain properties. We state below the abstract version of these previous results, omitting their proofs since they follow exactly as before.

Theorem 1 *Suppose $V \subset H \subset V'$ with continuous and dense inclusions, and let A be a monotonic and hemicontinuous operator $A : V \to V'$ for which there are constants $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta \geq 0$, $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$ such that for any $u, v \in V$ it holds $\langle Au - Av, u - v\rangle_{V'V} \geq \alpha_1 |u - v|_V^p$, $\langle Au, u\rangle_{V'V} \geq \alpha_2 |u|_V^p - \beta$ and $|Au|_{V'} \leq \gamma_1 |u|_V^{p-1} + \gamma_2$.*
Let also X be a Banach space such that $V \subset X \subset V'$ with dense and continuous inclusions and $V \subset X$ compactly. Moreover, let $F : L^p(0,T;X) \to L^{p'}(0,T;V')$ be a continuous and bounded map satisfying

$$|F(u)|_{L^{p'}V'} \leq k_1 |u|^s_{L^pX} + k_2,$$

for some non negative constants k_1 and k_2. Then the problem

$$\begin{gathered} u_t + Au = F(u)\,, \\ u(0) = u(T) \end{gathered}$$

has a weak solution when one of the following conditions is satisfied:

(i) *When $0 \leq s < p - 1$.*

(ii) *When $s = p - 1$ and k_1 is sufficiently small.*

(iii) *$V = W_0^{1,p}(\Omega)$, $X = L^q(\Omega)$ with $1 - (N/p) > -(N/q)$, k_1 is small enough and $p - 1 < s < p - 1 + \Gamma$, with either $\Gamma = 2p/N$ when $N > p$ or $\Gamma = 2$ when $N \leq p$.*

Examples of problems for which the abstract result applies:

(a) Let p and m, g, h as in Theorem 1 and a function $c \in C^1(\mathbb{R})$ satisfying $c'(t) \geq 0$ and also $\delta t^{p-1} \leq c(t^p)t^{p-1} \leq \alpha t^{p-1} + \beta$ for all $t \in \mathbb{R}$ and certain positive constants α, β, δ. Then, consider the problem:

$$\begin{gathered} u_t - \operatorname{div}(c(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = m(t)g(u) + h(t,x)\,, \\ u|_{\partial\Omega} = 0\,, \\ u(\cdot,0) = u(\cdot,T)\,. \end{gathered}$$

(b) Let m, g, h as in Theorem 1 and $p > p_2 > p_1 > 2$ and $\lambda, \lambda_1, \lambda_2 > 0$ such that $p = 2p_2 - p_1$, $\lambda_2^2 \leq 4\lambda_1\lambda$ and $[\lambda_2(p_2 - 1)]^2 - 4\lambda\lambda_1(p - 1)(p_1 - 1) \geq 0$. Then, consider the problem:

$$\begin{gathered} u_t - \lambda\Delta_p u + \lambda_2\Delta_{p_2} u - \lambda_1\Delta_{p_1} u = m(t)g(u) + h(x,t) \\ u|_{\partial\Omega} = 0\,, \\ u(\cdot,0) = u(\cdot,T)\,. \end{gathered}$$

To prove that the operators corresponding to above two problems are strongly monotone, one can use Lemma 1 in Section 2 in Tolksdorf [23]. The other properties required in the last theorem are easily proved. The above smallness condition on k_1, when it applies, corresponds in the examples to a smallness condition on $|m|_{L^\infty}$, as in the results of the previous sections. Thus, for the above examples we can obtain the corresponding existence results of T-periodic weak solutions.

5 Existence of Solutions of the Model with Abrupt Changes

Now we will consider the Problem 3.

Theorem 1 *Let $\bar{t} \in (0,T)$, $I_1 = [0,\bar{t}]$, and $I_2 = [\bar{t},T]$. Suppose that for each $i = 1,2$, $m_i \in L^\infty(I_i)$, $h_i \in L^{p_i'}(I_i\,;W^{-1,p_i'}(\Omega))$ and g_i is a continuous real function satisfying $|g_i(v)| \le a(|v_i|^{s_i}+1)$. Suppose also that one of the following three conditions is true.*

(i) $0 \le s_i < p_i - 1$, *for* $i = 1,2$.

(ii) $p_i - 1 \le s_i < p_i + 1$ *for* $i = 1,2$, $N \le p_i$ *and* $\max\limits_{i=1,2} |m_i|_{L^\infty}$ *is small enough.*

(iii) $p_i - 1 \le s_i < p_i - 1 + 2\dfrac{p_i}{N}$ *for* $i = 1,2$, $N > p_i$ *and* $\max\limits_{i=1,2} |m_i|_{L^\infty}$ *is small enough.*

Then (3) has a weak solution.

Proof: We will just sketch the proof since after constructing a suitable solution operator it uses arguments similar to those used in the previous theorems.

One starts by defining the following spaces: $L_i = L^{p_i}(I_i\,;W_0^{1,p_i}(\Omega))$, $L_i' = L^{p_i'}(I_i\,;W^{-1,p_i'}(\Omega))$, $W_i = \{w \in L_i\,;\, w_t \in L_i'\}$.

Then one considers the following problem: given $(f_1,f_2) \in L_1' \times L_2'$, find $(u_1,u_2) \in W_1 \times W_2$ such that

$$\begin{array}{c} (u_1)_t(t,x) - \Delta_{p_1} u_1(t,x) = f_1(t,x), \quad \text{for} \quad (t,x) \in I_1 \times \Omega\,, \\ u_1(t,\cdot)|_{\partial\Omega} = 0, \quad \text{for} \quad t \in I_1\,, \\ (u_2)_t(t,x) - \Delta_{p_2} u_2(t,x) = f_2(t,x), \quad \text{for} \quad (t,x) \in I_2 \times \Omega\,, \\ u_2(t,\cdot)|_{\partial\Omega} = 0, \quad \text{for} \quad t \in I_2\,, \\ u_1(0,x) = u_2(T,x), \quad \text{for} \quad x \in \Omega\,, \\ u_1(\bar{t},x) = u_2(\bar{t},x). \quad \text{for} \quad x \in \Omega\,. \end{array} \tag{1}$$

Observe that the last condition make sense because

$$\begin{aligned} W_1 \times W_2 \quad &\subset C(I_1; W^{-1,p_i'}(\Omega)) \times C(I_2; W^{-1,p_2'}(\Omega)) \\ &\subset C(I_1; W^{-1,\min\{p_1',p_2'\}}(\Omega)) \times C(I_2; W^{-1,\min\{p_1',p_2'\}}(\Omega)). \end{aligned}$$

To prove that Problem 1 has a unique solution, one can proceed as follows: consider the Poincare map $\mathcal{K}$ associated to $(u_i)_t - \Delta_{p_i} u = f_i$ for $t \in I_i$, $u_i(0) = u_0 \in L^2(\Omega)$, $u_2(\bar{t}) = u_1(\bar{t})$. By using the monotonicity and coercivity of Δ_{p_i}, one proves that $\mathcal{K}$ is a non-strict contraction and that there is a sufficiently large $R > 0$ so that $\mathcal{K}(\bar{B}_R(0)) \subset \bar{B}_R(0)$. So according to Lemma 2, $\mathcal{K}$ has a fixed point. Using the monotonicity again, one obtains that is fixed point is unique.

This defines a solution operator for problem 1: $S : L_1' \times L_2' \to W_1 \times W_2$, where $S(h_1, h_2) = (u_1, u_2)$. Like in the previous theorems, it is then possible to show that S is completely continuous and, moreover, that for $i = 1, 2$, with suitable positive constants C_i, α_i, $i = 1, 2$, the following estimates are true:

$$\begin{gathered}
\textstyle\sum_{i=1,2} \alpha_i/p_i |u_i - v_i| \le \sum C_i |f_i - h_i|^{p_i'}\,, \\
\textstyle\sum_{i=1,2} \alpha_i/p_i |u_i|_i^p \le \sum C_i |f_i|^{p_i'}\,, \\
|\frac{d}{dt} u_i| \le |f_i|_{L_i'} + (p_i/\alpha_i)^{1/p_i'} (C_1 |f_1|^{p_1'} + C_2 |f_2|^{p_2'})^{1/p_i'}\,.
\end{gathered}$$

By introducing the natural Nemytskii operators, N_{H_1} and N_{H_2}, corresponding respectively to the second members of (3), with the help of the above estimates, one proves that $N_{H_1} \times N_{H_2} : W_1 \times W_2 \to L_1' \times L_2'$ is continuous.

Now, working with $S \circ (N_{H_1} \times N_{H_2})$ and repeating the arguments done in the proofs of the previous theorems, we obtain a fixed point of $S \circ (N_{H_1} \times N_{H_2})$ and then the stated result. □

Remarks:

(i) As in Section 4, it is possible to state a result on the existence of periodic solutions for abstract operators A_i having properties similar to the ones of Δ_{p_i}.

(ii) Adapting ideas of Kawohl and Rühl [16], it is possible to prove existence of periodic solutions of the following problem involving abrupt changes:

$$\begin{gathered}
u_t - \Delta_{p_i} + B_i(t, u) = 0, \quad \text{for} \quad (t, x) \in I_i \times \Omega = Q_i\,, \quad i = 1, 2\,, \\
-\frac{\partial u}{\partial \eta} \in \beta_i(u), \quad \text{for} \quad (t, x) \in I_i \times \partial\Omega\,, \\
u(0, x) = u(T, x)\,, \quad \text{for} \quad x \in \Omega\,,
\end{gathered}$$

where the B_i, $i = 1, 2$, are coercive maximal monotonic operators such that $B_i(t, \cdot)$, $i = 1, 2$, are locally Lipschitizian in $L_2(\Omega)$. We remark that the presence of such operators in the above equations prevents the system to be dissipative all the time. For the proof of this result, see [5].

6 Commentaries on Strong Solutions

With further restrictions on the problem data, it is possible to obtain strong solutions of the previous problems. Let us comment on this.

When $h \in L^2(0,T;L^2(\Omega))$ instead of $L^{p'}(0,T;W^{-1,p'}(\Omega))$, one can repeat the arguments done in the proof of Theorem 3 to find exponents $k(\theta)$ and $r(\theta,q)$ (as in (6)) so that $L^{k/s}(0,T;L^{r/s}(\Omega)) \subset L^2(0,T;L^2(\Omega))$. Then, similarly as before, we could look for a fixed point of $S \circ N_H : L^k(0,T;L^r(\Omega)) \to L^k(0,T;L^r(\Omega))$, with $S(v) = u$. However, by considering $S \circ N_H$ acting on such $L^k(0,T;L^r(\Omega))$ one does not obtain the best possible result concerning superior bounds for s.

To obtain better results, we proceed as follows: we observe that, when $h \in L^2(0,T;L^2(\Omega))$, Brezis [2], Chapter 3, Theorem 3.6 furnishes the existence of a strong solution of(7) such that $u_t \in L^2(0,T;L^2(\Omega))$ and, moreover, the mapping $[0,T] \ni t \mapsto \Phi(u(t)) = \int_\Omega |\nabla u(t,x)|^p\,dx$ is continuous. In particular, we conclude that for $1-(N/p) > -(N/q)$, the solution belongs to $\widetilde{W} = \left\{w \in L^q(0,T;W^{1,p}(\Omega))\,,\, w_t \in L^2(0,T;L^2(\Omega))\right\}$. Hence, for all s so that $0 \le s \le q/2$, we can consider $S \circ N_H : \widetilde{W} \mapsto \widetilde{W}$

¿From this argument, we have gained regularity for u; but the operator $S \circ N_H$ is not completely continuous. In fact, it is not even continuous when one considers the natural norm topology of $\widetilde{W}$. Therefore, it is not possible proceed exactly as before to obtain a fixed point. However, as in Ôtani [21], by considering weak topologies and using Lemma 4, it is possible to show that $S \circ N_H$ is semi-continuous and satisfies one can verify that SoN_H does satisfy the hypotheses of the Tychonoff-Schauder Fixed Point Theorem (Lemma 3) and conclude that:

Theorem 1 *Let $h \in L^2(0,T;L^2(\Omega))$ and suppose that either $0 \le s \le p/2$ or $p/2 < s \le q/2$ with $1-(N/p) > -(N/q)$ and $|m|_{L^\infty}$ is small enough. Then (1) has a strong solution.*

Remarks:

(i) With the obvious modifications, there holds a result similar to Theorem 1 concerning strong solutions of (3).

(ii) When $p = 2$ (that is, when the principal part of the operator is linear), Ω is of class C^2, for instance, and $h \in L^2(0,T;L^2(\Omega))$, it is possible to prove that there are strong periodic solutions when $1 \le s \le \frac{N+2}{N-2}$, in the case $N > 2$. This result, which gives an improved upper bound for s as compared to that given in Theorem 1, can be obtained by using the same arguments as above, together with the better regularizing properties the Laplacian operator (Dautray, Lions [9], for instance) to embed the solution in higher order Sobolev spaces.

(iii) We recall that in the proofs of the existence results for weak solutions presented in the previous sections, we used the usual Schauder Fixed Point Theorem instead of using the Tychonov-Schauder Fixed Point Theorem as we did above. Hence, one could wonder whether the use of Lemma 2.2 and weak topologies could give increased upper bounds for s in the results in the previous sections. This would be indeed the case if we had a $L^\infty(0,T;W_0^{1,p}(\Omega))$-estimate for $u = S \circ N_H(v)$. Unfortunately, we were not able to prove such estimate in the case of weak solutions, and, only with the presented estimates, the use of Tychonov-Schauder Fixed Point does not improve the values of s.

(iv) As we commented in the Introduction, Ôtani in [21] established a rather general existence result on existence of strong periodic solutions of abstract equations. This could be particularized to equation (1), furnishing thus the existence of certain strong periodic solutions. However, compared to our result, Ôtani's require more regular conditions on h and, moreover, a certain smallness condition on it. This implies that the amplitude of periodic solutions found in [21] are also small (in suitable norms). On the other hand, in our results h can be less regular (in which case we obtain a weak solution), and we do not impose any smallness condition on h (to be fair, we impose instead a smallness condition on $|m|_{L^\infty}$, but only in the cases where it is necessary). Thus, our periodic solutions (both weak and strong) are found in a large enough ball, and in fact, it is easy to prove that when h is large, our solutions are also large (in suitable norms). Therefore, our solutions, even in the case of strong ones, are different from the ones obtained by the particularization of Ôtani's results.

Another aspect that deserve attention is that since Ôtani's results allow for certain time-dependent operators, one could try to apply them to Problem 3. However, due to the required hypotheses in Ôtani's theorem, this is only true in the special case that $h_i \in L^2((0,T)\times\Omega)$, with small enough norms (thus furnishing a strong periodic solution), and, moreover, $p1 = p2$. Therefore, the results in [21] rules out situations where there are changes in the type of internal dissipation, as in (3).

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