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On multi-lump solutions to the non-linear Schrödinger equation *

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Abstract

We present a new approach to proving the existence of semi-classical bound states of the non-linear Schrödinger equation which are concentrated near a finite set of non-degenerate critical points of the potential function. The method is based on considering a system of non-linear elliptic equations. The positivity of the solutions is considered. It is shown how the same method yields "multi-bump" solutions "homoclinic" to an equilibrium point for non-autonomous Hamiltonian equations. The method provides a calculable asymptotic form for the solutions in terms of a small parameter.

1 Introduction

In this paper we study a system of non-linear elliptic equations which can yield the existence of multilump solutions to the non-linear Schrödinger equation (NLS)

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi\,.$$

If we seek standing wave solutions over the whole of n-dimensional Euclidean space $\mathbb{R}^n$ of the form $\psi = e^{-iEt/\hbar}v(x)$, we find that the function $v(x)$ satisfies the equation

$$-\frac{\hbar^2}{2}\Delta v + (V(x)-E)v - \gamma|v|^{p-1}v = 0.$$

Now set $\hbar^2/2 = \epsilon^2$, rename $V(x)-E$ as $V(x)$ and put $\gamma = 1$ to obtain the equation

$$-\epsilon^2\Delta v + V(x)v - |v|^{p-1}v = 0\,. \tag{1}$$

We shall assume that $V(x)$ is bounded from below by a positive constant.

Floer and A. Weinstein [4] showed that, given a non-degenerate critical point b of $V(x)$, equation (1) has a positive solution for all sufficiently small ϵ which concentrates at b, in the sense that, as $\epsilon \to 0$ the solution tends to 0 uniformly

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in the complement of any given neighbourhood of b. (We omit some technical conditions on $V(x)$ which the reader can look up in [4]). Y. G. Oh [9] showed that a similar result holds in which b is replaced by finitely many non-degenerate critical points $b_1, \ldots, b_m$. These are the multi-lump (-bump or -hump) solutions. Oh showed that they are positive, and furthermore, that the corresponding standing wave solution is unstable if there is more than one hump.

The method used by Oh is a generalization of the method of Floer and Weinstein. Although the main idea, based on Liapunov-Schmidt splitting, is simple enough, the details are rather difficult, involving many subtle estimates. In this paper we propose an alternative method. This has several points of contact with the previous method, but we hope the reader will agree that it is somewhat simpler. Moreover it provides a computable asymptotic form for the solution. This will be clarified in the course of the paper.

In a recent paper [2], Ambrosetti, Badiale and Cingolani showed how to obtain single-hump states by an attractive method which is simpler than that of Floer and Weinstein. It is not clear whether their method can be used to prove the results presented in this paper. A number of treatments have appeared based on variational principles and they typically do not require non-degeneracy of the critical points of the potential function [5, 6, 10, 11, 12].

As in previous approaches we transform the independent variable, setting $y = x/\epsilon$. Thus, renaming y as x and dropping the absolute-value signs, we have the equation in the form in which we shall treat it

$$-\Delta v + V(\epsilon x)v - v^p = 0. \tag{2}$$

Let $b_1, \ldots, b_m$ be non-degenerate critical points of $V(x)$. Now we seek solutions that are concentrated near to the points $b_1/\epsilon, \ldots, b_m/\epsilon$. These points draw apart as ϵ tends to 0. Let $v_1, \ldots, v_m$ be approximate single-hump solutions with v_k concentrated near to b_k/ϵ. Then approximately

$$-\Delta\Big(\sum v_i\Big) + V(\epsilon x)\Big(\sum v_i\Big) - \Big(\sum v_i\Big)^p \approx \sum \Big(-\Delta v_i + V(\epsilon x)v_i - v_i^p\Big) \approx 0\,.$$

for small ϵ. We have approximate additivity of the non-linear operator as the products of the m functions $v_1, \ldots, v_m$ grow small due to their maxima drawing apart. We exploit this by writing m equations, one for each v_i, and coupling them by products of the variables $v_1, \ldots, v_m$, in such a way that the sum $v_1 + \cdots + v_m$ satisfies (2).

In fact there is an obvious way to write m equations so that the sum $v_1 + \cdots + v_m$ satisfies (2), namely

$$-\Delta v_i + V(\epsilon x)v_i - \Big(\sum_{k=1}^m v_k\Big)^{p-1} v_i = 0, \quad i = 1, \ldots, m.$$

Unfortunately there is a hidden degeneracy here which causes technical problems (see section 2.1 and the condition ND). These can be overcome by distributing the polynomial $\Big(\sum_{k=1}^m v_k\Big)^p$ in a different fashion over the right-hand sides of the m equations.

We propose therefore to study a system of elliptic equations

$$-\Delta v_i + V(\epsilon x)v_i - G_i(v) = 0, \quad i = 1, \ldots, m,$$

where $v = (v_1, \ldots, v_m)$, and the functions G_i are homogeneous polynomials of degree p in the m variables $v_1, \ldots, v_m$ with real coefficients. The results which we obtain concern existence of solutions of such systems. We also investigate the positivity of the solutions in a systematic way. To obtain multilump solutions of the non-linear Schrödinger equation we can choose the functions G_i so that

$$\sum_{i=1}^{m} G_i(v) = \Big(\sum_{i=1}^{m} v_i\Big)^p$$

and such that the non-degeneracy condition referred to above is satisfied.

We are going to assume a polynomial non-linearity throughout this paper. This means that p will be an integer greater than 1, and imposes some restriction as we will also need the upper bound $p < (n+2)/(n-2)$. The reason for taking p to be an integer is to facilitate an algebraic treatment of functions of sums of the form $G_i(u+v)$. It is likely that a homogeneous non-linearity of non-integer degree can be treated by an analytic method (similar to a Taylor expansion). We shall not attempt this here. However it means that p is not restricted from above if $n = 1$ or 2; for $n = 3$ we have $p = 2$, 3 or 4; for $n = 4$ or 5 we have only $p = 2$; while for $n > 5$ there are no cases.

We shall show how the same approach can be used to study the equation

$$-\Delta v + v - (1 + \epsilon h(x))v^p = 0$$

where h is a bounded measurable function. We view this as a small perturbation of

$$-\Delta v + v - v^p = 0\,,$$

which possesses a spherically symmetric positive solution ϕ in the space $H^2(\mathbb{R}^n)$. All translates of the function ϕ are solutions. However the perturbation breaks the translational symmetry of the equation and by [8] we should look for solutions near to a translate $\phi(x - c)$ where $c \in \mathbb{R}^n$ is a critical point of the function

$$F(s) = \int h(x)\phi(x + s)^{p+1}\, d^n x.$$

Such a solution has one hump. We assume that h is periodic and find solutions near a linear combination of translates $\sum_{i=1}^{m} \phi(x - c_i)$ provided the separations $\|c_i - c_j\|$ are large enough. In one dimension ($n = 1$) these are the homoclinic solutions investigated in [3] and [14] by variational methods. Our solutions are like homoclinic solutions in that they decay at infinity and we are able to give calculable asymptotic forms.

Here is a brief summary of the contents of the sections. In section 2 we study the existence of solutions to the system of elliptic equations; in section 3 we apply this to the non-linear Schrödinger equation; in section 4 we consider positivity of the solutions of the system of elliptic equations studied in section 2, with an application to the non-linear Schrödinger equation; in section 5 we study multi-dimensional “homoclinics”; in section 6 we provide a few technical results needed in previous sections, including a version of the implicit function theorem adapted to our needs.

2 A system of equations

2.1 Hypotheses and statement of theorem

We consider the system

$$-\Delta v_i + V(\epsilon x)v_i - G_i(v) = 0, \quad i = 1, \dots, m \tag{3}$$

where $v = (v_1, \dots, v_m)$ and the functions G_i are homogeneous polynomials of degree p in the m variables $v_1, \dots, v_m$ with real coefficients. We seek solutions $v_i(x)$ belonging to the real Hilbert space $H^2(\mathbb{R}^n)$, hereafter abbreviated to H^2, of real-valued functions on all of $\mathbb{R}^n$ which have square-integrable derivatives up to the second order.

We assume that V is a bounded, C^2, real-valued function, with bounded first and second derivatives. These conditions can be relaxed somewhat (Oh employs weaker conditions), but they allow our method to run smoothly. We assume that there exists $\delta > 0$ such that $V(x) > \delta$ for all x.

We emphasize that we are concerned with real-valued solutions. If the non-linear term in (2) appears as $|v|^{p-1}v$ then for every solution v we obtain another by multiplying by a phase-factor $e^{i\alpha}$. It then makes sense to seek complex-valued solutions which look like a sum of lumps with possibly different phase factors. This may be taken up in another paper.

If we assume that $2 \le p < \infty$ in case $n = 1$, 2, 3 or 4 and $1 \le p \le n/(n-4)$ in case $n \ge 5$, then the Sobolev Embedding Theorem [1] guarantees that the space H^2 is continuously embedded in L^{2p}. Thus if $v \in H^2$ then $G_i(v) \in L^2$.

However we have to make the further restriction that $p < (n+2)/(n-2)$. This is to ensure the existence of the positive, radially symmetric solution $v = \phi(x)$ to the equation

$$-\Delta v + v - v^p = 0\,.$$

The existence and uniqueness of this solution was shown by M. Weinstein [16] in the case $n = 1$ and in the cases $n = 3$, $1 < p \le 2$. The missing material needed for the complete proof was then supplied by M. K. Kwong [7].

Let b_k, $k = 1, \dots, m$, be distinct, non-degenerate critical points of V (one for each equation). We set $a_k = V(b_k)$. Let

$$u_i(x) = a_i^{\frac{1}{p-1}} \phi(\sqrt{a_i}x), \quad i = 1, \dots, m.$$

Then u_i is the unique, positive, radially-symmetric solution of

$$-\Delta v + a_i v - v^p = 0.$$

It is known that the space of all solutions in H^2 to the equation

$$-\Delta v + v - \phi(x)^{p-1}v = 0$$

is one-dimensional and spanned by $\phi(x)$; whilst the space of all solutions to

$$-\Delta v + v - p\phi(x)^{p-1}v = 0$$

is n-dimensional and spanned by the partial derivatives $D_k\phi(x)$, $k = 1, \ldots, n$. For proofs or further references see [9, 7, 16]. The Fredholm alternative is valid here and tells us that if $h \in L^2$ then the equation

$$-\Delta v + v - p\phi(x)^{p-1}v = h$$

has a solution if and only if h is orthogonal to the partial derivatives of $\phi(x)$.

Let Σ denote the set of complex numbers λ for which the equation

$$-\Delta v + v - \lambda\phi(x)^{p-1}v = 0$$

has a non-trivial solution in H^2. It is known that Σ is a real sequence tending to ∞. The lowest member of Σ is 1, and certainly in the case $n = 1$ it is clear that its next lowest member is p. (The problem can be viewed as an eigenvalue problem for a compact, self-adjoint operator; see Appendix section 6.3.)

The equations are coupled by means of the functions G_i. Concerning them we assume that

$$G_i(v) = v_i^p + g_i(v)v_i$$

and

$$g_i(v) = \sum_{j=1}^{m} \lambda_{ij} v_j^{p-1} + \text{mixed terms.}$$

Mixed terms are monomials involving two or more different variables. We assume that $\lambda_{ii} = 0$. This means that $G_i(v)$ consists of v_i^p plus mixed terms. We shall require λ_{ij} to satisfy the following non-degeneracy condition:

$$\lambda_{ij} \notin \Sigma \quad \text{for each } i \text{ and } j. \tag{ND}$$

Note that we are assuming that the mixed terms in G_i all contain the factor v_i. This assumption can be dropped at the expense of a more complicated non-degeneracy condition. It has not seemed worthwhile to treat this here but it could be examined in a later paper.

Theorem 1 *Under the above assumptions the system of equations* (3) *possesses a solution $v = (v_1, \ldots, v_m)$ for each sufficiently small $\epsilon > 0$, which depends continuously on ϵ in the H^2-norm, and is such that $v_i(\cdot + \frac{b_i}{\epsilon})$ tends to u_i in the H^2-norm as $\epsilon \to 0$.*

We have moreover the following asymptotic information. The solution has the form

$$v_i(x) = u_i\Big(x - \frac{b_i}{\epsilon} + s_i\Big) + \epsilon^2 w_i\Big(x - \frac{b_i}{\epsilon} + s_i\Big)$$

where s_i is a vector in $\mathbb{R}^n$ and the function w_i is orthogonal in L^2 to the partial derivatives of u_i. Both s_i and w_i depend on ϵ in such a way that:

(i) $\lim_{\epsilon\to 0+} s_i = 0$;

(ii) $\lim_{\epsilon\to 0+} w_i = \eta_i$ *in* H^2;

where η_i is the unique solution of

$$-\Delta\eta_i(x) + a_i\eta_i(x) - pu_i(x)^{p-1}\eta_i(x) = -\frac{1}{2}V''(b_i)(x,x)u_i(x)$$

which is orthogonal to the partial derivatives of u_i.

Note that the function η_i is in principle calculable and it is to that extent that we consider the asymptotic form of the solution calculable.

2.2 First part of the proof

In this subsection we give the proof of Theorem 1 apart from a technical lemma, and derive much other useful information about the asymptotic form of v as $\epsilon \to 0$.

The system of equations (3) defines a non-linear operator

$$v \mapsto F(\epsilon, v) : (H^2)^m \to (L^2)^m$$

which depends on a parameter $\epsilon > 0$. Our strategy is to solve the equations for sufficiently small ϵ by means of a substitution, careful consideration of the limiting problem as $\epsilon \to 0$, and the implicit function theorem.

Introduce the subspaces

$$W_k = \left\{ w \in H^2 : \int wD_j u_k = 0, \quad j = 1, \dots, n \right\}$$

and set

$$W = W_1 \times \cdots \times W_m \,.$$

Introduce the variable vectors $s_1, \dots, s_m$, each in $\mathbb{R}^n$, and let

$$\xi_k = -\frac{b_k}{\epsilon} + s_k \,.$$

Note that $\|\xi_i - \xi_j\| \to \infty$ as $\epsilon \to 0$ provided $i \neq j$. We let $s = (s_1, \dots, s_m)$ but we emphasize that each component of the m-tuple s is a vector in $\mathbb{R}^n$.

We shall now use the substitution

$$v_i = u_i(x + \xi_i) + \epsilon^2 w_i(x + \xi_i), \quad i = 1, \dots, m$$

where $w_i \in W_i$. The independent variables are the functions $w_i \in W_i$ and the vectors s_i implicit in ξ_i. We shall prove the existence of a solution for each sufficiently small ϵ, and as $\epsilon \to 0$ we shall see that $s \to 0$ and $w_i \to \eta_i$ in the norm topology of H^2, where η_i are the functions referred to in Theorem 1.

We make the substitution and, for each i, translate the i^{th} equation, replacing x by $x - \xi_i$. Divide each equation by ϵ^2. The result is a new operator equation $f(\epsilon, s, w) = 0$ involving an operator

$$(s, w) \mapsto f(\epsilon, s, w) : (\mathbb{R}^n)^m \times W \to (L^2)^m$$

which we proceed to describe. To ease the exposition we split the description into three parts:

(A) terms not involving $w = (w_1, \ldots, w_m)$;

(B) terms linear in w;

(C) terms quadratic or higher in w.

We consider each part separately with a view to taking the limit as $\epsilon \to 0$.

(A) Before division by ϵ^2 the i^{th} component is as follows:

$$\begin{aligned} &-\Delta u_i(x) + V(\epsilon(x-\xi_i))u_i(x) - G_i\Big(u_1(x+\xi_1-\xi_i), \ldots, u_m(x+\xi_m-\xi_i)\Big) \\ &= \Big(V(\epsilon(x-\xi_i)) - a_i\Big)u_i(x) + u_i(x)^p - G_i\Big(u_1(x+\xi_1-\xi_i), \ldots, u_m(x+\xi_m-\xi_i)\Big) \\ &= \Big(V(\epsilon(x-\xi_i)) - a_i\Big)u_i(x) - g_i\Big(u_1(x+\xi_1-\xi_i), \ldots, u_m(x+\xi_m-\xi_i)\Big) \cdot u_i(x). \end{aligned}$$

The second term in the last line consists of sums of monomials. In each we have the unshifted factor $u_i(x)$ together with at least one other *shifted* factor of the form $u_k(x + \xi_k - \xi_i)$ for which $k \neq i$. Such a monomial tends to 0 in the L^2-norm as $\epsilon \to 0$. In fact the convergence to 0 is faster than that of any power of ϵ because of the exponentially fast decrease of the function ϕ at infinity.

Division by ϵ^2 leads therefore to the limit

$$\frac{1}{2}V''(b_i)(x - s_i, x - s_i)u_i(x),$$

where the second derivative $V''(b_i)$ is regarded as a symmetric, bilinear form. Note that the limit is attained in the L^2-norm thanks to the boundedness of the second derivatives of V. But in fact, owing to the rapid decrease of $u_i(x)$ we get the same result if the second derivatives of V have polynomial growth.

The expression (A) defines a mapping

$$f_0 : \mathbb{R}_+ \times (\mathbb{R}^n)^m \to (L^2)^m$$

where $\mathbb{R}_+$ denotes the interval $[0, \infty[$ and the i^{th} component of f_0 is given by

$$\begin{aligned} \Big(f_0(\epsilon, s)\Big)_i &= \epsilon^{-2}\Big(V(\epsilon(x-\xi_i)) - a_i\Big)u_i(x) \\ &\quad - \epsilon^{-2} g_i\Big(u_1(x+\xi_1-\xi_i), \ldots, u_m(x+\xi_m-\xi_i)\Big) \cdot u_i(x) \end{aligned}$$

if $\epsilon > 0$, and

$$\Big(f_0(0,s)\Big)_i = \frac{1}{2}V''(b_i)(x-s_i, x-s_i)u_i(x)\,.$$

Note that the derivative of $f_0(\epsilon, s)$ with respect to s converges to the corresponding derivative of $f_0(0,s)$ as $\epsilon \to 0$. Convergence occurs in the uniform operator topology (the operator norm) thanks to the boundedness of the second derivatives of V (and, as before, polynomial growth would suffice).

(B) After division by ϵ^2 the i^{th} component comprises the following terms:

$$\begin{aligned}
&-\Delta w_i(x) + V(\epsilon(x-\xi_i))w_i(x) - pu_i(x)^{p-1}w_i(x) \\
&\qquad\qquad - g_i\Big(u_1(x+\xi_1-\xi_i), \dots, u_m(x+\xi_m-\xi_i)\Big)w_i(x) \\
&-\sum_{k=1}^{m}\Big(D_{v_k}g_i\Big)\Big(u_1(x+\xi_1-\xi_i), \dots, u_m(x+\xi_m-\xi_i)\Big)u_i(x)w_k(x+\xi_k-\xi_i).
\end{aligned}$$

(Note that $D_{v_k}g_i$ denotes the partial derivative of g_i with respect to the variable v_k.) This expression may be thought of as

$$f_1(\epsilon, s)w$$

where $f_1(\epsilon,s)$ is a linear mapping from W to $(L^2)^m$ for each $\epsilon > 0$ and s.

The main difficulty we have to face is the fact that $f_1(\epsilon,s)$ does not behave well in the operator norm as $\epsilon \to 0$, as we now proceed to see.

The last two terms in the expression consist of sums of monomials of the form

$$u_{k_1}(x+\xi_{k_1}-\xi_i)\cdots u_{k_{p-1}}(x+\xi_{k_{p-1}}-\xi_i)w_i(x)$$

and of the form

$$u_{k_1}(x+\xi_{k_1}-\xi_i)\cdots u_{k_{p-2}}(x+\xi_{k_{p-2}}-\xi_i)u_i(x)w_k(x+\xi_k-\xi_i)$$

Let us consider these as linear maps acting on the functions w_i. A monomial of the first kind defines a linear map that tends to 0 with ϵ in the operator norm provided at least two distinct shifts are present to cause the function multiplying $w_i(x)$ to converge uniformly to 0. This occurs unless $k_1 = \cdots = k_{p-1}$. Similarly a monomial of the second kind defines a linear map that tends to 0 with ϵ in the operator norm unless $k_1 = \cdots = k_{p-2} = i$. Throwing out terms that tend to 0 in the operator norm leaves

$$\begin{aligned}
&-\Delta w_i(x) + V(\epsilon(x-\xi_i))w_i(x) - pu_i(x)^{p-1}w_i(x) \\
&\qquad\qquad - \sum_{k=1,k\neq i}^{m}\lambda_{ik}u_k(x+\xi_k-\xi_i)^{p-1}w_i(x) \\
&\qquad\qquad\qquad\qquad - \sum_{k=1,k\neq i}^{m}\gamma_{ik}u_i(x)^{p-1}w_k(x+\xi_k-\xi_i)
\end{aligned}$$

for certain constants γ_{ik}. This defines a linear map acting on the functions w_i but it is plain that it does not attain a limit in the norm topology, but only in the strong operator topology, as $\epsilon \to 0$.

In fact the strong operator limit is the linear mapping $f_1(0,s)$ given by

$$f_1(0,s)w = -\Delta w_i(x) + a_i w_i(x) - p u_i(x)^{p-1} w_i(x).$$

Note that it is independent of s.

(C) This may be written as $\epsilon^2 f_2(\epsilon, s, w)$ and tends to zero, along with any derivatives it possesses, as $\epsilon \to 0$. The convergence is uniform for s and w in bounded sets.

The limiting problem is the following.

$$\frac{1}{2}V''(b_i)(x-s_i, x-s_i)u_i(x) - \Delta w_i(x) + a_i w_i(x) - p u_i(x)^{p-1} w_i(x) = 0, \qquad i = 1, \ldots, m.$$

It has a non-degenerate solution $s_i = 0$, $w_i = \eta_i$, $i = 1, \ldots, m$, where $\eta_i(x)$ is the unique solution in W_i of

$$-\Delta \eta_i(x) + a_i \eta_i(x) - p u_i(x)^{p-1} \eta_i(x) = -\frac{1}{2}V''(b_i)(x,x)u_i(x)$$

(see section 6.2).

For $\epsilon > 0$ our problem takes the form

$$f(\epsilon, s, w) = f_0(\epsilon, s) + f_1(\epsilon, s)w + \epsilon^2 f_2(\epsilon, s, w) = 0\,. \tag{4}$$

At this point we would like to apply the implicit function theorem to derive a solution for all sufficiently small $\epsilon > 0$. But this requires that the derivative w.r.t. (s, w) converges in the operator-norm as $\epsilon \to 0$. This fails for terms (B). However we can still use the implicit function theorem via a modification which is discussed in the appendix (see Theorem 4 in section 6).

For convenience let us denote the space $(\mathbb{R}^n)^m \times W$ by E and the space $(L^2)^m$ by F. Define an operator-valued function $A : \mathbb{R}_+ \to L(E, F)$ given by

$$A(\epsilon)(\sigma, z) = D_s f_0(0,0)\sigma + f_1(\epsilon, 0)z. \tag{5}$$

To apply Theorem 4 we have to check the following properties of A.

(1) A is continuous for $\epsilon > 0$ w.r.t. the strong operator-topology. (This is needed to ensure that the solution depends continuously on ϵ; see Theorem 4.)

(2) The limit $\lim_{\epsilon\to 0} A(\epsilon) = D_{(s,w)} f(0,0,\eta)$ is attained in the strong operator topology. (The importance of $s = 0$, $w = \eta$ is that it is the solution of the limiting problem. Compare condition (c) of Theorem 4.)

(3) The limit $\lim_{\epsilon\to 0, s\to 0, w\to\eta}\Big(A(\epsilon) - D_{(s,w)}f(\epsilon,s,w)\Big) = 0$ is attained in the operator-norm topology. (Compare condition (d) of Theorem 4.)

(4) There exist $M > 0$ and $\epsilon_0 > 0$ such that $A(\epsilon)$ is invertible for $0 \le \epsilon < \epsilon_0$ and $\|A(\epsilon)^{-1}\| < M$. (Compare condition (e) of Theorem 4. Here, and in similar contexts, we say that a bounded operator from Banach space X to Banach space Y is invertible if it has a bounded inverse defined on all of Y.)

Properties 1 and 2 are fairly obvious. Let us check property 3. We have

$$\begin{aligned} A(\epsilon)(\sigma,z) - D_{(s,w)}f(\epsilon,s,w)(\sigma,z) &= D_s f_0(0,0)\sigma + f_1(\epsilon,0)z \\ &\quad - D_s f_0(\epsilon,s)\sigma - f_1(\epsilon,s)z - \Big(D_s f_1(\epsilon,s)\sigma\Big)w + O(\epsilon^2) \end{aligned}$$

where the remainder term is uniformly of order ϵ^2 for bounded s, σ, w and z. Since the variable s (and hence also σ) belongs to a finite-dimensional space, we see that the linear maps $\sigma \mapsto Df_0(0,0)\sigma - D_s f_0(\epsilon,s)\sigma$ and $\sigma \mapsto \Big(D_s f_1(\epsilon,s)\sigma\Big)w$ converge to 0 in the strong operator topology as $\epsilon \to 0$, $s \to 0$ and $w \to \eta$. This leaves the difference term $f_1(\epsilon,0)z - f_1(\epsilon,s)z$. This also tends to 0 as $s \to 0$, uniformly for bounded z, owing to the uniform continuity of the functions $u_k(x)$ and their products, and the boundedness of the first derivatives of V.

The proof of property 4 requires more effort. This will be carried out in a separate subsection. We note now the conclusion. For all sufficiently small $\epsilon > 0$ equation (4) has a unique solution (s,w) which tends to $(0,\eta)$ as $\epsilon \to 0$ and depends continuously on ϵ.

2.3 Proof of property 4

It suffices to show that if we have sequences $\epsilon_\nu \in \mathbb{R}_+$, $\sigma_\nu \in (\mathbb{R}^n)^m$ and $z_\nu \in W$ such that

$$\epsilon_\nu \to 0, \quad \|\sigma_\nu\|_{(\mathbb{R}^n)^m} + \|z_\nu\|_{(H^2)^m} \le 1$$

whilst

$$A(\epsilon_\nu)(\sigma_\nu, z_\nu) = D_s f_0(0,0)\sigma_\nu + f_1(\epsilon_\nu,0)z_\nu \to 0$$

in the norm topology of $(L^2)^m$, then a subsequence of (σ_ν, z_ν) tends to 0 in the norm topology of $(\mathbb{R}^n)^m \times W$. This will prove that $A(\epsilon)$ is injective for sufficiently small ϵ and that its inverse has a uniform bound. Using the fact that $A(\epsilon)$ is a Fredholm operator of index 0 we see that $A(\epsilon)$ is invertible in the normal sense.

The i^{th} component of $Df_0(0,0)\sigma + f_1(\epsilon,s)z$ can be written as

$$\begin{aligned} &-V''(b_i)(x,\sigma_i)u_i(x) - \Delta z_i(x) + V(\epsilon(x-\xi_i))z_i(x) \\ &\qquad - p u_i(x)^{p-1}z_i(x) - \sum_{k=1,k\ne i}^{m} \lambda_{ik}u_k(x+\xi_k-\xi_i)^{p-1}z_i(x) \\ &\qquad\qquad - \sum_{k=1,k\ne i}^{m} \gamma_{ik}u_i(x)^{p-1}z_k(x+\xi_k-\xi_i) + J_i(\epsilon,s)z \end{aligned} \tag{6}$$

where the remainder term is expressed in terms of an operator $J_i(\epsilon, s)$ which tends to 0 in norm with ϵ. This term may be safely discarded.

Introduce sequences as above with subscript $\nu \in \mathbb{N}$. Recalling that $s = 0$ we have

$$\xi_{\nu,k} = -\frac{b_k}{\epsilon_\nu}\,.$$

Recall that in a Hilbert space a bounded sequence has a weakly convergent subsequence. By going to a subsequence we may assume that

(a) $\lim_{\nu\to\infty} \sigma_\nu = \sigma_\infty$;

(b) $\lim_{\nu\to\infty} z_{\nu,i} = z_{\infty,i}$ weakly in W_i for each i;

(c) $\lim_{\nu\to\infty} z_{\nu,i}(\cdot + \xi_{\nu,i} - \xi_{\nu,j}) = y_{ij}$ weakly in H^2 for each i.

We recall the following facts. If a sequence is weakly convergent in H^2 it is convergent in the sense of distributions. Its restriction to a bounded set is norm convergent in L^2 on that set. If we multiply by a fixed function which tends to 0 at infinity the resulting sequence is norm convergent in L^2 over all of $\mathbb{R}^n$.

Let $j \neq i$ and translate the expression (6) by replacing x by $x + \xi_i - \xi_j$. The resulting expression

$$\begin{aligned}
&-V''(b_i)(x + \xi_{\nu,i} - \xi_{\nu,j}, \sigma_{\nu,i})u_i(x + \xi_{\nu,i} - \xi_{\nu,j})\\
&\quad - \Delta z_{\nu,i}(x + \xi_{\nu,i} - \xi_{\nu,j}) + V(\epsilon_\nu(x - \xi_{\nu,j}))z_{\nu,i}(x + \xi_{\nu,i} - \xi_{\nu,j})\\
&\qquad - pu_i(x + \xi_{\nu,i} - \xi_{\nu,j})^{p-1} z_{\nu,i}(x + \xi_{\nu,i} - \xi_{\nu,j})\\
&\qquad - \sum_{k=1,k\neq i}^{m} \lambda_{ik} u_k(x + \xi_{\nu,k} - \xi_{\nu,j})^{p-1} z_{\nu,i}(x + \xi_{\nu,i} - \xi_{\nu,j})\\
&\qquad\quad - \sum_{k=1,k\neq i}^{m} \gamma_{ik} u_i(x + \xi_{\nu,i} - \xi_{\nu,j})^{p-1} z_{\nu,k}(x + \xi_{\nu,k} - \xi_{\nu,j})
\end{aligned}$$

tends to 0 in L^2. It therefore tends to 0 in the sense of distributions; but, recalling that $j \neq i$, this implies that

$$-\Delta y_{ij}(x) + a_j y_{ij} - \lambda_{ij} u_j(x)^{p-1} y_{ij}(x) = 0\,.$$

This is where the assumption that $\lambda_{ij} \notin \Sigma$ is brought into play. It implies that $y_{ij} = 0$. In particular the distribution limit of $z_{\nu,k}(\cdot + \xi_{\nu,k} - \xi_{\nu,i})$ is 0 if $k \neq i$.

Next we consider the distribution limit without translation. Using what we have just proved we obtain

$$V''(b_i)(x, \sigma_{\infty,i})u_i(x) - \Delta z_{\infty,i}(x) + a_i z_{\infty,i}(x) - pu_i(x)^{p-1} z_{\infty,i}(x) = 0.$$

From this we deduce that $\sigma_{\infty,i} = 0$ and $z_{\infty,i} = 0$. (The non-degeneracy of b_i is needed here; the calculation needed to verify this is similar to that in section 6.2.)

We now have that the weak limit of $z_{\nu,i}(\cdot+\xi_{\nu,i}-\xi_{\nu,j})$ is 0 for any pair (i,j). Since u_i decays at infinity we have that both

$$u_i(\cdot)^{p-1}z_{\nu,j}(\cdot+\xi_{\nu,j}-\xi_{\nu,i})$$

and

$$u_i(\cdot+\xi_{\nu,i}-\xi_{\nu,j})^{p-1}z_{\nu,j}(\cdot)$$

tend to 0 in L^2. Hence, also

$$-\Delta z_{\nu,i}+V(\epsilon_\nu(x-\xi_{\nu,i}))z_{\nu,i}$$

tends to 0 in L^2. By Wang's Lemma [15] (see Appendix section 6.4) this implies that $z_{\nu,i}$ tends to 0 in L^2.

3 Deductions from Theorem 1

3.1 Multi-lump solutions of NLS

We seek a solution to

$$-\Delta v+V(\epsilon x)v+v^p=0$$

for which $v=\sum_{i=1}^m v_i$ and v_i is near $a_i^{\frac{1}{p-1}}\phi(\sqrt{a_i}(x-\frac{b_i}{\epsilon}))$.

We write

$$\Big(\sum_{i=1}^m v_i\Big)^p=\sum_{i=1}^m G_i(v)=\sum_{i=1}^m(v_i^p+g_i(v)v_i)$$

where the functions g_i are chosen so that the constants λ_{ij} fall outside Σ (see Section 2 for the definition of Σ). In fact if $p\geq 3$ we can arrange things so that $\lambda_{ij}=0$. For

$$\Big(\sum_{i=1}^m v_i\Big)^p=\sum_{i=1}^m v_i^p+\sum_{i\neq j}pv_iv_j^{p-1}+\text{ other monomials.}$$

If $p\geq 3$ we can split this into the sum of m polynomials. We group $v_iv_j^{p-1}$ with v_j^p. For any other monomial choose one of its variables v_k arbitrarily and group it with v_k^p.

If a concise prescription is required we could use the following, although it does not recommend itself above any other method. Using the usual multi-indices we write

$$\Big(\sum_{i=1}^m v_i\Big)^p=\sum_{|\alpha|=p}b_\alpha v^\alpha.$$

For each multi-index α let $m(\alpha)$ be the highest subscript at which the maximum coordinate occurs, that is,

$$m(\alpha)=\max\{j:\alpha_j=\max\alpha\}.$$

Then we set

$$G_i(v) = \sum_{m(\alpha)=i} b_\alpha v^\alpha \,.$$

This works only for $p \geq 3$ and so we are left with the case $p = 2$. To handle this we choose $g_i(v) = \sum_{j=1}^m \alpha_{ij} v_j$ where $\alpha_{ii} = 0$, $\alpha_{ij} + \alpha_{ji} = 2$ for $i \neq j$ and $\alpha_{ij} \notin \Sigma$.

3.2 Multilump solutions with sign

Now we seek real-valued solutions to

$$-\Delta v + V(\epsilon x)v + v^p = 0$$

for which $v = \sum_{i=1}^m y_i$ and y_i is near $\kappa_i a_i^{\frac{1}{p-1}} \phi(\sqrt{a_i}(x - \frac{b_i}{\epsilon}))$, where $\kappa_i = \pm 1$.

Let us assume that p is odd. Then we seek solutions v_i to the system

$$-\Delta v_i + V(\epsilon x)v_i - v_i^p - g_i(v)v_i = 0, \quad i = 1, \ldots, m$$

for which v_i is near $a_i^{\frac{1}{p-1}} \phi(\sqrt{a_i}(x - \frac{b_i}{\epsilon}))$, choosing the polynomials $g_i(v)$ so that

$$\Big(\sum_{i=1}^m \kappa_i v_i\Big)^p = \sum_{i=1}^m (\kappa_i v_i^p + \kappa_i g_i(v) v_i)$$

and so that $\lambda_{ij} \notin \Sigma$. This is clearly possible since p is odd.

The required solution is then $\sum_{i=1}^m \kappa_i v_i$.

4 Positivity

4.1 Positive solutions of the system

In this section we prove that solutions of (3) are positive under appropriate conditions. Throughout the section we let v_i be the solutions the existence of which were established in section 2. They depend on ϵ but this dependence will not be explicitly indicated. We maintain all the conditions of section 2. In particular we recall the non-degeneracy condition ND. In the following general result we impose a further restriction on the constants λ_{ij}.

Theorem 2 *Assume that the constants λ_{ij} all satisfy $\lambda_{ij} < 1$. Then the solutions v_i are all positive and without zeros.*

We begin by noting that v_i satisfies the linear differential equation

$$L_\epsilon y := -\Delta y + V(\epsilon x)y - (v_i^{p-1} + g_i(v))y = 0 \,.$$

Our strategy is the usual one of showing that v_i is the ground-state eigenfunction of the operator L_ϵ; in other words L_ϵ has no negative eigenvalues if ϵ is sufficiently small. For operators of this kind it is known that the ground state is positive up to a numerical factor (see, for example, [13]).

Lemma 1 *There exist $\epsilon_0 > 0$ and $\rho_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the eigenvalue 0 of L_ϵ is simple and is the only eigenvalue in the interval $]-\rho_0, \rho_0[$.*

Note: For the lemma the assumption that $\lambda_{ij} < 1$ is not needed.

Proof of Lemma 1. Consider the linear mapping

$$T_\epsilon : (\lambda, z) \mapsto \lambda v_i(x) + L_\epsilon z$$

from $\mathbb{R} \times Z_\epsilon$ to L^2 where

$$Z_\epsilon = \Big\{ y \in H^2 : \int y v_i = 0 \Big\}.$$

Here we have

$$v_i(x) = u_i(x + \xi_i) + \epsilon^2 w_i(x + \xi_i)$$

where, as before,

$$\xi_i = -\frac{b_i}{\epsilon} + s_i\,,$$

w_i and s_i having values, depending on ϵ, which give a solution of the system (3).

We shall show that T_ϵ is invertible for all sufficiently small ϵ and that its inverse satisfies a bound in norm independent of ϵ.

It suffices to show that if we have sequences $\epsilon_\nu \in \mathbb{R}_+$, $\lambda_\nu \in \mathbb{R}$ and $z_\nu \in Z_{\epsilon_\nu}$ such that

$$\epsilon_\nu \to 0, \quad |\lambda_\nu| + \|z_\nu\|_{H^2} \le 1$$

whilst

$$\lambda_\nu v_{\nu,i}(x) + L_{\epsilon_\nu} z_\nu \to 0 \tag{7}$$

in L^2 as $\nu \to \infty$, then a subsequence of (λ_ν, z_ν) converges to 0 in norm. Note that we have written v_ν for the solution of (3) with $\epsilon = \epsilon_\nu$; we use likewise the notation $\xi_{\nu,i}$. Going to a subsequence we may suppose that the following limits exist in the weak topology of H^2:

$$\lim_{\nu\to\infty} z_\nu(\cdot - \xi_{\nu,j}) = z_{\infty,j} \quad j = 1, \dots, m;$$

also that

$$\lim_{\nu\to\infty} \lambda_\nu = \lambda_\infty.$$

Shifting the left-hand side of (7) by replacing x by $x - \xi_{\nu,i}$, taking the distribution limit and recalling the form of $v(x)$ for small ϵ we find

$$\lambda_\infty u_i(x) - \Delta z_{\infty,i}(x) + a_i z_{\infty,i}(x) - u_i(x)^{p-1} z_{\infty,i}(x) = 0.$$

Since $\int z_\nu v_{\nu,i} = 0$ we have that $\int z_{\infty,i} u_i = 0$ and we deduce that $\lambda_\infty = 0$ and $z_{\infty,i} = 0$. Shifting the left-hand side of (7) by replacing x by $x - \xi_{\nu,j}$ where $j \neq i$, and taking the distribution limit we find

$$-\Delta z_{\infty,j}(x) + a_j z_{\infty,j}(x) - \lambda_{ij} u_{\infty,j}(x)^{p-1} z_{\infty,j}(x) = 0$$

which, since $\lambda_{ij} \notin \Sigma$, implies $z_{\infty,j} = 0$. Returning to (7) we see that

$$-\Delta z_\nu + V(\epsilon_\nu x) z_\nu \to 0$$

in the L^2-norm. By Wang's Lemma (section 6.4) this implies $z_\nu \to 0$ in the H^2-norm.

Thus there exists $\epsilon_0 > 0$ and $K > 0$ such that T_ϵ is invertible for $0 < \epsilon < \epsilon_0$ and $\|T_\epsilon^{-1}\| < K$. In particular it follows that the kernel of the operator L_ϵ from H^2 to L^2 is one-dimensional if ϵ is small enough.

By the last paragraph there exists $\rho_0 > 0$ such that the map

$$(\lambda, z) \mapsto T_\epsilon(\lambda, z) + \rho z$$

from $\mathbb{R} \times Z_\epsilon$ to L^2 is invertible for $0 < \epsilon < \epsilon_0$ and $|\rho| < \rho_0$. Suppose a value of ρ in this range is an eigenvalue of L_ϵ with eigenfunction y. Then

$$-\Delta y + V(\epsilon x) y - (v_i(x)^{p-1} + g_i(v(x))) y = \rho y\,.$$

Write

$$y = \lambda v_i + z$$

where $z \in Z_\epsilon$. Since $L_\epsilon v_i = 0$ we have that

$$T_\epsilon(-\rho\lambda, z) - \rho z = 0\,.$$

But then $\rho\lambda = 0$ and $z = 0$ whence $\rho = 0$. Thus the only eigenvalue in the range $|\rho| < \rho_0$ is 0 provided $0 < \epsilon < \epsilon_0$. This ends the proof.

Proof of Theorem 2. We prove that for all sufficiently small $\epsilon > 0$ the operator L_ϵ has no negative eigenvalues. Suppose the contrary holds. Then we can find sequences

$$\epsilon_\nu \to 0, \quad \lambda_\nu < 0, \quad y_\nu \in H^2$$

such that $\|y_\nu\|_{H^2} = 1$ and

$$L_{\epsilon_\nu} y_\nu = -\Delta y_\nu + V(\epsilon_\nu x) y_\nu - (v_{\nu,i}^{p-1} + g_i(v_\nu)) y_\nu = \lambda_\nu y_\nu.$$

It is clear that λ_ν is bounded below, and by the lemma $\lambda_\nu \le -\rho_0 < 0$ for sufficiently large ν. Going to a subsequence we may assume that

$$\lambda_\nu \to \lambda_\infty < 0, \quad y_\nu(\cdot - \xi_{\nu,k}) \to z_k$$

weakly in H^2, for $k = 1, \ldots, m$.

Replace x by $x - \xi_{\nu,k}$ and take the limit in the sense of distributions. For $k \ne i$ we obtain

$$-\Delta z_k + a_k z_k - \lambda_{ik} u_k(x)^{p-1} z_k = \lambda_\infty z_k.$$

But the operator $-\Delta + a_k - u_i(x)^{p-1}$ has no negative spectrum, $\lambda_{ik} < 1$ and $\lambda_\infty < 0$. We deduce that $z_k = 0$. (For future reference we note that this would also be true if $\lambda_{ik} = 1$.) For $k = i$ we obtain

$$-\Delta z_i + a_i z_i - u_i(x)^{p-1} z_i = \lambda_\infty z_i\,.$$

From this we deduce that $z_i = 0$ (note that here we need the fact that λ_∞ is strictly negative).

Now we consider the limit in L^2. We find that

$$-\Delta y_\nu + (V(\epsilon x) - \lambda_\nu) y_\nu \to 0$$

in the L^2-norm. By Wang's Lemma (section 6.4) this implies that $y_\nu \to 0$ in the H^2-norm, which is a contradiction.

For future reference we note that without using Lemma 1 the arguments of the last paragraphs show that if L_ϵ has negative eigenvalues then the lowest eigenvalue tends to 0 as $\epsilon \to 0$. This even works if $\lambda_{ij} = 1$.

4.2 Deductions from Theorem 2

Theorem 2 indicates that in the case $p \geq 3$ the multilump solution to

$$-\Delta v + V(\epsilon x) v + v^p = 0$$

for which $v = \sum_{i=1}^m v_i$ and v_i is near $a_i^{\frac{1}{p-1}} \phi(\sqrt{a_i}(x - \frac{b_i}{\epsilon}))$ is positive. In these cases we can arrange for λ_{ij} to be 0 and the individual components v_i are all positive.

The case $p = 2$ is somewhat different. For simplicity let us consider the case of two humps. Here we take $\lambda_{12} < 1$ and $\lambda_{21} > 1$. Theorem 2 indicates that v_1 is positive but says nothing about v_2.

In fact in this case the function v_2 cannot be positive. Recall that $v = (v_1, v_2)$ satisfies the system

$$\begin{aligned} -\Delta v_1 + V(\epsilon x) v_1 - (v_1 + \lambda_{12} v_2) v_1 &= 0 \\ -\Delta v_2 + V(\epsilon x) v_2 - (v_2 + \lambda_{21} v_1) v_2 &= 0 \end{aligned}$$

where $\lambda_{12} + \lambda_{21} = 2$ and $\lambda_{12} \neq 1$. Multiply the first equation by v_2, the second by v_1, subtract and integrate. The result is $\int v_1^2 v_2 + v_2^2 v_1 = 0$ so that v_2 cannot be everywhere positive.

4.3 The case $m = 2$, $p = 2$

Even though v_2 is not everywhere positive, more subtle arguments suffice to show that the sum $v_1 + v_2$ is positive. Similar arguments can handle the case of more than two humps.

For technical reasons we shall suppose that n (the dimension of the ambient space) is at most 3. This is because we need H^2 to be embedded in the space of bounded continuous functions.

Let $v = (v_1, v_2)$ be the solution of the system

$$\begin{aligned} -\Delta v_1 + V(\epsilon x) v_1 - (v_1 + \tfrac{1}{2} v_2) v_1 &= 0 \\ -\Delta v_2 + V(\epsilon x) v_2 - (v_2 + \tfrac{3}{2} v_1) v_2 &= 0 \end{aligned}$$

for small $\epsilon > 0$ given by Theorem 1. We shall not indicate the dependence on ϵ explicitly. The function $v_1 + v_2$ satisfies the equation $-\Delta y + V(\epsilon x)y + y^2 = 0$.

Define the operator

$$S_\epsilon y := -\Delta y + V(\epsilon x)y - (v_1(x) + v_2(x))y.$$

We shall show that S_ϵ has no negative eigenvalues if ϵ is sufficiently small. So $v_1 + v_2$ is positive, being the ground state of a Schrödinger operator.

Introduce the linear mapping (cf. the proof of Lemma 1)

$$T_\epsilon : (\lambda, \mu, z) \mapsto \lambda v_1 + \mu v_2 + S_\epsilon z$$

from $\mathbb{R}^2 \times Z_\epsilon$ to L^2, where Z_ϵ now denotes the space

$$Z_\epsilon = \Big\{ y \in H^2 : \int y v_1 = \int y v_2 = 0 \Big\}.$$

Lemma 2 *T_ϵ is invertible for all sufficiently small $\epsilon > 0$ and its inverse satisfies a bound in norm independent of ϵ.*

Proof. It suffices to show that if we have sequences $\epsilon_\nu \in \mathbb{R}_+$, $\lambda_\nu \in \mathbb{R}$, $\mu_\nu \in \mathbb{R}$ and $z_\nu \in Z_{\epsilon_\nu}$ such that

$$\epsilon_\nu \to 0, \quad |\lambda_\nu| + |\mu_\nu| + \|z_\nu\|_{H^2} \le 1$$

whilst

$$\lambda_\nu v_{\nu,1} + \mu_\nu v_{\nu,2} + S_{\epsilon_\nu} z_\nu \to 0$$

in L^2 as $\nu \to \infty$, then a subsequence of $(\lambda_\nu, \mu_\nu, z_\nu)$ converges to 0 in norm. We have written $(v_{\nu,1}, v_{\nu,2})$ for the solution corresponding to the value ϵ_ν. A similar notation, $\xi_{\nu,1}$, $\xi_{\nu,2}$, is used for the relevant shifts.

By going to a subsequence we may assume that the limits

$$\lim_{\nu\to\infty} z_\nu(\cdot + \xi_{\nu,1}) = z_{\infty,1}, \quad \lim_{\nu\to\infty} z_\nu(\cdot + \xi_{\nu,2}) = z_{\infty,2}$$

exist in the weak H^2-topology. Following the proof of Lemma 1 we now deduce that $\lambda_\nu \to 0$, $\mu_\nu \to 0$, $z_{\infty,1} = 0$ and $z_{\infty,2} = 0$. We then find $z_\nu \to 0$ in the norm topology of H^2 using the same argument as in Lemma 1. This ends the proof.

We note that the arguments of the last paragraphs of subsection 4.1 suffice to show that, if S_ϵ has negative eigenvalues, then the lowest eigenvalue must tend to 0 as $\epsilon \to 0$. It suffices therefore to prove the following lemma.

Lemma 3 *There exist $\epsilon_1 > 0$ and $\rho_1 > 0$ such that S_ϵ has no eigenvalues in the interval $]-\rho_1, 0[$ for $0 < \epsilon < \epsilon_1$.*

Proof. Suppose that S_ϵ has a negative eigenvalue. Let ρ be its *lowest* eigenvalue and choose a *positive* normalized eigenfunction y. Write $y = \lambda v_1 + \mu v_2 + z$ where $z \in Z_\epsilon$. Since $v_1 + v_2$ is an eigenfunction with eigenvalue 0 we have

$$0 = \int y(v_1 + v_2) = \lambda \int v_1(v_1 + v_2) + \mu \int v_2(v_1 + v_2).$$

If ϵ is sufficiently small both the integrals are positive; in fact

$$\lim_{\epsilon\to 0}\int v_i(v_1+v_2)=\int u_i^2,\quad i=1,2.$$

We conclude that λ and μ have opposite signs. Let $\beta(\epsilon)=\int v_1(v_1+v_2)$ and $\gamma(\epsilon)=\int v_2(v_1+v_2)$. Then $\lambda\beta(\epsilon)+\mu\gamma(\epsilon)=0$. We shall assume that $\lambda\geq 0$ and $\mu\leq 0$. A similar argument will dispose of the other possibility.

Substituting $y=\lambda v_1+\mu v_2+z$ into the equation $S_\epsilon y=\rho y$ we find

$$\frac{1}{2}(\lambda-\mu)v_1v_2+S_\epsilon z-\rho\lambda v_1-\rho\lambda v_2-\rho z=0.$$

Hence applying Lemma 2 and again assuming ϵ sufficiently small we can write

$$(-\rho\lambda,-\rho\mu,z)=T_\epsilon^{-1}\Big(\rho z-\frac{1}{2}(\lambda-\mu)v_1v_2\Big).$$

It follows that

$$|\rho\lambda|+|\rho\mu|+\|z\|_{H^2}\leq K\|\rho z\|_{L^2}+\frac{1}{2}K|\lambda-\mu|\cdot\|v_1v_2\|_{L^2}.$$

Now assume that $|\rho|<1/2K$. We deduce

$$\frac{1}{2}\|z\|_{L^2}\leq\Big(\frac{1}{2}K\|v_1v_2\|_{L^2}-|\rho|\Big)(|\lambda|+|\mu|)$$

If ϵ is small enough the right-hand side becomes negative, which is a contradiction. The problem is that how small ϵ should be depends on ρ. We need to deduce a contradiction from making ϵ small in a way not depending on ρ.

Dropping ρ from the inequality we may write

$$\frac{1}{2}\|z\|_{L^2}\leq\frac{1}{2}K\|v_1v_2\|_{L^2}(|\lambda|+|\mu|)=\frac{1}{2}K\|v_1v_2\|_{L^2}\left(\frac{\gamma(\epsilon)}{\beta(\epsilon)}+1\right)|\mu|.\tag{8}$$

Consider the ball I_ϵ of volume 1 centred at $-\xi_2$, the point of maximum of $u_2(x+\xi_2)$. We know that $v_i(\cdot-\xi_i)\to u_i$ in the H^2-norm. It is here that we need to limit the number of dimensions to 3, for this implies that $v_i(\cdot-\xi_i)\to u_i$ uniformly. So we can find $M>0$ such that $v_2(x)>M$ for all $x\in I_\epsilon$, and for any $\delta>0$ we can ensure that $|v_1(x)|<\delta$ for all $x\in I_\epsilon$ provided only that ϵ is small enough. Fix δ so that $M-\frac{\gamma(\epsilon)}{\beta(\epsilon)}\delta>0$ for sufficiently small ϵ. Since $y\geq 0$ we have

$$\lambda v_1(x)+\mu v_2(x)+z(x)\geq 0$$

which implies

$$z(x)\geq-\lambda v_1(x)+|\mu|v_2(x)\geq-\lambda\delta+|\mu|M=\Big(M-\frac{\gamma(\epsilon)}{\beta(\epsilon)}\delta\Big)|\mu|$$

for all $x\in I_\epsilon$. Integrating we deduce

$$\|z\|_{L^2}\geq\Big(M-\frac{\gamma(\epsilon)}{\beta(\epsilon)}\delta\Big)|\mu|.$$

But this is inconsistent with (8) if ϵ is small enough.

This concludes the proof that v_1+v_2 is positive.

5 Application to homoclinics

The equation

$$-\Delta v + v - v^p = 0$$

considered over all of $\mathbb{R}^n$ has a manifold of positive solutions. These may be described as the n-dimensional plane of functions $\phi(x-c)$ parametrized by $c \in \mathbb{R}^n$, where ϕ is the positive, radially-symmetric solution introduced in section 2. We perturb the equation to

$$-\Delta v + v - (1 + \epsilon h(x))v^p = 0 \tag{9}$$

where h is measurable and periodic in $\mathbb{R}^n$. Now we can seek multi-bump solutions looking like linear combinations of translates of ϕ. The method of section 2 carries through easily enough if we assume that h has bounded second derivatives. (This is much stronger than is needed. Another approach is possible which does not even require h to be continuous, and yet gives more precise asymptotic information. The calculations are too unwieldy to present here.)

Let

$$F(s) = \int h(x)\phi(x+s)^{p+1}\, d^n x$$

for $s \in \mathbb{R}^n$. Let $c_1, \dots, c_m$ be non-degenerate critical points of F (not necessarily distinct). We seek solutions to (9) near to $\sum_{i=1}^m \phi(x+k_i+c_i)$, where the vectors k_i are periods of h for which the separations $\|k_i - k_j\|$ are sufficiently large.

We consider therefore a system

$$-\Delta v_i + v_i - (1+\epsilon h(x))G_i(v) = 0, \quad i = 1, \dots, m,$$

where the polynomials $G_i(v)$ satisfy the same conditions as in section 2. We use the substitution

$$v_i = \phi(x + s_i + k_i) + \epsilon w_i(x + s_i + k_i)$$

where, for each i, s_i is a variable vector in $\mathbb{R}^n$, k_i is a period of h, and the function w_i belongs to the subspace

$$W_i = \Big\{ w \in H^2 : \int w D_j \phi = 0, \quad j = 1, \dots, n \Big\}.$$

We let

$$W = W_1 \times \dots \times W_m\,.$$

Make the substitution, translate the i^{th} equation by replacing x by $x - s_i - k_i$, divide by ϵ with a view to taking the limit as $\epsilon \to 0$ and $\|k_i - k_j\| \to \infty$, $i \neq j$. The result of this is

$$\begin{aligned}\epsilon^{-1} g_i\Big(\phi(x + s_1 + k_1 - s_i - k_i), \dots, \phi(x + s_n + k_n - s_i - k_i)\Big)\phi(x) \\ - h(x - s_i)\phi(x)^p - \Delta w_i(x) + w_i(x) - p\phi(x)^{p-1} w_i(x) \\ - \sum_{j=1, j\neq i}^{m} \lambda_{ij}\phi(x + s_j + k_j - s_i - k_i)^{p-1} w_i(x)\end{aligned}$$

$$- \sum_{j=1, j\neq i}^{m} \gamma_{ij}\phi(x)^{p-1} w_j(x + s_j + k_j - s_i - k_i)$$

where we have thrown out all terms of order ϵ and all terms containing a product of two distinct translates of ϕ, with the exception that we have retained the first term since it also involves division by ϵ.

As usual the details of the limit are a bit tricky. The first term will converge exponentially fast to 0 provided the separations $\|k_i - k_j\|$ do not grow too slowly compared to $1/\epsilon$. With this proviso we obtain the limiting problem in the variables s_i, w_i:

$$-h(x - s_i)\phi(x)^p - \Delta w_i + w_i - p\phi(x)^{p-1} w_i = 0, \quad i = 1, \dots, m\,.$$

This has the non-degenerate solution $s_i = c_i$, $w_i = \eta_i$, $(i = 1, \dots, m)$, where η_i is the unique solution in W_i of

$$-\Delta w_i + w_i - p\phi(x)^{p-1} w_i = h(x - c_i)\phi(x)^p.$$

The existence of solutions for sufficiently small ϵ follows much as in section 2. We have to use the implicit function theorem in the form given in section 6.1 (Theorem 4). Let $A(\epsilon)$ be the operator from $(\mathbb{R}^n)^m \times W$ to $(L^2)^m$ for which the i^{th} component of $A(\epsilon)(\sigma, z)$ is

$$\begin{aligned}
&(\nabla h(x - c_i) \cdot \sigma_i)\phi(x)^p - \Delta z_i(x) + z_i(x) - p\phi(x)^{p-1} z_i(x) \\
&\qquad - \sum_{j=1, j\neq i}^{m} \lambda_{ij}\phi(x + s_j + k_j - s_i - k_i)^{p-1} z_i(x) \\
&\qquad\qquad - \sum_{j=1, j\neq i}^{m} \gamma_{ij}\phi(x)^{p-1} z_j(x + s_j + k_j - s_i - k_i).
\end{aligned}$$

The condition that $\lambda_{ij} \notin \Sigma$ is used, as in section 2.3, to verify that $\|A(\epsilon)^{-1}\|$ has a uniform upper bound as $\epsilon \to 0$. Another detail to note is that as the k_i are periods of h they depend discontinuously on ϵ. The other conditions of Theorem 4 are straightforward to verify.

6 Appendix

6.1 The implicit function theorem

Let E and F be real Banach spaces and let $f : \mathbb{R}_+ \times E \to F$, where $\mathbb{R}_+$ denotes the interval $[0, \infty[$. We write $f_\epsilon(x) = f(\epsilon, x)$ to emphasize the distinct role of ϵ as a small parameter. Assume that f_ϵ is differentiable for each $\epsilon \geq 0$. For reasons which should be clear from section 5 we do not assume that f is a continuous function of ϵ.

We say that a solution x_0 of $f_\epsilon(x) = 0$ is non-degenerate if the derivative $Df_\epsilon(x_0)$ is an invertible linear mapping of E onto F. The following is just the implicit function theorem (in a slightly non-standard form but its proof is just the standard one).

Theorem 3 *Make the assumptions:*

(a) *the equation* $f_0(x) = 0$ *has a non-degenerate solution* x_0*;*

(b) *the limit* $\lim_{\epsilon\to 0, x\to x_0} f_\epsilon(x)$ *is* 0*;*

(c) *the limit* $\lim_{\epsilon\to 0, x\to x_0} Df_\epsilon(x) = Df_0(x_0)$ *is attained in the operator-norm topology.*

Then for each sufficiently small $\epsilon > 0$ *the equation* $f_\epsilon(x) = 0$ *has a unique solution near to* x_0*, which tends to* x_0 *as* $\epsilon \to 0$*. If* f *is jointly continuous in* ϵ *and* x *then the solution depends continuously on* ϵ*.*

In the problems treated in this paper the third condition (c) fails. The limit is only attained in the strong operator topology. In this case we can use the following.

Theorem 4 *Assume as before that:*

(a) *the equation* $f_0(x) = 0$ *has a non-degenerate solution* x_0*; and*

(b) *the limit* $\lim_{\epsilon\to 0, x\to x_0} f_\epsilon(x)$ *is* 0*.*

Assume in addition that there exists an operator-valued function $A : [0, \epsilon_0[\to L(E, F)$ *such that:*

(c) *the limit* $\lim_{\epsilon\to 0} A(\epsilon) = A(0) = Df_0(x_0)$ *is attained in the strong operator topology;*

(d) *the limit* $\lim_{\epsilon\to 0, x\to x_0} \Big(A(\epsilon) - Df_\epsilon(x)\Big) = 0$ *is attained in the operator-norm topology.*

(e) $A(\epsilon)$ *is invertible for* $0 \le \epsilon < \epsilon_0$ *and there exists a constant* M *such that its inverse satisfies* $\|A(\epsilon)^{-1}\| < M$*.*

Then for each sufficiently small $\epsilon > 0$ *the equation* $f_\epsilon(x) = 0$ *has a unique solution near to* x_0*, which tends to* x_0 *as* $\epsilon \to 0$*. If* A *is continuous in* $]0, \epsilon_0[$ *(w.r.t. the strong operator-topology) and* f *is jointly continuous in* ϵ *and* x *then the solution depends continuously on* ϵ*.*

To prove Theorem 4 we apply Theorem 3 to the problem

$$A(\epsilon)^{-1} f_\epsilon(x) = 0\,.$$

Condition (b) of Theorem 3 follows from the fact that the limit $\lim_{\epsilon\to 0} A(\epsilon)^{-1} = Df_0(x_0)^{-1}$ is attained in the strong operator topology. To verify (c) of Theorem 3 we have

$$\|A(\epsilon)^{-1} Df_\epsilon(x) - I\| \le \|A(\epsilon)^{-1}\| \cdot \|Df_\epsilon(x) - A(\epsilon)\| \le M\|Df_\epsilon(x) - A(\epsilon)\| \to 0$$

as $x \to x_0$ and $\epsilon \to 0$.

6.2 The solution of the limiting problem

Here we solve the limiting problem from section 2.2:

$$\frac{1}{2}V''(b_i)(x - s_i, x - s_i)u_i(x) - \Delta w_i(x) + a_i w_i(x) - p u_i(x)^{p-1} w_i(x) = 0, \qquad i = 1, \ldots, m.$$

Recall that s_i is a vector in $\mathbb{R}^n$ for each i. By the Fredholm alternative we must have

$$\frac{1}{2}\int V''(b_i)(x-s_i,x-s_i)u_i(x)D_ku_i(x)\,dx=0$$

for $k=1,\dots,n$, $i=1,\dots,m$. Since u_i is an even function we have that D_ku_i is an odd function and the condition reduces to

$$V''(b_i)\left(s_i,\int xu_i(x)D_ku_i(x)\,dx\right)=0\,.$$

The integral $\int xu_i(x)D_ku_i(x)\,dx$ is a vector whose j^{th} component is the integral $\int x_ju_i(x)D_ku_i(x)\,dx$. Now D_ku_i is an odd function of x_k, but an even function of x_j for $j\neq k$. Hence $\int x_ju_i(x)D_ku_i(x)\,dx=0$ unless $j=k$ in which case the integral is plainly non-zero. Hence the Fredholm alternative gives

$$V''(b_i)(s_i,e_k)=0$$

for $k=1,\dots,n$, $i=1,\dots,m$, where e_k is the k^{th} standard basis vector of $\mathbb{R}^n$. Since $V''(b_i)(x,y)$ is a non-degenerate, symmetric, bilinear form we obtain the non-degenerate solutions $s_i=0$, $i=1,\dots,m$. For these values of s_i we can solve the limiting problem for the functions w_i, uniquely, in the spaces W_i. The solutions are the functions η_i.

6.3 A weighted eigenvalue problem

The set Σ of complex numbers λ, for which the equation

$$-\Delta v+v-\lambda\phi(x)^{p-1}v=0$$

has a non-trivial solution in H^2, can be viewed as the set of reciprocals of the spectrum of a compact, self-adjoint operator. Let A be the operator $-\Delta+1$. Then A is a positive, self-adjoint operator in $L^2(\mathbb{R}^n)$ with spectrum $[1,\infty[$. Let $A^{\frac{1}{2}}$ be its positive, self-adjoint square-root and let $v=A^{-\frac{1}{2}}y$ where $y\in L^2(\mathbb{R}^n)$. We write the weighted eigenvalue problem as

$$y=\lambda\Big(A^{-\frac{1}{2}}\phi^{p-1}A^{-\frac{1}{2}}\Big)y,\quad 0\neq y\in L^2.$$

Now the operator $A^{-\frac{1}{2}}\phi^{p-1}A^{-\frac{1}{2}}$ is clearly self-adjoint and positive, but it is also compact. The reason is that $A^{-\frac{1}{2}}$ can be viewed as a bounded operator from L^2 to H^1, while the multiplication operator ϕ^{p-1} is compact from H^1 to L^2 since the function ϕ decays at infinity. It follows that Σ is a sequence of positive numbers tending to infinity.

We can easily show that the lowest eigenvalue is 1, corresponding to the eigenfunction ϕ. Let L be the self-adjoint operator

$$Lu=-\Delta u+u-\phi(x)^{p-1}u\,.$$

The lowest eigenvalue of L is 0 and corresponds to the positive ground state ϕ. Hence, using the inner-product and norm in L^2, we have

$$0 \leq \langle Lu, u\rangle = \langle Au, u\rangle - \langle \phi^{p-1}u, u\rangle$$

for all u in the domain of L. Putting $u = A^{-\frac{1}{2}}y$ and using self-adjointness we have

$$0 \leq \|y\|^2 - \langle A^{-\frac{1}{2}}\phi^{p-1}A^{-\frac{1}{2}}y, y\rangle.$$

From this it follows that the lowest eigenvalue of the weighted eigenvalue problem is 1.

6.4 Wang's Lemma

If $V(x)$ is bounded and satisfies $V(x) > \delta > 0$ for some constant δ, then

$$\| -\Delta v + V(\epsilon x)v\|_{L^2(\mathbb{R}^n)} \geq K\|v\|_{H^2(\mathbb{R}^n)}$$

for all $u \in H^2$ and for some constant $K > 0$ independent of u and ϵ. The proof is very short and is in the appendix of [15].

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