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Instability of discrete systems *

Raúl Naulin & Carmen J. Vanegas

Abstract

In this paper, we give criteria for instability and asymptotic instability for the null solution to the non-autonomous system of difference equations

$$y(t+1) = A(t)y(t) + f(t, y(t)), \quad f(t, 0) = 0\,,$$

when the system $x(t+1) = A(t)x(t)$ is unstable. In particular for A constant, we study instability from a new point of view. Our results are obtained using the method of discrete dichotomies, and cover a class of difference systems for which instability properties cannot be deduced from the classical results by Perron and Coppel.

1 Introduction

A classical result on Liapounov instability for the difference equation

$$y(t+1) = Ay(t) + f(t, y(t)), \quad f(t, 0) = 0, \quad t = 0, 1, 2, \dots \tag{1}$$

states that the null solution is unstable if the matrix A has an eigenvalue λ satisfying $|\lambda| > 1$, and the nonlinear term $f(t, y)$ satisfies

$$\lim_{|y|\to 0} \frac{f(t, y)}{|y|} = 0\,.$$

This result is known as Perron's Theorem on instability [10, 15], and has played an important role in the study of difference systems [6].

We are interested in the study of two questions related to Perron's Theorem. First, when the matrix A depends on t, and second, when above limit is replaced by condition (F) below. For the first question consider the non-autonomous difference system

$$y(t+1) = A(t)y(t) + f(t, y(t)), \quad f(t, 0) = 0, \tag{2}$$

where $f(t, y)$ is continuous in y and $A(t)$ is invertible at $t = 0, 1, 2, \dots$. We remark that instability of this system cannot be obtained through Perron's

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Theorem. Coppel [5] studied this problem for ordinary differential equations, and Agarwal [1] studied the difference equation case. Here we reproduce the result obtained in [1], which requires the concept of fundamental matrix. For the nonautonomous system of difference equations

$$x(t+1) = A(t)x(t), \quad t = 0, 1, 2, 3, \dots, \tag{3}$$

the fundamental matrix is defined as

$$\Phi(t) = \prod_{s=0}^{t-1} A(s) = A(t-1)\cdots A(1)A(0)\,,$$

where I denotes the identity matrix, and $\prod_{s=0}^{-1} A(s) = I$.

Theorem 1 ([1]) *Assume that $f(t,y)$ is continuous in the variable y, and that for some constant γ and $t = 0, 1, 2, \dots$,*

$$|f(t,y)| \le \gamma |y|\,.$$

Also assume that there is a projection matrix $P \ne I$, and a constant K such that

$$\sum_{s=t_0}^{t-1} |\Phi(t)P\Phi^{-1}(s+1)| + \sum_{s=t}^{\infty} |\Phi(t)(I-P)\Phi^{-1}(s+1)| \le K.$$

Then the null solution to (2) is unstable if $K\gamma < 1$.

This theorem is important because of its applications. For example, Perron's Theorem can be proven easily form Theorem 1. However, instability of a large class of difference systems cannot be obtained using Theorem 1. The aim of this paper is to provide a method for investigating the instability of (2), relying on the dichotomy properties of the non-autonomous system (3). According to Coppel [5], System (2) must inherit some kind of instability of (3) under certain conditions on $f(t,y)$. This idea was also proposed in [11] for ordinary differential systems, and in [7, 8] for difference equations.

For the second question about Perron's Theorem, we assume that $f(t,y)$ satisfies

Condition (F) There exists a sequence of positive numbers $\{\gamma(t)\}$, and $\alpha \ge 0$ such that

$$|f(t,y)| \le \gamma(t)|y|^{\alpha}\,, \forall t\,.$$

Assuming that the matrix A has an eigenvalue satisfying $|\lambda| > 1$ we formulate the question: Under what conditions on $\{\gamma(t)\}$ is the system (1) unstable?

Notice that the hypothesis in Perron's Theorem is implied by Condition (F) with $\alpha > 1$ and γ bounded. Also notice that the hypothesis of Theorem 1 is implied by Condition (F) with $\alpha = 1$ and γ bounded. In this article we show instability under the assumption that A has an eigenvalue with magnitude larger than 1, and f satisfies conditions weaker than those of Perron's Theorem. See the remark in §5. To the best of our knowledge, this is the first publication in response to the question above.

2 Notation and preliminaries

The summation (discrete integral) $\sum_{s=m}^{n} a_s$ is assumed to be equal to zero if $m > n$. The set of non-negative integers is denoted by $\mathbf{N}$, i.e., $\mathbf{N} = \{0, 1, 2, \ldots\}$. Functions $h(t)$ and $k(t)$ denote sequences of positive numbers. For $t_0 \in \mathbf{N}$, we put $\mathbf{N}_{t_0} = \{t \in \mathbf{N} : t \geq t_0\}$. For $m \leq n$, we define $\overline{m,n} = \{s : s \in \mathbf{N}, m \leq s \leq n\}$. The sequences $\{y(t, t_0, \xi)\}$ and $\{x(t, t_0, \xi)\}$, respectively, stand for the solutions to systems (2) and (3) with initial condition ξ at time t_0. The spaces $\mathbb{R}^r$ and $\mathbb{C}^r$ with the norm $|\cdot|$ are denoted by V. The term "sequential space" means the space of sequences with range in V. For a sequence $x : \mathbf{N} \to V$, we define

$$|x|_\infty = \sup\{|x(t)| : \ t \in \mathbf{N}\}, \quad |x|_k = |k(\cdot)^{-1}x(\cdot)|_\infty .$$

The space of sequences such that $|x|_\infty < \infty$ is denoted by ℓ^∞, and the space of sequences such that $|x|_k < \infty$ by ℓ_k^∞. In the space ℓ_k^∞, the closed ball with center 0 with radius ρ is denoted by $B_k[0, \rho] = \{x \in \ell_k^\infty : |x|_k \leq \rho\}$. On the set of initial conditions, we define

$$V_k = \{\xi \in V : \{k(t)^{-1}x(t, t_0, \xi)\} \in \ell^\infty\},$$
$$V_{k,0} = \{\xi \in V_k : \lim_{t\to\infty} k(t)^{-1}x(t, t_0, \xi) = 0\}.$$

Based on [11], solutions to (2) on an interval $\mathbf{N}_{t_0}$ are classified as follows:

h-stable: If for each positive ε there exists a positive δ such that for any initial condition y_0 satisfying $|h(t_0)^{-1}y_0| < \delta$, the solution $y(t, t_0, y_0)$ satisfies $|y(\cdot, t_0, y_0)|_h < \varepsilon$ on $\mathbf{N}_{t_0}$.

h-unstable: If the null solution is not h-stable.

Asymptotically h-stable: If for each positive ε there exists a positive δ such that any initial condition y_0 satisfying $|h(t_0)^{-1}y_0| < \delta$, the solution $y(t, t_0, y_0)$ satisfies $|y(\cdot, t_0, y_0)|_h < \varepsilon$ on $\mathbf{N}_{t_0}$, and

$$\lim_{t\to\infty} h(t)^{-1}y(t, t_0, y_0) = 0 . \tag{4}$$

Asymptotically h-unstable: If the null solution is not asymptotically h-stable.

We will assume that System (3) has a certain dichotomy behavior, but the analysis of instability would be restricted if we limited our attention to the dichotomy properties described by ordinary and exponential dichotomies only, [1]. Therefore, we use (h, k)-dichotomies [12, 14] to study system (3).

Definition System (3) has an (h, k)-dichotomy on $\mathbf{N}$, if there exist a constant K and a projection matrix P such that

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\leq Kh(t)h(s)^{-1}, \quad 0 \leq s \leq t, \\ |\Phi(t)(I - P)\Phi^{-1}(s)| &\leq Kk(t)k(s)^{-1}, \quad 0 \leq t \leq s. \end{aligned} \tag{5}$$

For short notation, (h,h)-dichotomies are called h-dichotomies. An important class of (h,k)-dichotomies is given by those having the following property.

Definition An ordered pair (h,k) is uniformly compensated [12] if there exists a positive constant C such that

$$h(t)h(s)^{-1} \leq Ck(t)k(s)^{-1}, \quad t \geq s.$$

Remark If System (3) has an (h,k)-dichotomy (with projection P and constant K), and the pair (h,k) is uniformly compensated (with constant C), then the system has both an h and a k-dichotomies with projection P and constant CK.

Uniformly compensated dichotomies have the following property [13].

Theorem 2 *Assume that (3) has an (h,k)-dichotomy, and that the pair (h,k) is compensated. Then (3) has an (h,k)-dichotomy with projection Q if and only if*

$$V_{h,0} \subset V_{k,0} \subset Q[V] \subset V_h \subset V_k.$$

We need the following version of the Schauder fixed point theorem [9] in a later proof.

Theorem 3 *Let E be a Banach space with norm $|\cdot|$, and let $\mathcal{T}$ be an operator, $\mathcal{T} : \Omega \to \Omega$, where Ω is a bounded, closed and convex subset of E. If $\mathcal{T}(\Omega)$ is precompact, and $\mathcal{T}$ is continuous, then there exists $x \in \Omega$, such that $\mathcal{T}(x) = x$.*

In discrete analysis, the application of the Schauder theorem frequently is accompanied by the following criterion for compactness.

Definition A subset Ω of the sequential space is equiconvergent to 0, if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $x \in \Omega$ and all $n \geq N$, $|x(n)| < \varepsilon$.

Theorem 4 *If $\Omega \subset S$ is bounded, closed and equiconvergent to 0, then Ω is compact.*

For a future use, we also define the operator

$$\mathcal{U}(y)(t) = \sum_{s=t_0}^{t-1} \Phi(t)P\Phi^{-1}(s+1)f(s,y(s)) - \sum_{s=t}^{\infty} \Phi(t)(I-P)\Phi^{-1}(s+1)f(s,y(s)).$$

3 Instability under contraction conditions

In this section we assume that the nonlinear term of (2) satisfies

Condition (L) Assume that for some positive ρ_0, and all $\rho \in (0, \rho_0)$ there exists a sequence $\gamma(t, \rho)$, such that

$$|f(t, h(t)y) - f(t, h(t)z)| \leq \gamma(t, \rho)|y - z|, \quad |z|, |y| \leq \rho\,.$$

Theorem 5 *Assume that (3) has an h-dichotomy and (2) satisfies Condition (L), with*

$$K \sum_{s=t_0}^{\infty} h(s+1)^{-1}\gamma(s, \rho) < 1. \tag{6}$$

If $V_h \neq V$, then the null solution of (2) is not h-stable.

Proof. Assume that the null solution is h-stable. Then for $\varepsilon = \rho \in (0, \rho_0)$ there exists a δ such that $|h(t_0)^{-1}y_0| < \delta$ implies $|y(\cdot, t_0, y_0)|_h < \rho$. We will show a contradiction to this statement. The estimate

$$\begin{aligned} |h(t)^{-1}\mathcal{U}(y)(t)| &\leq K \sum_{s=t_0}^{\infty} h(s+1)^{-1}|f(s, y(s))| \\ &\leq K \sum_{s=t_0}^{\infty} h(s+1)^{-1}\gamma(s, \rho)\rho \end{aligned} \tag{7}$$

implies that $\mathcal{U} : B_h[0, \rho] \to B_h[0, \rho]$. Moreover, we have the estimate

$$|h(t)^{-1}(\mathcal{U}(y)(t) - \mathcal{U}(z)(t))| \leq K \sum_{s=t_0}^{\infty} h(s+1)^{-1}\gamma(s, \rho)|y - z|_h. \tag{8}$$

Let us consider the sequence

$$x(t) = y(t, t_0, y_0) - \mathcal{U}(y(\cdot, t_0, y_0))(t),\ |h(t_0)^{-1}y_0| < \delta.$$

It is easy to see that x is an h-bounded solution of (3). Hence $x(t_0) \in \Phi(t_0)[V_h]$. From Theorem 2 we may assume that $x(t_0) \in \Phi(t_0)P[V]$. Let y_0 be chosen with the properties

$$y_0 \in \Phi(t_0)(I - P)[V], \quad y_0 \neq 0, \quad |h(t_0)^{-1}y_0| < \delta\,. \tag{9}$$

From the definition of the sequence x we obtain

$$x(t_0) = y_0 - \Phi(t_0)(I - P) \sum_{s=t_0}^{\infty} \Phi^{-1}(s+1)f(s, y(s, t_0, y_0))$$

that belongs to $\Phi(t_0)(I-P)[V]$, which implies $x(t_0) = 0$. In this case $y(\cdot, t_0, y_0)$ satisfies the integral equation

$$y(\cdot, t_0, y_0) = \mathcal{U}(y(\cdot, t_0, y_0)).$$

Thus, any solution $y(\cdot, t_0, y_0)$, where y_0 satisfies (9), is a fixed point of the dichotomy operator $\mathcal{U}$. But from (7) and (8) we see that operator $\mathcal{U}$ is a contraction acting from $B_h[0, \rho]$ to $B_h[0, \rho]$. Moreover $\mathcal{U}(0) = 0$, therefore $y(\cdot, t_0, y_0) = 0$ giving $y_0 = y(t_0, t_0, y_0) = 0$ which is a contradiction.

Theorem 6 *Under the hypotheses of Theorem 5, if $V_{h,0} \neq V_h$, then the null solution of (2) is not asymptotically h-stable on the interval $\mathbf{N}_{t_0}$.*

Proof. By contradiction assume that the null solution is asymptotically h-stable. Then for $\varepsilon = 1$ there exists a positive δ such that $|h(t_0)^{-1}y_0| < \delta$ implies (4). Let $0 < \rho < \min\{\rho_0, \delta\}$ and σ be a small number such that

$$\sigma + K\rho \sum_{s=t_0}^{\infty} h(s+1)^{-1}\gamma(s,\rho) \leq \rho. \tag{10}$$

Fixing a vector $x_0 \in V_h \setminus V_{h,0}$ with the property $|x(\cdot, t_0, x_0)|_h < \sigma$, we introduce the operator

$$\mathcal{T}(y)(t) = x(t, t_0, x_0) + \mathcal{U}(y)(t).$$

From the property (7) and (10) we obtain that $\mathcal{T} : B_h[0,\rho] \to B_h[0,\rho]$. From condition (L), it follows that

$$|\mathcal{T}(y) - \mathcal{T}(z)|_h \leq K \sum_{s=t_0}^{\infty} h(s+1)^{-1}\gamma(s,\rho)|y-z|_h.$$

Thus, condition (6) implies that the operator $\mathcal{T}$ is a contraction from the ball $B_h[0,\rho]$ into itself and therefore has a unique fixed point $y(\cdot)$. This fixed point is a solution of (2). From Theorem 2 we may assume that projection P defining the h-dichotomy satisfies the condition

$$\lim_{t\to\infty} h(t)^{-1}\Phi(t)P = 0. \tag{11}$$

From this property it follows the asymptotic formula

$$y(t) = x(t, t_0, x_0) + o(h(t)), \tag{12}$$

where "small o" is the standard Landau symbol. Inasmuch as the initial condition of the solution $y(\cdot)$ satisfies

$$|h(t_0)^{-1}y(t_0)| \leq \rho < \delta,$$

then $\lim_{t\to\infty} h(t)^{-1}y(t) = 0$. But $\lim_{t\to\infty} h(t)^{-1}x(t, t_0, x_0) \neq 0$ which contradicts (12).

4 General conditions for instability

The contraction property of $\mathcal{U}$ is implied by the stringent Condition (L), and it plays a very important role in proof of Theorem 5. A more general situation can be considered by a small modification to the monotone conditions imposed by Brauer and Wong in [4]. In this section we will assume that $f(\cdot, y)$ satisfies

Condition M There exists a scalar-valued function $\psi(t,r)$ defined for $t \in \mathbf{N}$, $r \geq 0$, which is continuous, and nondecreasing in r for each fixed t, such that

$$|f(t,y)| \leq \psi(t,|y|).$$

Theorem 7 *Assume that (3) has an (h,k)-dichotomy, with (h,k) a compensated pair, and $f(\cdot,y)$ satisfying Condition (M). Also assume that there exists ρ_0 such that for $0<\rho<\rho_0$,*

$$KC\sum_{s=t_0}^{\infty} k(s+1)^{-1}\psi(s,k(s)\rho) < \rho\,. \tag{13}$$

Then, if $V_h \neq V_k$, the null solution to (2) is h-unstable.

Proof. By contradiction, assume that the null solution to (2) is h-stable. Then for $\varepsilon > 0$, there exists a $\delta > 0$ such that $|y(\cdot,t_0,y_0)|_h < \varepsilon$ if $|h(t_0)^{-1}y_0| < \delta$. Let

$$\rho < \frac{h(t_0)}{k(t_0)}\delta\,. \tag{14}$$

Choose a positive σ satisfying

$$\sigma + KC\sum_{s=t_0}^{\infty} k(s+1)^{-1}\psi(s,k(s)\rho) \leq \rho,$$

and fix an initial value $x_0 \in \Phi(t_0)[V_k] \setminus \Phi(t_0)[V_h]$ such that $|x(\cdot,t_0,x_0)|_k \leq \sigma$. Let us consider the integral equation

$$y = \mathcal{T}(y),$$

where

$$\mathcal{T}(y)(t) = x(t,t_0,x_0) + \mathcal{U}(y)(t).$$

Step 1: Show that $\mathcal{T} : B_k[0,\rho] \to B_k[0,\rho]$. From (5) and (13), we obtain

$$\begin{aligned} |k(t)^{-1}\mathcal{T}(y)(t)| &\leq |k(t)^{-1}x(t,t_0,x_0)| + k(t)^{-1}|\mathcal{U}(y)(t)| \\ &\leq |k(t)^{-1}x(t,t_0,x_0)| + KC\sum_{s=t_0}^{\infty} k(s+1)^{-1}\psi(s,k(s)\rho) \leq \rho\,. \end{aligned}$$

Step 2: Prove that the operator $\mathcal{U}$ is continuous in the ℓ_k^∞ metric. Let $\mu > 0$, choose T large enough such that

$$KC\sum_{s=T}^{\infty} k(s+1)^{-1}\psi(s,k(s)\rho) \leq \mu/3\,.$$

Therefore, for all $n = 0,1,\ldots$, and all $t \geq T$ we have

$$|k(t)^{-1}\sum_{s=T}^{\infty}\Phi(t)(I-P)\Phi^{-1}(s+1)f(s,y_n(s))| \leq \mu/3\,.$$

From this estimate we obtain

$$(\mathcal{U})(y_n)(t) = \sum_{s=t_0}^{t-1} \Phi(t)P\Phi^{-1}(s+1)f(s,y_n(s)) - \sum_{s=t}^{T} \Phi(t)(I-P)\Phi^{-1}(s+1)f(s,y_n(s)) + k(t)O(\mu/3). \tag{15}$$

From this asymptotic formula, we observe that the uniform convergence of $\{y_n\}$ to y_∞ on the interval $\overline{0,T}$ implies the convergence of $\{\mathcal{U}(y_n)\}$ to $\mathcal{U}(y_\infty)$ in the metric of the space ℓ_k^∞.

Step 3: Prove that if $\{y_n\}$ is contained in $B_k[0,\rho]$, then $\{k(t)^{-1}\mathcal{U}(y_n)(t)\}$ is equiconvergent to zero. Notice that given a positive number μ, then there exists a $T \in \mathbf{N}$ such that (15) is valid. From Theorem 2, we may assume that

$$\lim_{t\to\infty} k(t)^{-1}\Phi(t)P = 0\,.$$

From this limit and (15), it follows that $\{k(t)^{-1}\mathcal{U}(y_n)(t)\}$ is equiconvergent to zero.

Because of steps 1–3, the conditions of Theorem 3 are fulfilled, and therefore the operator $\mathcal{T}$ has a fixed point $y(\cdot)$ in the ball $B_k[0,\rho]$. Since $|k(t_0)^{-1}y(t_0)| < \rho$, from (14) we obtain $|h(t_0)^{-1}y(t_0)| < \delta$, implying that $y(\cdot)$ is an h-bounded function. But condition (13) and the compensation of the (h,k)-dichotomy imply the h-boundedness of the sequence $\mathcal{U}(y)$. Since

$$y(t) = x(t,t_0,x_0) + \mathcal{U}(y)(t),$$

the sequence $x(\cdot,t_0,x_0)$ is h-bounded. But this contradicts the choice of x_0.

Theorem 8 *Assume that (3) has an h-dichotomy and $f(\cdot,y)$ satisfies Condition (M). Also assume that there exists a $\rho_0 > 0$ such that for $0<\rho<\rho_0$,*

$$K\sum_{s=t_0}^{\infty} h(s+1)^{-1}\psi(s,h(s)\rho) < \rho\,.$$

Then, if $V_{h,0} \neq V_h$, the null solution of (2) is asymptotically h-unstable.

Proof. By contradiction, assume that the null solution to (2) is asymptotically h-stable. Then, for $\varepsilon = 1$ there exists a positive δ such that $|h(t_0)^{-1}y_0| < \delta$ implies $\lim_{t\to\infty} h(t)^{-1}y(t,t_0,y_0) = 0$.

Let $0<\rho<\min\{\rho_0,\delta\}$, and choose σ positive such that

$$\sigma + K\sum_{s=t_0}^{\infty} h(s+1)^{-1}\psi(s,h(s)\rho) \le \rho.$$

For an initial condition $x_0 \in \Phi(t_0)[V_h] \setminus \Phi(t_0)[V_{h,0}]$ such that $|x(\cdot, t_0, x_0)|_h < \sigma$, we consider the operator

$$\mathcal{T}(y)(t) = x(t, t_0, x_0) + \mathcal{U}(y)(t).$$

For any $y \in B_h[0, \rho]$ we have the estimate

$$|\mathcal{T}(\dagger)|_h \leq \sigma + K \sum_{s=t_0}^{\infty} h(s+1)^{-1} \psi(s, h(s)\rho) \leq \rho\,,$$

which implies that $\mathcal{T} : B_h[0, \rho] \to B_h[0, \rho]$. By repeating the arguments given in the proof of Theorem 7, we conclude that this operator satisfies the conditions of Theorem 3. Therefore, there is a fixed point $y(\cdot)$ in the ball $B_h[0, \rho]$; hence

$$y = x(\cdot, t_0, x_0) + \mathcal{U}(y).$$

Because of Theorem 2, we assume that projection P defining the h-dichotomy satisfies the condition (11). Therefore,

$$y(t) = x(t, t_0, x_0) + o(h(t))$$

which contradicts $y(\cdot)$ satisfying (4) with $x_0 \in V_h \setminus V_{h,0}$.

5 A Perron like result

In this section we assume that the matrix A is constant and has an eigenvalue with magnitude greater than 1. We also assume that Conditions (F) is satisfied under two possible cases.

Case $0 \leq \alpha < 1$**:** Then there exists a real number r in $(0, 1)$, such that none of the eigenvalues has magnitude 1, and at least one eigenvalue λ of matrix rA satisfies $|\lambda| > 1$. The change of variable $y(t) = r^{-t}z(t)$ in (1) yields

$$z(t+1) = rAz(t) + r^{t+1}f(t, r^{-t}z(t)), \quad f(t, 0) = 0\,. \tag{16}$$

Let

$$R_1 = \min\{|\lambda| : |\lambda| > 1, \lambda \text{ is an eigenvalue of } rA\},$$

and $\Phi_r(t)$ be the fundamental matrix of the equation

$$x(t+1) = rAx(t)\,.$$

Let R be a positive number satisfying $R_1^{\alpha} < R < R_1$. Then is is easy to prove the existence of a projection matrix P and a constant $K \geq 1$, such that

$$|\Phi_r(t)P\Phi_r^{-1}(s)| \leq KR^{t-s}, \quad 0 \leq s \leq t,$$
$$|\Phi_r(t)(I-P)\Phi_r^{-1}(s)| \leq KR_1^{t-s}, \quad 0 \leq t \leq s\,.$$

These estimates imply that the difference system $x(t+1) = rAx(t)$ has an (R^t, R_1^t)-dichotomy (This is not an exponential dichotomy). Since the condition $V_h \neq V_k$ is satisfied, then we aim to apply Theorem 7. If condition (F) is satisfied then the monotone condition (M) is valid with

$$\psi(t,s) = \gamma(t) r^{(1-\alpha)t+1} s^\alpha .$$

To satisfy (13) we need

$$K r R_1^{-1} \rho^\alpha \sum_{s=t_0}^{\infty} \left(\frac{R_1}{r}\right)^{(\alpha-1)s} \gamma(s) < \rho \,. \tag{17}$$

Because $(R_1/r)^{(\alpha-1)} < 1$, the series in the above inequality converges even for a $\gamma(t)$ of exponential growth, and (17) is satisfied for all ρ sufficiently small.

Then by Theorem 7 the null solution to (16) is R^t-unstable. This implies the instability of the null solution to (1).

Remark Instability of (1) has been obtained under conditions weaker than those in Theorem 1. In Condition (F) γ is unbounded ($\gamma(t) = R^t$ with $R > 1$), as opposed to γ being bounded in Theorem 1.

Case $1 \leq \alpha$**:** This case can be reduced to the previous one, because stability of the null solution to (1) is equivalent to stability of the null solution to

$$y(t+1) = Ay(t) + F(t, y(t)), \quad F(t,0) = 0,$$

where $F(t,y)$ is defined by

$$F(t,y) = \begin{cases} f(t,y), & |y| < 2^{-1}, \\ f(t, \frac{y}{2|y|}), & |y| \geq 2^{-1}. \end{cases}$$

Notice that $F(t,y)$ satisfies Condition (F) with

$$|F(t,y)| \leq \gamma(t)|y|^\beta, \quad \forall \beta \in [0,1)\,.$$

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Raúl Naulin
Departamento de Matemáticas, Universidad de Oriente
Cumaná 6101 Apartado 285, Venezuela
Email address: rnaulin@sucre.udo.edu.ve

Carmen J. Vanegas
Departamento de Matemáticas, Universidad Simón Bolívar
Caracas, Apartado 89000, Venezuela
Email address: cvanegas@usb.ve