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C-INFINITY INTERFACES OF SOLUTIONS FOR ONE-DIMENSIONAL PARABOLIC p-LAPLACIAN EQUATIONS

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Abstract. We study the regularity of a moving interface $x = \zeta(t)$ of the solutions for the initial value problem

$$\begin{gathered} u_t = \left(|u_x|^{p-2}u_x\right)_x \\ u(x,0) = u_0(x)\,, \end{gathered}$$

where $u_0 \in L^1(\mathbb{R})$ and $p > 2$. We prove that each side of the moving interface is C^∞.

1. Introduction

We consider the Cauchy problem of the form

$$\begin{gathered} u_t = \left(|u_x|^{p-2}u_x\right)_x \text{ in } S := \mathbb{R}\times(0,\infty) \\ u(x,0) = u_0(x) \end{gathered} \tag{1.1}$$

where $p > 2$. This equation has application to many physical situations, and has been studied by many authors; see for example [2] and references therein. In the study of this equation, the velocity of propagation, $V(x,t)$, is very important, and can be obtained in terms of u by writing (1.1) as the conservation law

$$u_t + (uV)_x = 0\,.$$

In this way we obtain $V = -v_x|v_x|^{p-2}$, where the nonlinear potential $v(x,t)$ is

$$v = \frac{p-1}{p-2}u^{(p-2)/(p-1)}. \tag{1.2}$$

By a direct computation, we realize that

$$v_t = (p-2)v|v_x|^{p-2}v_{xx} + |v_x|^p. \tag{1.3}$$

In [2], it is shown that V satisfies $V_x \le \frac{1}{2(p-1)t}$ which can also be written as

$$(v_x|v_x|^{p-2})_x \ge -\frac{1}{2(p-1)t}\,. \tag{1.4}$$

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Without loss of generality, we assume that u_0 vanishes on $\mathbb{R}^-$ and that u_0 is a continuous positive function on an interval $(0, a)$ with $a > 0$. Let

$$P[u] = \{(x,t) \in S : u(x,t) > 0\}$$

be the positivity set of a solution u. Then $P[u]$ is bounded from the left in the (x,t)-plane by the left interface curve $x = \zeta(t)$, where

$$\zeta(t) = \inf\{x \in \mathbb{R} : u(x,t) > 0\}\,.$$

Moreover, there is a time $t^* \in [0, \infty)$, called the waiting time, such that $\zeta(t) = 0$ for $0 \le t \le t^*$ and $\zeta(t) < 0$ for $t > t^*$. It is shown in [2] that t^* is finite (possibly zero) and $\zeta(t)$ is a non-increasing C^1 function on (t^*, ∞). Actually it is shown that $\zeta'(t) < 0$ for every $t > t^*$, i.e., a moving interface never stops.

On the other hand, D. G. Aronson and J. L. Vazquez [1] established Theorem 1.1 below.

Let $D = \{(x,t) : t > t^*, \zeta(t) \le x \le 0\}$, and let v be the pressure for the solution of the porous medium equation

$$u_t = (u^m)_{xx} \quad in \quad Q_T = \mathbb{R} \times (0,T). \tag{1.5}$$

Theorem 1.1. *v is a C^∞ function on D, and $\zeta(t)$ is a C^∞ function on (t^*, ∞).*

This theorem is proven by finding bounds for $v^{(k)}$ with $k \ge 2$.

The purpose of this paper is to discuss the C^∞ regularity of the moving part of the interface of the solution to (1.1). To accomplish this end, we use some ideas from [1].

2. Upper and Lower Bounds for v_{xx}

Let $q = (x_0, t_0)$ be a point on the left interface, so that $x_0 = \zeta(t_0)$, $v(x, t_0) = 0$ for all $x \le \zeta(t_0)$, and $v(x, t_0) > 0$ for all sufficiently small $x > \zeta(t_0)$. We assume the left interface is moving at q. Thus $t_0 > t^*$. We shall use the notation

$$R_{\delta,\eta} = R_{\delta,\eta}(t_0) = \{(x,t) \in \mathbb{R}^2 : \zeta(t) < x \le \zeta(t) + \delta, t_0 - \eta \le t \le t_0 + \eta\}.$$

Proposition 2.1. *Let q be the point as above. Then there exist positive constants C, δ, and η depending only on p, q, and u such that*

$$v_{xx} \ge C \quad in\ R_{\delta,\eta/2}.$$

Proof. From (1.4) we have, $v_{xx} \ge -\dfrac{1}{2(p-1)^2|v_x|^{p-2}t}$. However, from Lemma 4.4 in [2], v_x is bounded away and above from zero near q, where $u(x,t) > 0$. □

Proposition 2.2. *Let $q = (x_0, t_0)$ be as above. Then there exist positive constants C_2, δ, and η depending only on p, q, and u such that*

$$v_{xx} \le C_2 \quad in\ R_{\delta,\eta/2}\,.$$

Proof. From Theorem 2 and Lemma 4.4 in [2] we have

$$\zeta'(t_0) = -v_x|v_x|^{p-2} = -v_x^{p-1} = -a \tag{2.1}$$

and

$$v_t = |v_x|^p \tag{2.2}$$

on the moving part of the interface $\{x=\zeta(t), t>t^*\}$. Choose $\epsilon>0$ such that

$$(p-1)a-5p\epsilon \geq 4[(p-2)^2+(p-1)^2](a+\epsilon)\epsilon. \tag{2.3}$$

Then by Theorem 2 in [2], there exists a $\delta=\delta(\epsilon)>0$ and $\eta=\eta(\epsilon)\in(0,t_0-t^*)$ such that $R_{\delta,\eta}\subset P[u]$,

$$(a-\epsilon)^{\frac{1}{p-1}} < v_x < (a+\epsilon)^{\frac{1}{p-1}} \tag{2.4}$$

and

$$vv_{xx} \leq (a-\epsilon)^{\frac{2}{p-1}}\epsilon \tag{2.5}$$

in $R_{\delta,\eta}$. Then from (2.4) we have

$$(a-\epsilon)^{\frac{1}{p-1}}(x-\zeta) < v(x,t) < (a+\epsilon)^{\frac{1}{p-1}}(x-\zeta) \tag{2.6}$$

in $R_{\delta,\eta}$ and

$$-(a+\epsilon) < \zeta'(t) < -(a-\epsilon) \quad \text{in } [t_1,t_2] \tag{2.7}$$

where $t_1=t_0-\eta$ and $t_2=t_0+\eta$. We set

$$\zeta^*(t)=\zeta(t_1)-b(t-t_1) \tag{2.8}$$

where $b=a+2\epsilon$. Then clearly $\zeta(t)>\zeta^*(t)$ in $(t_1,t_2]$. On $P[u]$, $w\equiv v_{xx}$ satisfies

$$\begin{aligned} L(w) &= w_t-(p-2)v|v_x|^{p-2}w_{xx}-(3p-4)|v_x|^{p-2}v_xw_x \\ &\quad -[(p-2)^2+2(p-1)^2]|v_x|^{p-2}w^2 \\ &\quad -3(p-2)^2v|v_x|^{p-4}v_xww_x-(p-2)^2(p-3)v|v_x|^{p-4}w^3 \\ &= 0\,. \end{aligned}$$

We shall construct a barrier for w in $R_{\delta,\eta}$ of the form

$$\phi(x,t)\equiv\frac{\alpha}{x-\zeta(t)}+\frac{\beta}{x-\zeta^*(t)},$$

where α and β will be decided later.

By a direct computation we have

$$\begin{aligned} L(\phi) &= \frac{\alpha}{(x-\zeta)^2}\{\zeta'-(p-2)v|v_x|^{p-2}\frac{2}{x-\zeta}+(3p-4)|v_x|^{p-2}v_x\} \\ &\quad +\frac{\beta}{(x-\zeta^*)^2}\{\zeta^{*'}-(p-2)v|v_x|^{p-2}\frac{2}{x-\zeta^*}+(3p-4)|v_x|^{p-2}v_x\} \\ &\quad -[(p-2)^2+2(p-1)^2]|v_x|^{p-2}\phi^2+\bar{G} \end{aligned}$$

where

$$\begin{aligned} &\bar{G} \\ &= -3(p-2)^2vv_x|v_x|^{p-4}\phi\phi_x-(p-2)^2(p-3)v|v_x|^{p-4}\phi^3 \\ &= (p-2)^2v|v_x|^{p-4}\phi\left(3v_x[\frac{\alpha}{(x-\zeta)^2}+\frac{\beta}{(x-\zeta^*)^2}]-(p-3)[\frac{\alpha}{x-\zeta}+\frac{\beta}{x-\zeta*}]^2\right). \end{aligned}$$

If we choose α and β satisfying

$$v_x\geq|p-3|\max(\alpha,\beta),$$

then $\bar{G} \geq 0$ in $R_{\delta,\eta}$. Now set $\bar{A} = \frac{\alpha}{(x-\zeta)^2}$ and $\bar{B} = \frac{\beta}{(x-\zeta^*)^2}$. Then we have

$$\begin{aligned} & L(\phi) \\ \geq\ & \bar{A}\left\{\zeta' + |v_x|^{p-2}\{-(p-2)v\frac{2}{x-\zeta} + (3p-4)v_x - 2[(p-2)^2 + 2(p-1)^2]\alpha\}\right\} \\ & +\bar{B}\left\{{\zeta^*}' + |v_x|^{p-2}\{-(p-2)v\frac{2}{x-\zeta^*} + (3p-4)v_x - 2[(p-2)^2 + 2(p-1)^2]\beta\}\right\} \\ \geq\ & \bar{A}\left\{(p-1)a - (5p-7)\epsilon - 2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}\alpha\right\} \\ & +\bar{B}\left\{(p-1)a - (5p-6)\epsilon - 2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}\beta\right\}. \end{aligned}$$

Set

$$0 < \alpha \leq \frac{(p-1)a - (5p-7)\epsilon}{2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}} = \alpha_0$$

and

$$\beta = \frac{(p-1)a - (5p-6\epsilon)}{2[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}}\,. \tag{2.9}$$

Then from (2.3), $\beta > 0$ and $L(\phi) \geq 0$ in $R_{\delta,\eta}$ for all $\alpha \in (0, \alpha_0]$ and β.

Let us now compare w and ϕ on the parabolic boundary of $R_{\delta,\eta}$. In view of (2.5) and (2.6) we have

$$v_{xx} \leq \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{x-\zeta} \quad \text{in } R_{\delta,\eta}$$

and in particular

$$v_{xx}(\zeta(t)+\delta, t) \leq \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{\delta} \text{ in } [t_1, t_2]\,.$$

By the Mean Value Theorem and (2.7), we have that for some $\tau \in (t_1, t_2)$

$$\begin{aligned} \zeta(t) + \delta - \zeta^*(t) &= \delta + (a+2\epsilon)(t-t_1) + \zeta'(\tau)(t-t_1) \\ &\leq \delta + 3\epsilon(t-t_1) \leq \delta + 6\epsilon\eta. \end{aligned}$$

Now set

$$\eta = \min\{\eta(\epsilon), \delta(\epsilon)/6\epsilon\}.$$

Since ϵ satisfies (2.3) and β is given by (2.9) it follows that

$$\phi(\zeta+\delta, t) \geq \frac{\beta}{2\delta} \geq \frac{(p-1)a - (5p-6\epsilon)}{4[(p-2)^2 + 2(p-1)^2](a+\epsilon)^{\frac{p-2}{p-1}}\delta} \geq \frac{(a+\epsilon)^{\frac{1}{p-1}}}{\delta}\epsilon \geq v_{xx}\,,$$

on $[t_1, t_2]$. Moreover from (3.5) and (2.9)

$$\phi(x, t_1) \geq \frac{\beta}{x - \zeta(t_1)} > \frac{\epsilon(a-\epsilon)^{\frac{1}{p-1}}}{x-\zeta(t_1)} > v_{xx}(x, t_1) \text{ on } (\zeta(t_1), \zeta(t_1)+\delta]\,.$$

Let $\Gamma = \{(x,t) \in \mathbb{R}^2 : x = \zeta(t), t_1 \leq t \leq t_2\}$. Clearly Γ is a compact subset of $\mathbb{R}^2$. Fix $\alpha \in (0, \alpha_0)$. For each point $s \in \Gamma$ there is an open ball B_s centered at s such that

$$(vv_{xx})(x,t) \leq \alpha(a-\epsilon)^{\frac{1}{p-1}} \quad \text{in } B_s \cap P[u]\,.$$

In view of (2.6) we have

$$\phi(x,t) \geq \frac{\alpha}{x-\zeta} \geq v_{xx}(x,t) \quad \text{in } B_s \cap P[u]\,.$$

Since Γ can be covered by a finite number of these balls it follows that there is a $\gamma = \gamma(\alpha) \in (0,\delta)$ such that
$$\phi(x,t) \geq w(x,t) \quad \text{in } R_{\delta,\eta}.$$
Thus for every $\alpha \in (0,\alpha_0)$, ϕ is a barrier for w in $R_{\delta,\eta}$. By the comparison principle for parabolic equations [4] we conclude that
$$v_{xx}(x,t) \leq \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad \text{in } R_{\delta,\eta}\,,$$
where β is given by (2.9) and $\alpha \in (0,\alpha_0)$ is arbitrary. Now as α approaches zero, we obtain
$$v_{xx}(x,t) \leq \frac{\beta}{x-\zeta^*} \leq \frac{2\beta}{\epsilon\eta} \quad \text{in } \mathbb{R}.$$

3. Bounds for $\left(\frac{\partial}{\partial x}\right)^3 v$

In this section we find the estimates of the derivatives of the form
$$v^{(3)} \equiv \left(\frac{\partial}{\partial x}\right)^3 v.$$
By a direct computation we have,
$$\begin{aligned} (3.1)\ L_3(v^{(3)}) \;&=\; v_t^{(3)} - (p-2)vv_x^{p-2}v_{xx}^{(3)} - (A+B)v_x^{(3)} - Cv^{(3)} - D(v^{(3)})^2 \\ &\quad -Ev_x^{p-3}v_{xx}^3 - (p-2)^2(p-3)(p-4)vv_x^{p-5}v_{xx}^4 = 0\,, \end{aligned}$$
where
$$\begin{aligned} A &= (p-2)v_x^{p-1} + (p-2)^2vv_x^{p-3}v_{xx}\,, \\ B &= (3p-4)v_x^{p-1} + 3(p-2)^2vv_x^{p-3}v_{xx}\,, \\ C &= v_{xx}v_x^{p-2}\{(3p-4)(p-1) + 2[(p-2)^2 \\ &\quad +2(p-1)^2] + 6(p-2)^2(p-3)vv_x^{-2}v_{xx} + 3(p-2)^2\}\,, \\ D &= 3(p-2)^2vv_x^{p-3}\,, \\ E &= [(p-2)^2 + 2(p-1)^2](p-2) + (p-2)^2(p-3)\,. \end{aligned}$$

Suppose that $q = (x_0,t_0)$ is a point on the left interface for which (2.1) holds. Fix $\epsilon \in (0,a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0,\eta_0}(t_0) \subset P[u]$ and (2.5) holds. Thus we also have (2.6) and (2.7) in R_0. Then by rescaling and interior estimate we have

Proposition 3.1. *There are constants $K \in \mathbb{R}^+$, $\delta \in (0,\delta_0)$, and $\eta \in (0,\eta_0)$ depending only on p, q, and C_2 such that*
$$|v^{(3)}(x,t)| \leq \frac{K}{x-\zeta(t)} \quad \textit{in } R_{\delta,\eta}\,.$$

Proof. Set
$$\delta = \min\{\frac{2\delta_0}{3}, 2s\eta_0\}, \quad \eta = \eta_0 - \frac{\delta}{4s}\,,$$
and define
$$R(\overline{x},\overline{t}) \equiv \left\{(x,t) \in \mathbb{R}^2 : |x-\overline{x}| < \frac{\lambda}{2}, \overline{t} - \frac{\lambda}{4s} < t \leq \overline{t}\right\}$$

for $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, where $s = a+\epsilon$ and $\lambda = \overline{x} - \zeta(\overline{t})$. Then $(\overline{x},\overline{t}) \in R_{\delta,\eta}$ implies that $R(\overline{x},\overline{t}) \subset R_0$. Since $\delta_0 \geq \frac{3\delta}{2}$, $\lambda < \delta$ and ζ is non-increasing, we have

$$\begin{aligned} t_0 - \eta_0 = t_0 - \eta - \tfrac{\lambda}{4s} < t < t_0 + \eta < t_0 + \eta_0\,,\\ \overline{x} - \tfrac{\lambda}{2} = \overline{x} - \tfrac{\overline{x}+\zeta(\overline{t})}{2} = \tfrac{\overline{x}+\zeta(\overline{t})}{2} > \zeta(t_0+\eta_0)\,,\\ \zeta(t_0-\eta) + \delta + \tfrac{\lambda}{2} < \zeta(t_0-\eta_0)\,. \end{aligned}$$

Also observe that for each $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, $R(\overline{x},\overline{t})$ lies to the right of the line $x = \zeta(\overline{t}) + s(\overline{t}-t)$. Next set $x = \lambda\xi + \overline{x}$ and $t = \lambda\tau + \overline{t}$. The function

$$W(\xi,\tau) \equiv v_{xx}(\lambda\xi+\overline{x}, \lambda\tau+\overline{t}) = v_{xx}(x,t)$$

satisfies the equation

$$\begin{aligned} W_\tau \;=\;& \Big\{(p-2)\frac{v}{\lambda}v_x^{p-2}W_\xi + (3p-4)v_x^{p-1}W\Big\}_\xi \\ &+[2(p-2)^2 v v_x^{p-3}v_{xx} - (p-2)v_x^{p-1}]W_\xi \\ &+\lambda[(p-2)^2(p-3)v v_x^{p-4}(v_{xx})^3 - (p-2)v_x^{p-2}(v_{xx})^2] \end{aligned} \tag{3.2}$$

in the region

$$B \equiv \left\{(\xi,\tau) \in \mathbb{R}^2 : |\xi| \leq \frac{1}{2}, -\frac{1}{4s} < \tau \leq 0\right\},$$

and $|W| \leq C_2$ in B. In view of (2.6) and (2.7)

$$(a-\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda} \leq \frac{v(x,t)}{\lambda} \leq (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda}$$

and

$$\zeta(\overline{t}) \leq \zeta(t) \leq \zeta(\overline{t}) + s(\overline{t}-t) \leq \zeta(\overline{t}) + \frac{\lambda}{4}\,.$$

Therefore,

$$\frac{\lambda}{4} = \overline{x} - \frac{\lambda}{2} - \zeta(\overline{t}) - \frac{\lambda}{4} \leq x - \zeta(t) \leq \overline{x} + \frac{\lambda}{2} - \zeta(\overline{t}) = \frac{3\lambda}{2}$$

which implies

$$\frac{(a-\epsilon)^{\frac{1}{p-1}}}{4} \leq \frac{v}{\lambda} \leq \frac{3(a+\epsilon)^{\frac{1}{p-1}}}{2}\,.$$

Hence by (2.4) equation (3.2) is uniformly parabolic in B. Moreover, it follows from Proposition 2.2 that W satisfies all of the hypotheses of Theorem 5.3.1 of [4]. Thus we conclude that there exists a constant $K = K(a,p,C_2) > 0$ such that

$$\left|\frac{\partial}{\partial\xi}W(0,0)\right| \leq K;$$

that is,

$$|v^{(3)}(\overline{x},\overline{t})| \leq \frac{K}{\lambda}\,.$$

Since $(\overline{x},\overline{t}) \in R_{\delta,\eta}$ is arbitrary, this proves the proposition. □

We now turn to the barrier construction. If $\gamma \in (0,\delta)$ we will use the notation

$$R^\gamma_{\delta,\eta} = R^\gamma_{\delta,\eta}(t_0) \equiv \{(x,t) \in \mathbb{R}^2 : \zeta(t)+\gamma \leq x \leq \zeta(t)+\delta, t_0-\eta \leq t \leq t_0+\eta\}.$$

Proposition 3.2. *Let R_{δ_1,η_1} be the region constructed in the proof of Proposition 2.2 with*

$$0 < \delta_1 < \frac{(p-1)a^{\frac{1}{p-1}}}{12(p-2)^2 K}\,. \tag{3.3}$$

For $(x,t) \in R^{\gamma}_{\delta_1,\eta_1}$, let

$$\phi_\gamma(x,t) \equiv \frac{\alpha}{x-\zeta(t)-\gamma/3} + \frac{\beta}{x-\zeta^*(t)}\,, \tag{3.4}$$

where ζ^* is given by (2.8), and α and β are positive constant less than $K/2$. Then there exist $\delta \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ depending only on a, p and C_2 such that

$$L_3(\phi_\gamma) \ge 0 \quad \text{in } R^{\gamma}_{\delta,\eta}$$

for all $\gamma \in (0,\delta)$.

Proof. Choose ϵ such that

$$0 < \epsilon < \frac{(p-1)a}{13p-23}\,. \tag{3.5}$$

There exist $\delta_2 \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ such that (2.4), (2.6) and (2.7) hold in $R_{\delta_2,\eta}$. Fix $\gamma \in (0,\delta_2)$. For $(x,t) \in R^{\gamma}_{\delta_2,\eta}$, we have

$$\begin{aligned} L_3(\phi_3) \quad = \quad & \frac{\alpha}{(x-\zeta-\gamma/3)^2}\left\{\zeta' - \frac{2(p-2)vv_x^{p-2}}{x-\zeta-\gamma/3} + A + B\right\} \\ & + \frac{\alpha}{(x-\zeta^*)^2}\left\{\zeta^{*\prime} - \frac{2(p-2)vv_x^{p-2}}{x-\zeta^*} + A + B\right\} \\ & - C\phi_3 - D(\phi_3)^2 - Ev_x^{p-3}v_{xx}^3 - (p-2)^2(p-3)(p-4)vv_x^{p-5}v_{xx}^4 \end{aligned}$$

where A, B, C, D, and E are as above.

¿From (2.6), together with the fact that $x-\zeta^* \ge x-\zeta-\gamma/3$ we have

$$\frac{v}{x-\zeta^*} \le \frac{v}{x-\zeta-\gamma/3} \le (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta}{x-\zeta-\gamma/3} \le (a+\epsilon)^{\frac{1}{p-1}}\frac{\gamma}{\gamma-\gamma/3} = \frac{3}{2}(a+\epsilon)^{\frac{1}{p-1}}\,.$$

¿From (3.3), we have

$$D\alpha, D\beta < 1/2DK < DK \le \frac{(p-1)a}{4} + \frac{(p-1)\epsilon}{4}\,. \tag{3.6}$$

Then since $|C|$ is bounded and from (2.4) and (2.6), we have

$$\begin{aligned} & L_3(\phi_3) \\ & \ge \quad \frac{\alpha}{Y^2}\left\{(p-1)a - (7p-11)\epsilon - |C|Y - 2D\alpha - \overline{E}\frac{Y^2}{\alpha}\right\} \\ & \quad + \frac{\beta}{(x-\zeta^*)^2}\left\{(p-1)a - (7p-10)\epsilon - |C|(x-\zeta^*) - 2D\beta - \overline{E}\frac{(x-\zeta^*)^2}{\beta}\right\} \\ & \ge \quad \frac{\alpha}{Y^2}\left\{\frac{(p-1)a}{2} - \frac{13p-23}{2}\epsilon - \delta_2(|C| - \overline{E}\frac{Y}{\alpha})\right\} \\ & \quad + \frac{\beta}{(x-\zeta^*)^2}\left\{\frac{(p-1)a}{2} - \frac{13p-21}{2}\epsilon - \delta_2(|C| - \overline{E}\frac{x-\zeta^*}{\beta})\right\} \end{aligned}$$

where $Y = x-\zeta-\gamma/3$ and $\overline{E} = |E|v_x^{p-3}v_{xx}^3$. Since ϵ satisfies (3.5) we can choose $\delta = \delta_2(\epsilon,p,a,C_2) > 0$ so small that $L_3(\phi_3) \ge 0$ in $R^{\gamma}_{\delta,\eta}$. $\square$

Remark 3.1. ¿From (3.6) the Proposition 3.2 will be true for any $\alpha, \beta \in (0, K)$.

Proposition 3.3. *(Barrier Transformation). Let δ and η be as in Proposition 3.2 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}, \tag{3.7}$$

where ϵ is as in Proposition 3.2. Suppose that for some nonnegative constant β

$$v^{(3)}(x,t) \le \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad \textit{in } R_{\delta,\eta}. \tag{3.8}$$

Then $v^{(3)}$ also satisfies

$$v^{(3)}(x,t) \le \frac{2\alpha/3}{x-\zeta(t)} + \frac{\beta+2\alpha/3}{x-\zeta^*(t)} \quad \textit{in } R_{\delta,\eta}. \tag{3.9}$$

Proof. By Remark 3.1, for any $\gamma \in (0,\delta)$ since $\beta + 2\alpha/3 \le K$ the function

$$\phi_3(x,t) = \frac{2\alpha/3}{x-\zeta-\gamma/3} + \frac{\beta+2\alpha/3}{x-\zeta^*}$$

satisfies $L_3(\phi_3) \ge 0$ in $R^{\gamma}_{\delta,\eta}$. On the other hand, on the parabolic boundary of $R^{\gamma}_{\delta,\eta}$ we have $\phi_3 \ge v^{(3)}$. In fact, for $t = t_1$ and $\zeta_1 + \gamma \le x \le \zeta_1 + \delta$, with $\zeta_1 = \zeta(t_1)$, we have

$$\phi_3(x,t_1) = \frac{2\alpha}{x-\zeta_1-\gamma/3} + \frac{\beta+2\alpha/3}{x-\zeta_1} > \frac{4\alpha/3}{x-\zeta_1} + \frac{\beta}{x-\zeta_1} > v^{(3)}(x,t_1)$$

while for $x = \zeta + \delta$ and $t_1 \le t \le t_2$ we get, in view of (3.7),

$$\begin{aligned} \phi_3(\zeta+\delta,t) &\ge \frac{2\alpha/3}{\delta-\gamma/3} + \frac{\beta}{\zeta+\delta-\zeta^*} + \frac{2\alpha/3}{\delta+6\epsilon\eta} \\ &\ge \frac{2\alpha/3}{\delta} + \frac{\delta}{\zeta+\delta-\zeta^*} + \frac{\alpha/3}{\delta} \ge v^{(3)}(\zeta+\delta,t). \end{aligned}$$

Finally, for $x = \zeta + \gamma$, $t_1 \le t \le t_2$ we have

$$\phi_3(\zeta+\delta,t) = \frac{2\alpha/3}{\gamma-\gamma/3} + \frac{\beta+2\alpha/3}{\zeta+\gamma-\zeta^*} \ge \frac{\alpha}{\gamma} + \frac{\beta}{\zeta+\gamma-\zeta^*} \ge v^{(3)}(\zeta+\gamma,t).$$

By the comparison principle we get

$$\phi_3 \ge v^{(3)} \quad \text{in } R^{\gamma}_{\delta.\eta}$$

for any $\gamma \in (0,\delta)$, and (3.9) follows by letting $\gamma \downarrow 0$. □

Proposition 3.4. *Let $q = (x_0, t_0)$ be a point on the interface for which (2.1) holds. Then there exist constants C_3, δ and η depending only on p, q and u such that*

$$\left| \left(\frac{\partial}{\partial x}\right)^3 v \right| \le C_3 \quad \textit{in } R_{\delta,\eta/2}.$$

Proof. By Proposition 3.1 we have, by letting $\alpha = 0$,

$$v^{(3)}(x,t) \le \frac{\beta}{x-\zeta^*} \le \frac{2\beta}{\epsilon\eta} \quad \text{in } R_{\delta,\eta/2}.$$

Even though the equation (3.1) is not linear for $v^{(3)}$, a lower bound can be obtained in a similar way. □

4. Main Result

In this section we prove the interface is a C^∞ function in (t^*, ∞). We follow the methods in [1]. First we find the estimates of the derivatives of the form

$$v^{(j)} \equiv \left(\frac{\partial}{\partial x}\right)^j v$$

for $j \geq 4$. For the porous medium equation, we have [1] the following equation:

$$\begin{aligned} L_j v^{(j)} &\equiv v_t^{(j)} - (m-1)vv_{xx}^{(j)} - (2+j(m-1))v_x v_x^{(j)} - c_{mj} v_{xx} v^{(j)} \\ &\quad - \sum_{l=3}^{j^*} d_{mj}^l v^{(l)} v^{(j+2-l)} = 0 \end{aligned}$$

for $j \geq 3$ in $P[u]$, where $j^* = [j/2]+1$, and the c_{mj} and d_{mj}^l are constants which depend only on their indices, but whose precise values are irrelevant. Note that L_j is linear in $v^{(j)}$. On the other hand for the p-Laplacian equation by a direct computation we have the following equation for $j \geq 4$,

$$\begin{aligned} L_j v^{(j)} &= v_t^{(j)} - (p-2)vv_x^{p-2} v_{xx}^{(j)} - ((j-2)A+B)v_x^{(j)} - C_{pj} v^{(j)} \\ &\quad - F(v, v_x, \dots, v^{(j-1)}) = 0 \end{aligned} \tag{4.1}$$

where A and B are as before, and C_{pj} involves only v and derivatives of order $< j$. Note that equation (4.1) is linear in $v^{(j)}$. We also follow the method in [1]. Hence our result is

Proposition 4.1. *Let $q = (x_0, t_0)$ be a point on the interface for which (2.1) holds. For each integer $j \geq 2$ there exist constants C_j, δ and η depending only on p, j, q and u such that*

$$\left|\left(\frac{\partial}{\partial x}\right)^j v\right| \leq C_j \quad \textit{in } R_{\delta,\eta/2}.$$

The proof is done by induction on j. Suppose that $q = (x_0, t_0)$ is a point on the left interface for which (2.1) holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0,\eta_0}(t_0) \subset P[u]$ and (2.5) holds. Thus we also have (2.6) and (2.7) in R_0. Assume that there are constants $C_k \in \mathbb{R}^+$ for $k = 3, \dots, j-1$ such that

$$|v^{(k)}| \leq C_k \quad \text{on} R_0 \quad for \quad k = 3, \dots, j-1. \tag{4.2}$$

Observe that (4.2) hold for $k = 3$ by Proposition 3.4.

By rescaling and interior estimates, we have

Proposition 4.2. *There are constants $K \in \mathbb{R}^+$, $\delta \in (0, \delta_0)$, and $\eta \in (0, \eta_0)$ depending only on p,q and C_k for $k \in [2, j-1]$ with $j \geq 4$ such that*

$$|v^{(j)}(x,t)| \leq \frac{K}{x - \zeta(t)} \quad \textit{in } R_{\delta,\eta}.$$

Proof. Set

$$\delta = \min\{\frac{2\delta_0}{3}, 2s\eta_0\}, \quad \eta = \eta_0 - \frac{\delta}{4s},$$

and define

$$R(\overline{x}, \overline{t}) \equiv \left\{(x,t) \in \mathbb{R}^2 : |x - \overline{x}| < \frac{\lambda}{2}, \overline{t} - \frac{\lambda}{4s} < t \leq \overline{t}\right\}$$

for $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, where $s = a+\epsilon$ and $\lambda = \overline{x} - \zeta(\overline{t})$. Then $(\overline{x},\overline{t}) \in R_{\delta,\eta}$ implies that $R(\overline{x},\overline{t}) \subset R_0$. Since $\delta_0 \geq \frac{3\delta}{2}$, $\lambda < \delta$ and ζ is non-increasing, we have

$$\begin{gathered} t_0 - \eta_0 = t_0 - \eta - \tfrac{\lambda}{4s} < t < t_0 + \eta < t_0 + \eta_0\,, \\ \overline{x} - \tfrac{\lambda}{2} = \overline{x} - \tfrac{\overline{x}+\zeta(\overline{t})}{2} = \tfrac{\overline{x}+\zeta(\overline{t})}{2} > \zeta(t_0+\eta_0)\,, \\ \zeta(t_0-\eta) + \delta + \tfrac{\lambda}{2} < \zeta(t_0-\eta_0)\,. \end{gathered}$$

Also observe that for each $(\overline{x},\overline{t}) \in R_{\delta,\eta}$, $R(\overline{x},\overline{t})$ lies to the right of the line $x = \zeta(\overline{t}) + s(\overline{t}-t)$. Next set $x = \lambda\xi + \overline{x}$ and $t = \lambda\tau + \overline{t}$. The function

$$V^{(j-1)}(\xi,\tau) \equiv v^{(j-1)}(\lambda\xi+\overline{x},\lambda\tau+\overline{t}) = v^{(j-1)}(x,t)$$

satisfies the equation

$$\begin{aligned} V^{(j-1)}_\tau \;=\; & \Big\{(p-2)\frac{v}{\lambda}v_x^{p-2}V^{(j-1)}_\xi + [(j-2)A+B]v_x^{p-1}V^{(j-1)}\Big\}_\xi \\ & -[(p-2)v_x^{p-1} + (p-2)^2 v v_x^{p-3}v_{xx} + (j-2)A+B]V^{(j-1)}_\xi \qquad (4.3) \\ & +\lambda[C_{pj} - ((j-2)A_x+B_x)]V^{(j-1)} + \lambda F(v,\dots,v^{(j-2)} \end{aligned}$$

in the region

$$B \equiv \left\{(\xi,\tau)\in\mathbb{R}^2 : |\xi| \leq \frac{1}{2}, -\frac{1}{4s} < \tau \leq 0\right\},$$

and $|W| \leq C_2$ in B. In view of (2.6) and (2.7)

$$(a-\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda} \leq \frac{v(x,t)}{\lambda} \leq (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta(t)}{\lambda}$$

and

$$\zeta(\overline{t}) \leq \zeta(t) \leq \zeta(\overline{t}) + s(\overline{t}-t) \leq \zeta(\overline{t}) + \frac{\lambda}{4}.$$

Therefore

$$\frac{\lambda}{4} = \overline{x} - \frac{\lambda}{2} - \zeta(\overline{t}) - \frac{\lambda}{4} \leq x - \zeta(t) \leq \overline{x} + \frac{\lambda}{2} - \zeta(\overline{t}) = \frac{3\lambda}{2}$$

which implies

$$\frac{(a-\epsilon)^{\frac{1}{p-1}}}{4} \leq \frac{v}{\lambda} \leq \frac{3(a+\epsilon)^{\frac{1}{p-1}}}{2}\,.$$

Hence by (2.4) equation (3.2) is uniformly parabolic in B. Moreover, it follows from Proposition 2.2 that W satisfies all of the hypotheses of Theorem 5.3.1 of [4]. Thus we conclude that there exists a constant $K = K(a,p,C_1,\dots,C_{j-1}) > 0$ such that

$$\left|\frac{\partial}{\partial\xi}V^{(j-1)}(0,0)\right| \leq K;$$

that is,

$$|v^{(j)}(\overline{x},\overline{t})| \leq \frac{K}{\lambda}\,.$$

Since $(\overline{x},\overline{t}) \in R_{\delta,\eta}$ is arbitrary, this proves the proposition. $\square$

We now turn to the barrier construction. If $\gamma \in (0,\delta)$ we will use the notation

$$R^\gamma_{\delta,\eta} = R^\gamma_{\delta,\eta}(t_0) \equiv \{(x,t)\in\mathbb{R}^2 : \zeta(t)+\gamma \leq x \leq \zeta(t)+\delta, t_0-\eta \leq t \leq t_0+\eta\}\,.$$

Proposition 4.3. *Let R_{δ_1,η_1} be the region constructed in the proof of Proposition 2.2 with For $j \geq 4$ and $(x,t) \in R^{\gamma}_{\delta_1,\eta_1}$, let*

$$\phi_j(x,t) \equiv \frac{\alpha}{x-\zeta(t)-\gamma/3} + \frac{\beta}{x-\zeta^*(t)} \tag{4.4}$$

where ζ^* is given by (2.8), and α and β are positive constant. Then there exist $\delta \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ depending only on a, p, $C_1,\dots,C_{j-1}$ such that

$$L_j(\phi_j) \geq 0 \quad \text{in } R^{\gamma}_{\delta,\eta}$$

for all $\gamma \in (0,\delta)$.

Proof. Choose ϵ such that

$$0 < \epsilon < \frac{a}{(j-2)(p-2)+6p-8}\,. \tag{4.5}$$

There exist $\delta_2 \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ such that (2.4), (2.6) and (2.7) hold in $R_{\delta_2,\eta}$. Fix $\gamma \in (0,\delta_2)$. For $(x,t) \in R^{\gamma}_{\delta_2,\eta}$, we have

$$\begin{aligned} L_j(\phi_j) \;=\; & \frac{\alpha}{(x-\zeta-\gamma/3)^2}\left\{\zeta' - \frac{2(p-2)vv_x^{p-2}}{x-\zeta-\gamma/3} + (j-2)A + B\right\} \\ & -\frac{\alpha}{(x-\zeta-\gamma/3)^2}\left\{C_{pj}(x-\zeta-\gamma/3) - \frac{(x-\zeta-\gamma/3)^2}{\alpha}F\right\} \\ & +\frac{\beta}{(x-\zeta^*)^2}\left\{\zeta^{*\prime} - \frac{2(p-2)vv_x^{p-2}}{x-\zeta^*} + (j-2)A + B - C_{pj}(x-\zeta^*)\right\} \end{aligned}$$

where A, B, C_{pj} and F are as before. ¿From (2.6), together with the fact that $x-\zeta^* \geq x-\zeta-\gamma/3$ we have

$$\frac{v}{x-\zeta^*} \leq \frac{v}{x-\zeta-\gamma/3} \leq (a+\epsilon)^{\frac{1}{p-1}}\frac{x-\zeta}{x-\zeta-\gamma/3} \leq (a+\epsilon)^{\frac{1}{p-1}}\frac{\gamma}{\gamma-\gamma/3} = \frac{3}{2}(a+\epsilon)^{\frac{1}{p-1}}\,.$$

Then from (2.4), (2.6) and (4.2), we have

$$\begin{aligned} L_j(\phi_j) \;\geq\; & \frac{\alpha}{(x-\zeta-\gamma/3)^2}\left\{a - ((j-2)(p-2)+6p-9)\epsilon - \delta_2(|C_{pj}| + \frac{\delta}{\alpha}|F|\right\} \\ & +\frac{\beta}{(x-\zeta^*)^2}\left\{a - ((j-2)(p-2)+6p-8) - \delta_2(|C_{pj}|\right\} \end{aligned}$$

Since ϵ satisfies (4.5) we can choose $\delta = \delta_2(\epsilon,p,a,C_2) > 0$ so small that $L_3(\phi_3) \geq 0$ in $R^{\gamma}_{\delta,\eta}$. ☐

Hence as in we have the following proposition whose proof can be found in [1].

Proposition 4.4. *(Barrier Transformation). Let δ and η be as in Proposition 4.3 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}\,, \tag{4.6}$$

where ϵ is as in Proposition 4.3. Suppose that for some nonnegative constant β

$$v^{(j)}(x,t) \leq \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad \text{in} \quad R_{\delta,\eta}\,. \tag{4.7}$$

Then $v^{(j)}$ also satisfies

$$v^{(j)}(x,t) \leq \frac{2\alpha/3}{x-\zeta(t)} + \frac{\beta+2\alpha/3}{x-\zeta^*(t)} \quad \text{in } R_{\delta,\eta}\,. \tag{4.8}$$

Then as in [1], we can prove the C^∞ regularity of the interface.

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