

PARTIAL REGULARITY FOR FLOWS OF H -SURFACES, II

Changyou Wang

Abstract

We study the regularity of weak solutions to the heat equation for H -surfaces. Under the assumption that the function $H : \mathbb{R}^3 \rightarrow \mathbb{R}$ is bounded and Lipschitz, we show that the solution is $C^{2,\alpha}$ on its domain, except for a set of measure zero.

§1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let H be a bounded Lipschitz function on \mathbb{R}^3 . A map $u \in C^2(\Omega, \mathbb{R}^3)$ is called an H -surface (parametrized over Ω) if u satisfies

$$-\Delta u = 2H(u)u_{x_1} \wedge u_{x_2}. \quad (1.1)$$

It is well known that if $u = (u^1, u^2, u^3)$ is a conformal representation of a surface $S \subset \mathbb{R}^3$, then the mean curvature of S , at the point u , is $H(u)$ (see [S3]). The existence of surfaces with constant mean curvature under various boundary conditions has been studied by Hildebrandt [Hs], Wente [W], Struwe [S1] [S2] [S3], and Brezis-Coron [BC]. The regularity of weak solutions to (1.1) has also been studied; see for example Wente [W], Heinz [He], Tomi [T], and Bethuel-Ghidaglia [BG] for earlier results. Moreover, Bethuel [B] proved that weak solutions to (1.1) are $C^{2,\alpha}$ for any bounded Lipschitz function H .

The heat equation of H -surfaces is defined by

$$\partial_t u - \Delta u = 2H(u)u_{x_1} \wedge u_{x_2}, \quad \text{in } \Omega \times \mathbb{R}^+. \quad (1.2)$$

This equation describes an evolution process of (1.1), which models the deformation of a surface into another surface with mean curvature H at time infinity. The existence of global smooth solutions to (1.2), under special conditions on the H -function, has been studied in [R] and [S2]. In particular, Struwe [S2] considered free boundary conditions of (1.2), with constant H , and obtained a global weak solution to (1.2), which is smooth except for finitely many singular points. Rey [R] has established the existence of a global smooth solutions to (1.1) with the Dirichlet boundary conditions $u = \phi$, provided that $\phi \in H^1 \cap L^\infty(\Omega, \mathbb{R}^3)$ and

$$\|\phi\|_{L^\infty(\Omega)} \|H\|_{L^\infty(\mathbb{R}^3)} < 1. \quad (1.3)$$

1991 *Subject Classification*: 35B65, 35K65.

Key words and phrases: H -surfaces, Lorentz space, Hardy space.

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Submitted August 29, 1998. Published March 4, 1999.

Motivated by the notion of weak solution to (1.1), we say that $u : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ is a weak solution to (1.2), if $\partial_t u, Du \in L^2_{\text{loc}}(\Omega \times \mathbb{R}^+)$ and u satisfies (1.2) in the sense of distributions.

In this note, we consider the partial regularity of weak solutions to (1.2). The motivation is two folds. First, (1.2) is a parabolic counterpart of the elliptic system (1.1) which exhibits full regularity, and in the case of a single equation we know that the parabolic equation roughly has the same regularity as its elliptic counterpart. Second, the nonlinear term in (1.2) is of the same order as that in the flows of harmonic maps from surfaces (see [S3]), and the best regularity for heat equations of harmonic maps from surfaces is that there are finitely many singular points (see Freire [F] or Wang [Wa]). This suggests that weak solutions to (1.2) may have regularity similar to that of heat equations of harmonic maps from surfaces. However, the heat equation of harmonic maps is the negative gradient flow of the Dirichlet energy functional, which satisfies the energy inequality property, but it is not clear whether smooth solutions to (1.2) satisfy

$$\int_{\Omega} |Du|^2(\cdot, t) \leq \int_{\Omega} |Du|^2(\cdot, s), \quad 0 \leq s \leq t < \infty. \quad (1.4)$$

This makes the study of the size and the dimension of singular sets of weak solutions to (1.2) much more difficult. In fact, we are only able to show in Theorem 1 that the singular set has zero Lebesgue measure, which is far from the conjecture that the singular set has (parabolic) Hausdorff dimension at most 1.

In [Wa1], we studied the partial regularity of weak solutions to (1.2) under the condition that H is a bounded Lipschitz function depending only on two variables. A uniqueness result can be found in Chen [Ch].

Theorem 1. *Let $H(p) : \mathbb{R}^3 \rightarrow \mathbb{R}$ be bounded and Lipschitz continuous, and let $u \in H^1(\Omega \times \mathbb{R}^+, \mathbb{R}^3)$ be a weak solution of (1.2). Then there exists a closed subset $\Sigma = \cup_{t>0} \Sigma_t \subset \Omega \times \mathbb{R}^+$, with $\Sigma_t \subset \Omega \times \{t\}$ finite for almost all $t > 0$, such that $u \in C^{2,\alpha}(\Omega \times \mathbb{R}^+ \setminus \Sigma, \mathbb{R}^3)$. In particular, Σ has Lebesgue measure zero.*

§2. Proof of the main theorem

The goal of this section is to prove Theorem 1. First we show that the solution $u : B_1 \times (0, 1] \rightarrow \mathbb{R}^3$ has spatial Hölder continuity in $B_{1/2}$ uniformly with respect to $t \in [1/2, 1]$, under the assumption that $\int_{B_1} |Du|^2$ is small and $\int_{B_1} |\partial_t u|^2$ is bounded, uniformly with respect to $t \in [0, 1]$. Then based on the spatial continuity of u , and a simple observation, we obtain the continuity of u in the time direction. Finally, by elementary covering and suitable rescaling arguments, we show that u has regularity almost everywhere.

To make the proof clear, we review a few concepts. First, we recall the definition of Lorentz spaces [Z]. For an open set $W \subset \mathbb{R}^2$ and $1 \leq q \leq \infty$, let

$$L^{2,q}(W) = \{f : W \rightarrow \mathbb{R} \text{ measurable}, \|f\|_{L^{2,q}(W)} < \infty\}.$$

The norm in this space is defined by

$$\|f\|_{L^{2,q}(W)} = \begin{cases} (\int_0^\infty [t^{1/2} f^*(t)]^q \frac{1}{t} dt)^{1/q}, & \text{if } 1 \leq q < \infty; \\ \sup_{t>0} t^{1/2} f^*(t), & \text{if } q = \infty, \end{cases}$$

where $f^*(t) := \inf\{s > 0 : |\{x \in W : |f(x)| > s\}| \leq t\}$ is the the rearrangement of f . Notice that $L^{2,1} \subset L^{2,2}(\equiv L^2) \subset L^{2,\infty}$, and that $L^{2,1}$ and $L^{2,\infty}$ are dual of each other. For $x_0 \in \mathbb{R}^2$, $0 < r < \infty$, let $B(x_0, r) = \{y \in \mathbb{R}^2 : |y - x_0| \leq r\}$.

Lemma 2.1. For $f, g \in H^1(B(x_0, r))$, let $v \in H_0^1(B(x_0, r))$ be the solution to

$$\begin{aligned} -\Delta v &= f_x g_y - f_y g_x, \quad \text{in } B(x_0, r), \\ v &= 0, \quad \text{on } \partial B(x_0, r). \end{aligned} \tag{2.1}$$

Then $Dv \in L^{2,1}(B(x_0, r))$ and

$$\|Dv\|_{L^{2,1}(B(x_0, r))} \leq C \|Df\|_{L^2(B(x_0, r))} \|Dg\|_{L^2(B(x_0, r))}. \tag{2.2}$$

$$\|Dv\|_{L^{2,\infty}(B(x_0, r))} \leq C \|Df\|_{L^2(B(x_0, r))} \|Dg\|_{L^{2,\infty}(B(x_0, r))}. \tag{2.3}$$

The proof of the Lemma above can be found in Hélein [Hf1], Theorems 3.33–3.38, page 146-155.

Lemma 2.2. For $f \in L^1(B(x_0, r))$, let $v \in H^1(B(x_0, r))$ be the solution to

$$-\Delta v = f, \quad \text{in } B(x_0, r).$$

Then there exists a $C > 0$ such that, for any $\theta \in (0, 1/4)$,

$$\|Dv\|_{L^{2,\infty}(B(x_0, \theta r))} \leq C\theta \|Dv\|_{L^{2,\infty}(B(x_0, r))} + C \|f\|_{L^1(B(x_0, r))}. \tag{2.4}$$

Proof. Let $\bar{f} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an extension of f such that $\bar{f} = 0$ outside $B(x_0, r)$. Let $\bar{v} \in W^{1,1}(\mathbb{R}^2)$ be a solution to

$$-\Delta \bar{v} = \bar{f}, \quad \text{in } \mathbb{R}^2.$$

Then

$$D(\bar{v})(z) = \int_{\mathbb{R}^2} DK(z-x)\bar{f}(x) dx,$$

where $K(z) = \frac{1}{2\pi} \log(|z|^{-1})$. It is well known (cf. [Z]) that $DK \in L^{2,\infty}(\mathbb{R}^2)$. Hence, it follows from the convolution property that $D\bar{v} \in L^{2,\infty}(\mathbb{R}^2)$, and that

$$\|D\bar{v}\|_{L^{2,\infty}(\mathbb{R}^2)} \leq C \|DK\|_{L^{2,\infty}(\mathbb{R}^2)} \|f\|_{L^1(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)}. \tag{2.5}$$

Since $v - \bar{v}$ is a harmonic function on $B(x_0, r)$, an estimate of harmonic functions in [Hf1] implies that

$$\|D(v - \bar{v})\|_{L^{2,\infty}(B(x_0, \theta r))} \leq C\theta \|D(v - \bar{v})\|_{L^{2,\infty}(B(x_0, r))}, \tag{2.6}$$

for any $\theta \in (0, 1/4)$. Hence

$$\|Dv\|_{L^{2,\infty}(B(x_0, \theta r))} \leq C\theta \|Dv\|_{L^{2,\infty}(B(x_0, r))} + C \|D\bar{v}\|_{L^{2,\infty}(B(x_0, r))},$$

this, combined with (2.5), implies (2.4). \diamond

The key part of the proof of Theorem 1 is the following decay property.

Lemma 2.3. *There exist $\epsilon_0 > 0$ and $\theta_0 \in (0, 1/4)$ such that if $u \in H^1(B_1 \times (0, 1], \mathbb{R}^3)$ is a weak solution of (1.2) and $\sup_{t \in (0, 1]} \int_{B_1} |Du|^2 \leq \epsilon_0^2$ then, for $x_0 \in B_{1/2}$, $0 < r < 1/4$, $\theta \in (0, \theta_0)$, we have*

$$\|Du\|_{L^{2,\infty}(B(x_0, \theta r))} \leq \frac{1}{2} \|Du\|_{L^{2,\infty}(B(x_0, r))} + C \|\partial_t u\|_{L^2(B(x_0, r))} r, \quad (2.7)$$

for almost all $t \in [1/2, 1]$.

Proof. The argument here is inspired by that of Bethuel [Bf]. For $x_0 \in B_{1/2}$, $r \in (0, 1/4)$, and $t \in [1/2, 1]$. We apply the L^2 -Hodge decomposition theorem to get that there exist $A \in H^1(B(x_0, r), \mathbb{R}^3)$ and $B \in H_0^1(B(x_0, r), \mathbb{R}^3)$ such that

$$(2H(u)u_x^i, 2H(u)u_y^i) = (A_x^i, A_y^i) + (B_y^i, -B_x^i), \quad i = 1, 2, 3, \quad (2.8)$$

and

$$\|DA\|_{L^2(B(x_0, r))} + \|DB\|_{L^2(B(x_0, r))} \leq C \|Du\|_{L^2(B(x_0, r))}. \quad (2.9)$$

Then we have

$$\begin{aligned} \Delta B &= 2(H(u)_y u_x - H(u)_x u_y), \quad \text{in } B(x_0, r), \\ B &= 0, \quad \text{on } \partial B(x_0, r). \end{aligned} \quad \text{?10}$$

Hence Lemma 2.1 implies

$$\|DB\|_{L^{2,1}(B(x_0, r))} \leq C \|Du\|_{L^2(B(x_0, r))}^2. \quad (2.11)$$

Using (2.8), we can write (1.2) as

$$\Delta u = \partial_t u - A_x \wedge u_y - B_y \wedge u_x, \quad \text{in } B(x_0, r). \quad (2.12)$$

Let $w \in H_0^1(B(x_0, r), \mathbb{R}^3)$ be the solution to

$$\begin{aligned} \Delta w &= -A_x \wedge u_y, \quad \text{in } B(x_0, r), \\ w &= 0, \quad \text{on } \partial B(x_0, r). \end{aligned} \quad \text{?13}$$

Hence, by Lemma 2.1,

$$\begin{aligned} \|Dw\|_{L^{2,\infty}(B(x_0, r))} &\leq C \|DA\|_{L^2(B(x_0, r))} \|Du\|_{L^{2,\infty}(B(x_0, r))} \\ &\leq C \|Du\|_{L^2(B(x_0, r))} \|Du\|_{L^{2,\infty}(B(x_0, r))}. \end{aligned} \quad (2.14)$$

Now we can apply Lemma 2.2, to $u - w$ on $B(x_0, r)$, to conclude that, for $\theta \in (0, 1/4)$,

$$\begin{aligned} \|D(u - w)\|_{L^{2,\infty}(B(x_0, \theta r))} &\leq C\theta \|D(u - w)\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + \|\partial_t u\|_{L^1(B(x_0, r))} + \|B_y \wedge u_x\|_{L^1(B(x_0, r))} \\ &\leq C\theta \|D(u - w)\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + \|\partial_t u\|_{L^2(B(x_0, r))} r + \|DB\|_{L^{2,1}(B(x_0, r))} \|Du\|_{L^{2,\infty}(B(x_0, r))} \\ &\leq C\theta \|D(u - w)\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + \|\partial_t u\|_{L^2(B(x_0, r))} r + C \|Du\|_{L^2(B(x_0, r))}^2 \|Du\|_{L^{2,\infty}(B(x_0, r))} \end{aligned}$$

This, combined with (2.14), imply that

$$\begin{aligned} \|Du\|_{L^{2,\infty}(B(x_0, \theta r))} &\leq (C\theta + C \|Du\|_{L^2(B(x_0, r))}^2) \|Du\|_{L^{2,\infty}(B(x_0, r))} \\ &\quad + C \|Dw\|_{L^{2,\infty}(B(x_0, r))} + \|\partial_t u\|_{L^2(B(x_0, r))} r \\ &\leq (C\theta + C\epsilon_0) \|Du\|_{L^{2,\infty}(B(x_0, r))} + \|\partial_t u\|_{L^2(B(x_0, r))} r. \end{aligned}$$

Therefore, if we choose $\theta_0 \in (0, 1/4)$ and $\epsilon_0 > 0$ sufficiently small then (2.7) follows.

◇

A direct consequence of Lemma 2.3 is the following.

Corollary 2.4. *There exist $\epsilon_0 > 0$ and $\alpha_0 \in (0, 1)$ such that if $u \in H^1(B_1 \times (0, 1], \mathbb{R}^3)$ is a weak solution to (1.2) with $\sup_{t \in (0, 1]} \int_{B_1} |Du|^2 \leq \epsilon_0^2$ and $\Lambda = \sup_{t \in (0, 1]} \int_{B_1} |\partial_t u|^2 < \infty$, then $u(t, \cdot) \in C^{\alpha_0}(B_{1/2}, \mathbb{R}^3)$ for $t \in [1/2, 1]$, and*

$$\sup_{t \in [1/2, 1]} \|u(t, \cdot)\|_{C^{\alpha_0}(B_{1/2})} \leq C(\epsilon_0, \Lambda). \quad (2.15)$$

Proof. Notice that the $L^{2, \infty}$ -norm is conformally invariant. Hence we can iterate (2.7) of Lemma 2.3 to conclude that there exists $\theta_0 \in (0, 1/4)$ such that for any $x_0 \in B_{1/2}$, $r \in (0, 1/4)$, and $t \in [1/2, 1]$,

$$\|Du\|_{L^{2, \infty}(B(x_0, \theta_0^k r))} \leq 2^{-k} \|Du\|_{L^{2, \infty}(B(x_0, r))} + C(1 - \theta_0)^{-1} \Lambda r, \quad (2.16)$$

for all $k \geq 1$. This certainly implies (see for example [GT] Lemma 8.23) that there exists $\alpha_0 \in (0, 1)$ such that for all $t \in [1/2, 1]$

$$\|Du\|_{L^{2, \infty}(B(x, r))} \leq Cr^{\alpha_0} \|Du\|_{L^{2, \infty}(B(x, 1/4))} + C\Lambda r^{\frac{1}{2}}, \quad (2.17)$$

for any $x \in B_{1/2}$ and $0 < r \leq 1/4$. On the other hand, we know that $L^{2, \infty} \subset L^{1, p}$ for any $p \in (1, 2)$. In particular,

$$\begin{aligned} r^{2-p} \int_{B(x, r)} |Du|^p &\leq C \|Du\|_{L^{2, \infty}(B(x, r))}^p \\ &\leq Cr^{p\alpha_0} \|Du\|_{L^{2, \infty}(B(x, 1/4))}^p + C\Lambda r^{p/2}, \end{aligned} \quad (2.18)$$

for any $x \in B_{1/2}$, $r \in (0, 1/4)$, and $t \in [1/2, 1]$. This, combined with Morrey Lemma (see [Mc]), imply $u(t, \cdot) \in C^{\alpha_0}(B_{1/2}, \mathbb{R}^3)$ for $t \in [1/2, 1]$, and (2.15). \diamond

Based on Corollary 2.3 and a simple observation, we actually get the Hölder continuity of u in the time direction as follows.

Corollary 2.5. *There exist $\epsilon_0 > 0$ and $\alpha_1 \in (0, 1)$ such that if $u \in H^1(B_1 \times [0, 1], \mathbb{R}^3)$ is a weak solution to (1.2) with*

$$\sup_{t \in (0, 1]} \int_{B_1} |Du|^2 \leq \epsilon_0^2,$$

and $\Lambda = \sup_{t \in (0, 1]} \int_{B_1} |\partial_t u|^2 < \infty$, then $u \in C^{\alpha_1}(B_{1/2} \times [1/2, 1], \mathbb{R}^3)$.

Proof. For any $x \in B_{1/2}$, $r \in (0, 1/4)$, and $\frac{1}{2} \leq t_1 < t_2 \leq 1$. We have

$$\begin{aligned} |u(x, t_1) - u(x, t_2)| &\leq \left| u(x, t_1) - \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y, t_1) \right| \\ &\quad + \left| u(x, t_2) - \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y, t_2) \right| \\ &\quad + \frac{1}{|B(x, r)|} \int_{B(x, r)} |u(y, t_1) - u(y, t_2)| \\ &\leq \text{osc}_{B(x, r)} u(\cdot, t_1) + \text{osc}_{B(x, r)} u(\cdot, t_2) \\ &\quad + \frac{1}{|B(x, r)|} \int_{B(x, r)} dy \int_{t_1}^{t_2} |\partial_t u(y, t)| dt \\ &\leq C(\epsilon_0, \Lambda) r^{\alpha_0} + \|u_t\|_{L^2(B_1 \times (0, 1])} \frac{\sqrt{t_2 - t_1}}{r}. \end{aligned}$$

Here osc denotes the oscillation and we have used Hölder inequality and $\alpha_0 \in (0, 1)$ is given by Corollary 2.4. Now if we choose t_1, t_2 such that $|t_1 - t_2| \leq 4^{1/(2(1+\alpha_0))}$, and let $r = |t_1 - t_2|^{1/(2(1+\alpha_0))} (\leq \frac{1}{4})$, then

$$|u(x, t_1) - u(x, t_2)| \leq (C(\epsilon_0, \Lambda) + \|\partial_t u\|_{L^2(B_1 \times (0, 1])}) |t_1 - t_2|^{\alpha_0/(2(1+\alpha_0))}, \quad \forall x \in B_{1/2}. \quad (2.19)$$

Let $\alpha_1 = \alpha_0/(2(1 + \alpha_0))$. Then (2.15) and (2.19) imply that $u \in C^{\alpha_1}(B_{1/2} \times [1/2, 1], \mathbb{R}^3)$. \diamond

Completion of the proof of Theorem 1.

Define the parabolic metric: $\delta((x, t), (y, s)) = \max\{|x - y|, \sqrt{|t - s|}\}$. For $(x, t) \in \Omega \times \mathbb{R}^+$ and $R \in (0, \delta((x, t), \partial(\Omega \times \mathbb{R}^+)))$. Define

$$M_R^1(x, t) = \limsup_{s \uparrow t} \int_{B_R(x)} |Du|^2(x, s)$$

$$M_R^2(x, t) = \limsup_{s \uparrow t} \int_{B_R(x)} |\partial_t u|^2(x, s),$$

for the weak solution u of (1.2). It is easy to see that $M_R^i(x, t)$ is non-decreasing with respect to R so that $M^i(x, t) = \lim_{R \downarrow 0} M_R^i(x, t)$ exists and is upper semi-continuous exists for any $(x, t) \in \Omega \times \mathbb{R}^+$, for $i = 1, 2$. Let $\epsilon_0 > 0$ be as same as Corollary 2.5. For $t > 0$, define $\Sigma_t \equiv \Sigma_t^1 \cup \Sigma_t^2 (\subset \Omega)$, where

$$\Sigma_t^1 = \{x \in \Omega : M^1(x, t) \geq \epsilon_0^2\}$$

$$\Sigma_t^2 = \{x \in \Omega : M^2(x, t) = \infty\},$$

let $\Sigma = \cup_{t>0} \Sigma_t$. Then it follows that Σ is a closed subset of $\Omega \times \mathbb{R}^+$.

Claim. $u \in C^{2,\alpha}(\Omega \times \mathbb{R}^+ \setminus \Sigma, \mathbb{R}^3)$ for some $\alpha \in (0, 1)$. To prove this claim, Let $(x_0, t_0) \in \Omega \times \mathbb{R}^+ \setminus \Sigma$. By definition, there exists $r_0 > 0$ such that

$$M_{r_0}^1(x_0, t_0) < \epsilon_0^2, \quad \Lambda_0 \equiv M_{r_0}^2(x_0, t_0) < \infty.$$

For such r_0 , there exists $0 < \delta_0 \leq r_0$ such that

$$\sup_{[t_0 - \delta_0^2, t_0]} \int_{B_{r_0}(x_0)} |Du|^2(x, t) dx \leq \epsilon_0^2,$$

and

$$\sup_{[t_0 - \delta_0^2, t_0]} \int_{B_{r_0}(x_0)} |\partial_t u|^2(x, t) dx \leq 2\Lambda_0.$$

If we define the rescaled mappings $u_{\delta_0} : B_1 \times (-1, 0] \rightarrow \mathbb{R}^3$ by $u_{\delta_0}(x, t) = u(x_0 + \delta_0 x, t_0 + \delta_0^2 t)$, then u_{δ_0} is a weak solution to (1.2) on $B_1 \times (-1, 0]$ and satisfies

$$\sup_{(-1, 0]} \int_{B_1} |Du_{\delta_0}|^2(x, t) dx \leq \epsilon_0^2,$$

and

$$\sup_{(-1, 0]} \int_{B_1} |\partial_t u_{\delta_0}|^2(x, t) dx \leq 2\Lambda_0.$$

Hence Corollary 2.5 implies

$$u_{\delta_0} \in C^\alpha(B_{1/2} \times [-\frac{1}{2}, 0], \mathbb{R}^3),$$

for some $\alpha \in (0, 1)$. This means that $u \in C^\alpha(B(x_0, \delta_0) \times (t_0 - \delta_0^2, t_0 + \delta_0^2), \mathbb{R}^3)$. Since (x_0, t_0) is arbitrary in $\Omega \times \mathbb{R}^+ \setminus \Sigma$, this shows that $u \in C^\alpha(\Omega \times \mathbb{R}^+ \setminus \Sigma, \mathbb{R}^3)$. It is well known that C^α solutions to (1.2) is in $C^{2,\alpha}$ as well.

Now we estimate the size of Σ_t for a.e. $t > 0$. Since $u \in H_{\text{loc}}^1(\Omega \times \mathbb{R}^+)$, the set

$$A = \{t_0 \in \mathbb{R}^+ : \liminf_{t \uparrow t_0} \int_{\Omega} |Du|^2(x, t) + |\partial_t u|^2 dx = +\infty\}$$

has Lebesgue measure equal to zero, $|A| = 0$. For any $t_1 \in \mathbb{R}^+ \setminus A$, it is easy to see that $\Sigma_{t_1}^2 = \emptyset$. We claim that $\Sigma_{t_1}^1$ is finite. In fact, let $\{x_1, \dots, x_N\}$ be a finite subset of $\Sigma_{t_1}^1$. Then we can choose $R_0 > 0$ such that $\{B_{R_0}(x_i)\}_{i=1}^N$ are mutually disjoint and

$$\limsup_{t \uparrow t_1} \int_{B_{R_0}(x_i)} |Du|^2(x, t) dx \geq \epsilon_0^2, \quad 1 \leq i \leq N.$$

Therefore,

$$\begin{aligned} \liminf_{t \uparrow t_1} \int_{\Omega \setminus \cup_{i=1}^N B_{R_0}(x_i)} |Du|^2 &\leq \liminf_{t \uparrow t_1} \int_{\Omega} |Du|^2 - \sum_{i=1}^N \limsup_{t \uparrow t_1} \int_{B_{R_0}(x_i)} |Du|^2 \\ &\leq \liminf_{t \uparrow t_1} \int_{\Omega} |Du|^2 - N\epsilon_0^2. \end{aligned}$$

Hence $N \leq \epsilon_0^{-2} \liminf_{t \uparrow t_1} \int_{\Omega} |Du|^2$, which implies $\Sigma_{t_1}^1$ is finite. It then follows from Fubini's theorem that Σ Lebesgue measure equal to zero. \diamond

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Changyou Wang

Department of Mathematics, Loyola University of Chicago, Chicago, IL 60626. USA

E-mail address: wang@math.luc.edu