Electronic Journal of Differential Equations, Vol. **1998**(1998), No. 16, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

PERSISTENCE OF INVARIANT MANIFOLDS FOR PERTURBATIONS OF SEMIFLOWS WITH SYMMETRY

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Abstract. Consider a semiflow in a Banach space, which is invariant under the action of a compact Lie group. Any equilibrium generates a manifold of equilibria under the action of the group. We prove that, if the manifold of equilibria is normally hyperbolic, an invariant manifold persists in the neighborhood under any small perturbation which may break the symmetry. The Liapunov-Perron approach of integral equations is used.

I. Introduction

In the study of dynamical systems in finite-dimensional spaces, the theory of invariant manifolds has proved to be an important tool. Invariant manifolds, along with invariant foliations, can be used to construct coordinate systems in which the differential equations are partially decoupled. These coordinate systems are very useful in tracking the asymptotic behavior of orbits in neighborhoods of equilibria. In recent years, the theory of invariant manifolds has been generalized to semiflows in Banach spaces. See, for example, [BJ], [Ca], [CH], [CL1], [CL2], [H], [He], [Ke], [MS], [BLZ] and others. Here we extend some of these results to the case where an infinite-dimensional dynamical system is invariant under the action of a smooth Lie group in such a way that an equilibrium gives rise to a manifold of equilibria through the group action. The principal question addressed here is, what happens to this manifold when the system is perturbed, possibly breaking the symmetry in the system?

Let X be a Banach Space. Suppose $S(t)$ is a semiflow generated by a semilinear equation in X and suppose that it is invariant under the action in X of a connected compact symmetry group G. If the origin 0 is an equilibrium and the group G acts at 0 in a nondegenerate way, then the image of 0 under the group action is a manifold of equilibria, diffeomorphic to G. Here we establish the persistence of this manifold under small perturbations of the system provided the manifold is normally hyperbolic. One can find many examples of systems of PDE's which have an inherent symmetry arising from an idealized model. One is interested in the structural stability of such systems and in the behavior of solutions to a perturbed system. An example may be found in the work by Bates [Ba] and Barrow & Bates

1991 Mathematics Subject Classifications: 58F15, 58F35, 58G30, 58G35, 34C35.
Key words and phrases: Semiflow, invariant manifold, symmetry.

Manuscript received in April 1995, revised version April 6, 1999. Published May 18, 1999.
The inordinate delay was due to an oversight by editor P.W. Bates, for which he offers his apologies.

[BB1], [BB2], [BB3], where periodic traveling waves for a Ginzburg-Landau system are considered and the unperturbed system is invariant under the group $O(2)\times O(2)$.

In order to prove the persistence, we require that the center subspace of the linearized equation at 0 coincides with the tangent space of the manifold of equilibria, that is, the manifold is normally hyperbolic. For finite-dimensional dynamical systems, Fenichel [F1] and, independently, Hirsch, Pugh, and Shub [HPS] proved that compact normally hyperbolic invariant manifolds persist under small perturbations. Mañé [Mn] proved that normal hyperbolicity is also a necessary condition. In infinite-dimensional spaces, Henry [He] proved the persistence of normally hyperbolic invariant manifolds which are graphs of maps from closed linear subspaces to their complementary subspaces. A more general result can be found in [BLZ].

Traditionally, there are two methods dealing with invariant manifolds. One dates back to Hadamard [Ha] and the other to Liapunov [Ly] and Perron [Pe]. The Hadamard approach, which is also called the graph-transform method, is more geometric, while the Liapunov-Perron method is more analytic and the strategy is to finds the manifolds as fixed points of some integral equations. In this work, we use the analytic method of Liapunov-Perron.

Consider the equation

$$u_t = F(u), \tag{1}$$

where

$$F(u) = Au + f(u)$$

and $t > 0$. The operator A, defined on a dense subspace $D(A)$ of X, is the generator of a strongly continuous semigroup $T(t)$ on X. Assume that f is Lipschitz on X and such that

$f(0) = 0$ and for any $\theta > 0$, there exists a neighborhood U of 0, such that Lip $f|_U < \theta$.

Thus, A is the linear part of the right side of (1). Following a standard result, (See, for example, page 184, [Pa]) (1) determines a C_0 semiflow $S(t)$ on X, i.e. $S : [0,\infty)\times X \to X$ is continuous in both variables and

$$S(t_1)\circ S(t_2) = S(t_1+t_2)$$

for all $t_1, t_2 \in [0,\infty)$.

Let G be an n-dimensional connected compact Lie group and assume that G acts smoothly (C^2) on X and $D(A)$ is invariant under the action of G. Furthermore, assume that the semiflow generated by (1) is also G-invariant, i.e.,

$$S(t)(gu) = g(S(t)u) \tag{2}$$

for all $t \geq 0$, $g \in G$ and $u \in X$. Throughout this paper, we shall use u, v and so on to denote elements in X and g, h and so on to denote elements in G. With a slight abuse of notation, for an element $g \in G$, we also use g to represent the transformations on X defined by the group action and the left transformations L_g on G defined by $L_g h = gh$ for all $h \in G$. The invariance of the semiflow $S(t)$ under action of G can be written in another form: for all $u \in D(A)$ and $g \in G$,

$$F(gu) = Dg(u)F(u),$$

where $Dg(u)v \equiv lim_{h\to 0}\frac{g(u+hv)-g(u)}{h}$ is the derivative of the action of g on X.

Let $\bar\phi : G\times X \to X$ be the action, i.e. $\bar\phi(g,x) = gx$, which is C^2 on $G\times X$. Let $\phi_x = \bar\phi|_{G\times\{x\}}$ for $x \in X$ so that ϕ_x is a smooth map from G to X.

Assume that ϕ_0 is one-to-one and $D\phi_0(e)$ is of rank n (where $D\phi_x = D_G\bar\phi(g,x)$).

Lemma 1. *$G(0) = \phi_0(G)$ is a C^2 compact submanifold of X, which is composed of equilibria of the semiflow $S(t)$.*

Proof. Since
$$\phi_x \circ g = g \circ \phi_x : G \to X,$$
then,
$$D\phi_x(g) \circ Dg(e) = Dg(x)D\phi_x(e),$$
thus,
$$D\phi_x(g) = Dg(x)D\phi_x(e)(Dg(e))^{-1},$$
therefore for all $g \in G, D\phi_0(g)$ is of rank n.

Combining this result with the fact that ϕ_0 is one-to-one gives us the first conclusion.

Since 0 is an equilibrium of equation (1), (2) implies $\phi_0(G)$ is composed of equilibria, so it is invariant under the action of $S(t)$. ■

Since G is compact and $D\phi_x(g)$ is continuous in $G \times X$, there exists $\delta > 0$, such that $D\phi_x(g)$ is of rank n for all $g \in G$ and $x \in B_\delta(0)$, the ball of radius δ in X. So, for all $h \in G$, $D\phi_{h(x)}(g) = D\phi_x(gh) \circ DR_h(g)$ is of rank n for the above g and x, where R_h is the right translation in G. Also, there exists $\bar{M}$ such that for all $g, h \in G$ and $x \in B_\delta(h(0))$, we have $1/\bar{M} < \|Dg(x)\| < \bar{M}$. (In fact, we will use $\bar{M}$ as a universal upper bound.)

Let $\sigma(A)$ be the spectrum of A. Let $\sigma_s = \{\lambda \in \sigma(A) \,|\, \mathrm{Re}\ \lambda < 0\}$, $\sigma_c = \{\lambda \in \sigma(A) \,|\, \mathrm{Re}\ \lambda = 0\}$, $\sigma_u = \{\lambda \in \sigma(A) \,|\, \mathrm{Re}\ \lambda > 0\}$. Assume A satisfies:

(1) $\sigma_c = \{0\}$,
(2) σ_u is compact,
(3) There exists $\alpha > 0$ such that $\alpha < \inf \mathrm{Re}\ \sigma_u$ and $-\alpha > \sup \mathrm{Re}\ \sigma_s$.

Therefore, we have closed subspaces X_u, X_s, X_c corresponding to $\sigma_u, \sigma_s, \sigma_c$, invariant under A, and $X = X_u \oplus X_c \oplus X_s$, (see page 321 [TL]). Let P_u, P_s, P_c be the corresponding projections. Let $P_z = I - P_c = P_u + P_s$. Assume
$$X_c = D\phi_0(e)\mathcal{T}_e(G),$$
where $\mathcal{T}_e(G)$ is the tangent space of G at e. Because of the previous assumption, X_c is an n-dimensional subspace of X, which is the tangent space of the submanifold $\phi_0(G)$ at 0. Let $A_s = A|_{X_s}, A_u = A|_{X_u}$, and $A_c = A|_{X_c}$. Since $G(0)$ consists of equilibria, then $F|_{G(0)} = 0$. Note that f is differentiable at $u = 0$ and $f'(0) = 0$, so that $A_c = 0$. Since A_u has compact spectrum, so it is bounded and generates a group $e^{A_u t} = T_u(t) = T|_{X_u}$ satisfying $\|T_u(t)\| \le M_1 e^{\alpha t}$ for $t < 0$, where $M_1 \ge 1$. Also, A_s generates a C_0-semiflow $T_s(t) = T(t)|_{X_s}$ on X_s. Assume $\|T_s(t)\| \le M_2 e^{-\alpha t}$ for $t > 0$, where $M_2 \ge 1$. By renorming, we may assume $M_1 = M_2 = 1$.

Now we consider a perturbed equation:
$$u_t = Au + f(u) + \epsilon H(u) \equiv F_\epsilon(u), \tag{3}$$
where $H(\cdot)$ is Lipschitz on X with Lipschitz constant L. For the same reason as for equation (1), it is clear that (3) determines a C_0 semiflow $S_\epsilon(t)$ on X. Our main result is:

Theorem. *Under the above conditions, when ϵ is sufficiently small, there is a Lipschitz invariant manifold of the semiflow $S_\epsilon(t)$ near $G(0)$.*

Remark. *The same result is true with $H(u,\epsilon)$ in place of $\epsilon H(u)$ provided that $H(u,\epsilon)$ is continuous, $H(u,0)=0$, and H is Lipschitz in u with the Lipschitz constant converging to 0 as $\epsilon \to 0$.*

II. Proof of the Theorem.

Define $\|\cdot\|$ on $\mathcal{T}_g(G)$ as $\|v\| = \|D\phi_0(g)v\|$. We may use this norm to define a metric $d(\cdot,\cdot)$ on G as the infimum of the length of the C^1 curves lying in $\phi_0(G)$ joining two image points under ϕ_0 in X.

Clearly $d(g,h) \geq \|\phi_0 g - \phi_0 h\|$. If $\|\phi_0 g_k - \phi_0 g_0\| \to 0$ as $k \to +\infty$, there exists a neighborhood U of g_0 and a local coordinate $\psi : U \to B_1^n(0)$, the unit ball in $\mathbb{R}^n$, such that $\psi(g_0) = 0$. Since $\phi_0(G)$ is a compact submanifold of X, and in particular, a proper submanifold of X, therefore $\phi_0 g_k \to \phi_0 g_0$ in X, implies $g_k \to g_0$ in G. Suppose $g_k \in \psi^{-1}(B^n_{\frac{1}{2}}(0))$. Since $\phi_0 \circ \psi^{-1}$ is a diffeomorphism and $\|D\phi_0 \circ \psi^{-1}\|$ on $B^n_{\frac{1}{2}}(0)$ is bounded, it follows that $d(g_k, g_0) \to 0$.

So, $d(\cdot,\cdot)$ induces the same topology on $\phi_0(G)$ as that inherited from X and $\phi_0(G)$ is diffeomorphic to G. From this, there exits a constant C_0 such that $\|\phi_0 g - \phi_0 h\| \leq d(g,h) \leq C_0\|\phi_0 g - \phi_0 h\|$ for all $g,h \in G$.

Suppose the diameter of G under $d(\cdot,\cdot)$ is $M > 0$. Define $Y = G \times (X_s \oplus X_u)$, $\phi = \bar{\phi}|_Y$ for $(g,x) \in Y$, $v \in \mathcal{T}_g(G)$, $z \in X_s \oplus X_u$,

$$\begin{aligned} D\phi(g,x)(v,z) &= D\phi(g,x)(v,0) + D\phi(g,x)(0,z) \\ &= D\phi_x(g)v + Dg(x)(z) \\ &= Dg(x)D\phi_x(e)(Dg(e))^{-1}v + Dg(x)(z) \\ &= Dg(x)(D\phi_x(e)(Dg(e))^{-1}v + z), \end{aligned} \tag{4}$$

as in the proof of Lemma 1. Since

$$\begin{aligned} X &= X_c \oplus X_s \oplus X_u \\ &= D\phi_0(e)\mathcal{T}_e(G) \oplus X_s \oplus X_u \\ &= D\phi_0(e)(Dg(e))^{-1}\mathcal{T}_g(G) \oplus X_s \oplus X_u, \end{aligned}$$

so, by the argument following Lemma 1, there exists $\delta > 0$ such that for $(g,x) \in Y_\delta = \{(g,x) \in Y : \|x\| \leq \delta\}$, $D\phi(g,x)$ is one-to-one and onto from $\mathcal{T}_g G \times (X_u \oplus X_s)$ to X. $\mathcal{T}_g G \times (X_u \oplus X_s)$ may be identified with $\mathcal{T}_{(g,x)}Y$, the direct sum of $\mathcal{T}_g G$ and $X_u \oplus X_s$, with norm given by the sum of the two norms on $\mathcal{T}_g G$ and $X_u \oplus X_s$. Thus, $D\phi(g,x)$ is an isomorphism between $\mathcal{T}_{(g,x)}Y$ and X. With this norm on $\mathcal{T}_{(g,x)}Y$ we may extend the metric d on Y in the natural way. It is easy to verify that $d((g_1,x_1),(g_2,x_2)) = d(g_1,g_2) + \|x_1 - x_2\|$ is a metric on Y.

Next we prove that when δ is small enough $\phi : Y_\delta \to X$ is one-to-one. Otherwise, there exists $(g_k,x_k),(h_k,y_k) \in Y$ with $x_k \to 0$ and $y_k \to 0$ such that $g_k x_k = h_k y_k$, which implies $h_k^{-1} g_k x_k = y_k$. Since G is compact, without loss of generality, suppose that $h_k^{-1} g_k$ converges to g. Let $k \to \infty$, we get $g(0) = 0$, so, $g = e$, which implies $h_k^{-1} g_k \to e, h_k^{-1} g_k x_k = y_k$. But $D\phi(e,0)$ is an isomorphism so, by the Inverse Function Theorem, ϕ is a local diffeomorphism near $(e,0)$. So, for k sufficiently large, $h_k^{-1}g_k = e$ and $x_k = y_k$, which is a contradiction. Therefore, there exists $\delta > 0$ such that $\phi : Y_\delta \to X$ is one-to-one. By Inverse Function Theorem, ϕ is a diffeomorphism from Y_δ to $\phi(Y_\delta)$, an open subset containing $\phi_0(G)$.

For $g \in G$ define $g : Y \to Y$ as $g(h,x) = (gh,x)$. Note that $g \circ \phi = \phi \circ g$, where the g on the left side denotes the action on X and the g denotes the transformation on Y. Define

$$\begin{aligned}
&\pi_1 : Y \to G,\ \pi_1(g,x) = g,\\
&\pi_2^+ : Y \to X_u,\ \pi_2^+(g,x) = P_u x,\\
&\pi_2^- : Y \to X_s,\ \pi_s^-(g,x) = P_s x,\\
&\pi_2 = \pi_2^+ + \pi_2^-.
\end{aligned}$$

These projections are clearly smooth.

Since ϕ is smooth and G is compact, from (4), it is easy to find constants a_1, $\delta > 0$ such that for all $(g,x) \in Y_\delta$,

$$a_1 \geq \|D\phi(g,x)\|,\ \|D\phi(g,x)^{-1}\|. \tag{5}$$

So, for $(g_1,x_1),(g_2,x_2) \in Y_\delta$,

$$\|\phi(g_1,x_1) - \phi(g_2,x_2)\|/a_1 \leq d((g_1,x_1),(g_2,x_2)) \leq a_1\|\phi(g_1,x_1) - \phi(g_2,x_2)\|.$$

Now we pull back (1) and (3) through ϕ on Y_δ:

$$\tilde{F}(g,x) = (D\phi)^{-1}F(\phi(g,x)), \tag{6}$$

$$\tilde{F}_\epsilon(g,x) = (D\phi)^{-1}F_\epsilon(\phi(g,x)), \tag{7}$$

for $x \in X_s \oplus X_u \cap D(A)$ and $g \in G$. Let η_δ be a Lipschitz cut-off function such that η_δ: $[0,+\infty) \to [0,1]$, $\eta_\delta = 1$ on $[0,\frac{\delta}{2}]$, $\eta_\delta = 0$ on $[\delta,+\infty)$, and $0 \leq Lip\eta \leq \frac{4}{\delta}$. In fact, we will consider $\eta_\delta(\|x\|)\tilde{F}$ and $\eta_\delta(\|x\|)\tilde{F}_\epsilon$ instead of $\tilde{F}$ and $\tilde{F}_\epsilon$, but for simplicity, we will just write $\tilde{F}$ and $\tilde{F}_\epsilon$. With this notation, $\tilde{F}$ and $\tilde{F}_\epsilon$ are defined on all of Y.

By definition $\tilde{F}(g,0) = 0$. Since $g \circ \phi = \phi \circ g$,

$$\begin{aligned}
\tilde{F}(g,x) &= (D\phi)^{-1}F(gx) = (D\phi)^{-1}Dg(x)F(x)\\
&= D(\phi^{-1} \circ g)(x)F(x) = D(g \circ \phi^{-1})(x)F(x)\\
&= Dg(e,x)D\phi^{-1}(x)F(x) = Dg(e,x)\tilde{F}(e,x).
\end{aligned} \tag{8}$$

So, $\tilde{F}$ is invariant under G.

Let $\tilde{F}_1 = D\pi_1\tilde{F}$, $\tilde{F}_2 = D\pi_2\tilde{F}$, $\tilde{F}_2^+ = D\pi_2^+\tilde{F}$ and $\tilde{F}_2^- = D\pi_2^-\tilde{F}$. Similarly, define $\tilde{H}_1, \tilde{H}_2, \tilde{H}_2^+, \tilde{H}_2^-, \tilde{F}_{\epsilon,1}, \tilde{F}_{\epsilon,2}, \tilde{F}_{\epsilon,1}^+$ and $\tilde{F}_{\epsilon,2}^+$. Identity (8) and $\pi_1 \circ g = g \circ \pi_1$ imply that

$$\begin{aligned}
\tilde{F}_1(g,x) &= D\pi_1\tilde{F}(g,x) = D\pi_1 Dg\ \tilde{F}(e,x)\\
&= DgD\pi_1\tilde{F}(e,x) = Dg\tilde{F}_1(e,x),
\end{aligned} \tag{9}$$

and $\pi_2 \circ g = \pi_2$ implies that

$$\tilde{F}_2(g,x) = D\pi_2 Dg\tilde{F}(e,x) = \tilde{F}_2(e,x). \tag{10}$$

So F_2 is independent of the first component and we can write $\tilde{F}_2(x)$ for $x \in (X_s \oplus X_u) \cap D(A)$. Also,

$$\begin{aligned}\tilde{F}_2(x) &= D\pi_2(D\phi(e,x))^{-1}F(x) = D\pi_2(D\phi(e,x))^{-1}(Ax + f(x))\\ &= Ax + D\pi_2(D\phi(e,x))^{-1}f(x) = Ax + \tilde{f}_2(x), \end{aligned} \tag{11}$$

where $\tilde{f}_2(x) = D\pi_2(D\phi(e,x))^{-1}f(x)$. From (9),

$$\begin{aligned}\tilde{F}_1(g,x) &= Dg(e)D\pi_1(e,x)(D\phi(e,x))^{-1}F(x)\\ &= Dg(e)D\pi_1(e,x)(D\phi(e,x))^{-1}(Ax + f(x))\\ &= Dg(e)D\pi_1(e,x)(0,Ax) + Dg(e)\tilde{f}_1(x) = Dg(e)\tilde{f}_1(x), \end{aligned} \tag{12}$$

where $\tilde{f}_1(x) = D\pi_1(D\phi(e,x))^{-1}f(x)$. Let

$$Q_{\epsilon,1}(g,x) = \tilde{F}_1(g,x) + \epsilon\tilde{H}_1(g,x) = Dg(e)\tilde{f}_1(x) + \epsilon\tilde{H}_1(g,x), \tag{13}$$
$$Q_{\epsilon,2}(g,x) = \tilde{f}_2(x) + \epsilon\tilde{H}_2(g,x). \tag{14}$$

Similarly, define $Q^+_{\epsilon,2}$ $Q^-_{\epsilon,2}$. All this quantities are defined on Y_δ, the image of a tubular neighborhood of the manifold $G(0)$ under ϕ. In the rest of the paper, we shall only work in $Y_{\frac{\delta}{2}}$. Let $A_2 = A_u \oplus A_s$. Consider

$$g' = \tilde{F}_1(g,x) + \epsilon\tilde{H}_1(g,x) = Q_{\epsilon,1}(g,x), \tag{15}$$
$$x' = A_2x + \tilde{f}_2(x) + \epsilon\tilde{H}_2(g,x) = A_2x + Q_{\epsilon,2}(g,x). \tag{16}$$

This system is equivalent to (3) on $Y_{\delta/2}$. Equation (16) can be written as

$$\dot{x}^+ = A_ux^+ + Q^+_{\epsilon,2}(g,x^+,x^-), \tag{17}$$
$$\dot{x}^- = A_sx^- + Q^-_{\epsilon,2}(g,x^+,x^-), \tag{18}$$

where $x^+ = P_ux$ and $x^- = P_sx$. Notice by (5) that $\tilde{H}$, $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_2^+$, and $\tilde{H}_2^-$, are still Lipschitz functions in $Y_{\frac{\delta}{2}}$ and the Lipschitz constants are independent of ϵ and δ. Let $\bar{M}$ be a universal upper bound of $\mathrm{Lip}D\phi, \|Dg(e)\|, \|D\pi\|, \tilde{H}$, the norms of $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_2^+$, and $\tilde{H}_2^-$, and their Lipschitz constants on $Y_{\delta/2}$ and also bigger than a_1 in (5). In fact, we have used $\bar{M}$ as an upper bound of $\|Dg\|$ before. In the following $L(\delta)$ always will denote a quantity, which depends on x and g such that $L(\delta) \to 0$ as $\delta \to 0$ uniformly in x and g. Then we have

$$\|Q_{\epsilon,1}(g,x)\| \le \epsilon\bar{M} + L(\delta)\|x\|, \tag{19}$$
$$\|Q_{\epsilon,2}(g,x)\| \le \epsilon\bar{M} + L(\delta)\|x\|. \tag{20}$$

Since

$$\begin{aligned}\tilde{f}_2(x_1) - \tilde{f}_2(x_2) &= D\pi_2D\phi^{-1}(x_1)(f(x_1) - f(x_2))\\ &\quad + (D\pi_2D\phi^{-1}(x_1) - D\pi_2D\phi^{-1}(x_2))f(x_2),\end{aligned}$$

we have

$$\|\tilde{f}_2(x_1) - \tilde{f}_2(x_2)\| \le \|x_1 - x_2\| L(\delta), \tag{21}$$

In the same way we get

$$\begin{aligned}\|\tilde{H}_2(g_1, x_1) - \tilde{H}_2(g_2, x_2)\| &\le \bar{M}(d(g_1, g_2) + \|x_1 - x_2\|) \\ &= \bar{M} d((g_1, x_1), (g_2, x_2)).\end{aligned} \tag{22}$$

So,

$$\begin{aligned}\|Q_{\epsilon,2}(g_1, x_1) - Q_{\epsilon,2}(g_2, x_2)\| \le\ & (\epsilon\bar{M} + L(\delta))\|x_1 - x_2\| \\ & + \epsilon\bar{M} d(g_1, g_2),\end{aligned} \tag{23}$$

where all the above conclusions hold in $Y_{\delta/2}$.

Let

$$Z^+ = \left\{\gamma^+ : G \to X_u |\, \|\gamma^+(e)\| \le \frac{\delta}{8} \text{ and } \gamma^+ \text{ is Lipschitz with } Lip\gamma^+ \le \frac{\delta}{8M}\right\},$$
$$Z^- = \left\{\gamma^- : G \to X_s |\, \|\gamma^-(e)\| \le \frac{\delta}{8} \text{ and } \gamma^- \text{ is Lipschitz with } Lip\gamma^- \le \frac{\delta}{8M}\right\}.$$

Recall that M is the diameter of G. Define $|\cdot|$ on Z^+, Z^- as $|\gamma^\pm| = \max_{g\in G} \|\gamma^\pm(g)\|$. Define $|\cdot|$ on $Z^+ \times Z^-$ as $|\gamma| = |\gamma^+| + |\gamma^-|$. It is not hard to verify that Z^+, Z^-, $Z^+ \times Z^-$ are complete.

We shall define a contraction mapping E on $Z^+ \times Z^-$ so that the graph of its fixed point is the unique invariant manifold in $Y_{\frac{\delta}{2}}$ for system (15) and (16). The transformation $E = (E^+, E^-)$ is defined in the following way. For any fixed $\gamma \in Z^+ \times Z^-$, we substitute $\gamma(g)$ into equation (15) and obtain a vector field on G which depends on γ. For any initial point $g_0 \in G$, the trajectory $g(t)$ of this vector field exists for all $t \in (\infty, \infty)$. Next, substitute $g(t)$ and $\gamma(g(t))$ into the high order part of equations (17) and (18) and derive two unautonomous equations for x^+ and x^-, respectively. Solve equation (17) and (18) with zero value at ∞ and $-\infty$, respectively. Suppose $x^+(t)$ and $x^-(t)$ are solutions, then we define $E^+\gamma(g_0) = x^+(0)$ and $E^-\gamma(g_0) = x^-(0)$. Finally, we verify that E is a contraction and the graph of its fixed point is an invariant manifold.

Take any $\gamma \in Z^+ \times Z^-$. The argument before Lemma 4 depends on the choice of γ. Since $\gamma(g) \in Y_{\delta/2}$,

$$\dot{g} = Q_{\epsilon,1}(g, \gamma(g)) \tag{24}$$

defines a vector field on G.

Lemma 2. *Suppose $g_1, g_2 \in G$, and $g_1(t), g_2(t)$ are solutions of (24) with initial values g_1 and g_2, respectively. We have*

$$d(g_1(t), g_2(t)) \le e^{C_3(\epsilon,\delta)t} d(g_1, g_2),$$

where

$$C_3(\epsilon, \delta) = \epsilon\bar{M} + L(\delta)\delta + (\epsilon\bar{M} + L(\delta))\frac{\delta}{4M}$$

is a constant independent of γ, g_1, *and* g_2.

When (24) is vector field on R^n, this estimate follows directly from the Gronwall's Inequality. Here, the difficulty is that G only has a Finsler structure on it.

Proof. Let $g(t)$ be the solution of (24) with initial point g_0. Obviously, $g(t)$ depends on γ. By (19)

$$\begin{aligned} d(g(t),e) &\le d(g_0,e) + \int_0^t \|Q_{\epsilon,1}(g(s),\gamma(g(s)))\| ds \\ &\le d(g_0,e) + \int_0^t (\epsilon\bar{M} + L(\delta)\|\gamma(g(s))\|)\, ds \\ &\le d(g_0,e) + \epsilon\bar{M}t + \int_0^t L(\delta)\frac{\delta}{4M} d(g(s),e) + L(\delta)\|\gamma(e)\|\, ds \\ &\le d(g_0,e) + (\epsilon\bar{M} + L(\delta)\|\gamma(e)\|)t + \int_0^t L(\delta)\frac{\delta}{4M} d(g(s),e) ds. \end{aligned}$$

Let

$$C_1(\delta) = \frac{L(\delta)\delta}{4M}, \qquad C_2(\epsilon,\delta) = \epsilon\bar{M} + L(\delta)\frac{\delta}{4M}.$$

By Gronwall's inequality,

$$d(g(t),e) \le d(g_0,e)e^{C_1(\delta)t} + \frac{C_2(\epsilon,\delta)}{C_1(\delta)}(e^{C_1(\delta)t} - 1). \tag{25}$$

For $t < 0$ we have the same estimate by changing t to $|t|$.

For $g_1, g_2 \in G$, let $g_1(t), g_2(t)$ be solutions of (24) with initial values g_1 and g_2. (We use $Q_{\epsilon,1}(g)$ to denote $Q_{\epsilon,1}(g,\gamma(g))$.) Before we go further, we do some more estimates. For (g_1,x_1), $(g_2,x_2) \in Y_{\delta/2}$, we have

$$\begin{aligned} &\|D\phi_0(g_1)\tilde{H}_1(g_1,x_1) - D\phi_0(g_2)\tilde{H}_1(g_2,x_2)\| \\ &\quad = \|D\phi_0(g_1)D\pi_1(g_1,x_1)D\phi^{-1}(g_1x_1)H(g_1x_1) \\ &\quad - D\phi_0(g_2)D\pi_1(g_2,x_2)D\phi^{-1}(g_2x_2)H(g_2x_2)\| \\ &\quad \le \|D\phi_0(g_1)D\pi_1(g_1,x_1)D\phi^{-1}(g_1x_1) \\ &\qquad - D\phi_0(g_2)D\pi_1(g_2,x_2)D\phi^{-1}(g_2x_2)\| \cdot \|H(g_1x_1)\| \\ &\quad + \|D\phi_0(g_2)D\pi_1(g_2,x_2)D\phi^{-1}(g_2x_2)\|\ \|H(g_1x_1) - H(g_2x_2)\| \\ &\quad \le \bar{M}(d(g_1,g_2) + \|x_1 - x_2\|), \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\|D\phi_0(g_1)\tilde{F}_1(g_1,x_1) - D\phi_0(g_2)\tilde{F}_1(g_2,x_2)\| \\ &\qquad \le \|D\phi_0(g_1)Dg_1(e)D\pi_1(e,x_1)D\phi^{-1}(x_1)f(x_1) \\ &\qquad\quad - D\phi_0(g_2)Dg_2(e)D\pi_1(e,x_2)D\phi^{-1}(x_2)f(x_2)\| \\ &\qquad \le \bar{M}\|x_1 - x_2\|L(\delta) + L(\delta)\|x_1\| \cdot \\ &\qquad\quad \|D\phi_1(g_1)Dg_1(e)D\pi_1(e,x_1)D\phi^{-1}(x_1) \\ &\qquad\quad - D\phi_0(g_2)Dg_0(e)D\pi_1(e,x_2)D\phi^{-1}(x_2)\| \\ &\qquad \le \|x_1 - x_2\|L(\delta) + \bar{M}L(\delta)\delta(d(g_1,g_2) + \|x_1 - x_2\|) \\ &\qquad \le L(\delta)\|x_1 - x_2\| + L(\delta)\delta d(g_1,g_2). \end{aligned} \tag{27}$$

So,

$$\begin{aligned}\|D\phi_0 Q_{\epsilon,1}(g_1,x_1) - D\phi_0 Q_{\epsilon,1}(g_2,x_2)\| &\le (\epsilon\bar{M} + L(\delta)\delta)d(g_1,g_2)\\ &\quad + (\epsilon\bar{M} + L(\delta))\|x_1 - x_2\|. \end{aligned} \tag{28}$$

For all $a > 0$ take a smooth curve $c(r)$ on $G, r \in [0,1]$, such that $c(0) = g_1, c(1) = g_2$ and $d(g_1,g_2) \ge \int_0^1 \|c'(r)\|dr - a$. Let $g(t,r)$ denote the solution of (24) with initial value $c(r)$. If γ and $Q_{\epsilon,1}$ are smooth, by (28), we have

$$\begin{aligned}\int_0^1 \left\|\frac{\partial}{\partial r} g(t,r)\right\| dr &\le \int_0^1 \|c'(r)\|dr + \int_0^1 \left\|\int_0^t \frac{\partial}{\partial s}\frac{\partial}{\partial r} g(s,r)ds\right\| dr\\ &\le \int_0^1 \|c'(r)\|dr + \int_0^1\int_0^t \left\|\frac{\partial}{\partial r} Q_{\epsilon,1}(g(s,r))\right\| dsdr\\ &\le \int_0^1 \|c'(r)\|dr + \int_0^t\int_0^1 \left\|DQ_{\epsilon,1}\left(\frac{\partial}{\partial r} g(s,r), D\gamma(\frac{\partial}{\partial r} g(s,r))\right)\right\| drds\\ &\le \int_0^1 \|c'(r)\|dr + \int_0^t\int_0^1 \left(\epsilon\bar{M} + L(\delta)\delta + (\epsilon\bar{M} + L(\delta))\frac{\delta}{4M}\right)\left\|\frac{\partial}{\partial r} g(s,r)\right\| drds.\end{aligned}$$

Let $C_3(\epsilon,\delta) = \epsilon\bar{M} + L(\delta)\delta + (\epsilon\bar{M} + L(\delta))\dfrac{\delta}{4M}$. So,

$$\int_0^1 \left\|\frac{\partial}{\partial r} g(t,r)\right\| dr \le \int_0^1 \|c'(r)\|dr + C_3(\epsilon,\delta)\int_0^t\int_0^1 \left\|\frac{\partial}{\partial r} g(s,r)\right\| drds,$$

which implies,

$$\int_0^1 \left\|\frac{\partial}{\partial r} g(t,r)\right\| dr \le e^{C_3(\epsilon,\delta)t}\int_0^1 \|c'(r)\|dr \tag{29}$$

Now not all of them are smooth but they are Lipschitz and that is enough. We still have the same estimate. Therefore,

$$d(g_1(t),g_2(t)) \le \int_0^1 \left\|\frac{\partial}{\partial r} g(t,r)\right\| dr \le e^{C_3(\epsilon,\delta)t}(d(g_1,g_2) + a),$$

and thus,

$$d(g_1(t),g_2(t)) \le e^{C_3(\epsilon,\delta)t} d(g_1,g_2). \tag{30}$$

We will define a map $E = (E^+,E^-)$ on $Z^+ \times Z^-$. For brevity, below we use $Q(g(s))$ to denote $Q(g(s),\gamma(g(s)))$ and define

$$E^+\gamma(g_0) = -\int_0^{+\infty} T_u(-s)Q^+_{\epsilon,2}(g(s))ds, \tag{31}$$

$$E^-\gamma(g_0) = \int_{-\infty}^0 T_s(-s)Q^-_{\epsilon,2}(g(s))ds, \tag{32}$$

where $g(s)$ is the solution of (24) with initial value g_0 and recall that T_u, T_s are the semiflows generated by A_u, A_s.

Lemma 3. *E maps $Z^+ \times Z^-$ to itself if*

$$\frac{\epsilon \bar{M} + L(\delta)\delta}{\alpha} < \frac{\delta}{8} \tag{C1}$$

and

$$\frac{C_4(\epsilon, \delta)}{\alpha - C_3(\epsilon, \delta)} \leq \frac{\delta}{8M} \tag{C2}$$

hold.

Proof. First we verify that E^+ and E^- are well-defined. Since $|\gamma| \leq \dfrac{\delta}{2}$ and by (20), $\|Q^{\pm}_{\epsilon,2}(g(s))\|$ is bounded, along with the condition on T_s, T_u, we see that E is well-defined. Second we prove $E\gamma \in Z^+ \times Z^-$ under proper conditions. By (20),

$$\|E^+\gamma(e)\| \leq \int_0^{+\infty} e^{-\alpha s}\left(\epsilon \bar{M} + L(\delta)\frac{\delta}{2}\right) ds \leq \frac{\epsilon \bar{M} + L(\delta)\frac{\delta}{2}}{\alpha}.$$

Similarly

$$\|E^-\gamma(e)\| \leq \int_{-\infty}^0 e^{\alpha s}\left(\epsilon \bar{M} + L(\delta)\frac{\delta}{2}\right) ds \leq \frac{\epsilon \bar{M} + L(\delta)\frac{\delta}{2}}{\alpha}.$$

So, if condition (C1) holds, then $\|E^+\gamma(e)\| \leq \dfrac{\delta}{8}$ and $\|E^-\gamma(e)\| \leq \frac{\delta}{8}$. For $g_1, g_2 \in G$, let $g_1(t), g_2(t)$ denote the solution of (24) with initial data g_1, g_2, respectively. Then by (23),

$$\begin{aligned}
&\|E^+\gamma(g_1) - E^+\gamma(g_2)\| \leq \int_0^{+\infty} e^{-\alpha s}\|Q^+_{\epsilon,2}(g_1(s)) - Q^+_{\epsilon,2}(g_2(s))\| ds \\
&\leq \int_0^{+\infty} e^{-\alpha s}(\epsilon \bar{M}(d(g_1(s), g_2(s)) + (\epsilon \bar{M} + L(\delta))\frac{\delta}{4M} d(g_1(s), g_2(s)))ds \\
&= \int_0^{+\infty} e^{-\alpha s}\left(\epsilon \bar{M} + \epsilon\delta \frac{\bar{M}}{4M} + \frac{L(\delta)\delta}{4M}\right) d(g_1(s), g_2(s)) ds.
\end{aligned}$$

Let $C_4(\epsilon, \delta) = \epsilon \bar{M} + \epsilon\delta\, \dfrac{\bar{M}}{4M} + \dfrac{L(\delta)\delta}{4M}$. By (30)

$$\begin{aligned}
&\|E^+\gamma(g_1) - E^+\gamma(g_2)\| \\
&\qquad \leq \int_0^{+\infty} C_4(\epsilon, \delta) e^{-(\alpha - C_3(\epsilon,\delta))s} d(g_1, g_2) ds \\
&\qquad = \frac{C_4(\epsilon, \delta)}{\alpha - C_3(\epsilon, 2)} d(g_1, g_2).
\end{aligned} \tag{33}$$

The same is true for $\|E^-\gamma(g_1) - E^-\gamma(g_2)\|$. Therefore, if condition (C2) holds then $E\gamma$ satisfies the condition on the C_0 norm and Lipschitz constant and $E\gamma \in Z^+ \times Z^-$.

Finally, we prove

Lemma 4. *E is a contraction if conditions (C1), (C2), and*

$$\epsilon\bar{M} + L(\delta)\left(\frac{1}{\alpha} + \frac{C_4(\epsilon,\delta)}{C_3(\epsilon,\delta)}\,\frac{1}{\alpha - C_3(\epsilon,\delta)}\right) < \frac{1}{2} \tag{C3}$$

hold.

Proof. For $\gamma_1, \gamma_2 \in Z^+ \times Z^-$ let $g_1(t), g_2(t)$ be the solutions of (24), with initial value g_0, and with γ replaced by γ_1, γ_2, respectively. Then by (23),

$$\begin{aligned}
\|E^+\gamma_1(g_0) - E^+\gamma_2(g_0)\| &\le \int_0^{+\infty} e^{-\alpha s}\|Q^+_{\epsilon,2}(g_1(s),\gamma_1(g_1(s))) \\
&\quad - Q^+_{\epsilon,2}(g_2(s),\gamma_2(g_2(s)))\|ds \\
&\le \int_0^{+\infty} e^{-\alpha s}(\epsilon\bar{M}d(g_1(s),g_2(s)) + (\epsilon\bar{M} + L(\delta))(\|\gamma_1(g_1(s)) - \gamma_1(g_2(s))\| \\
&\quad + \|\gamma_1(g_2(s)) - \gamma_2(g_2(s))\|))ds \\
&\le \int_0^{+\infty} e^{-\alpha s}((\epsilon\bar{M} + (\epsilon\bar{M} + L(\delta))\frac{\delta}{4M})\, d(g_1(s),g_2(s)) \\
&\quad + (\epsilon\bar{M} + L(\delta))|\gamma_1 - \gamma_2|)ds.
\end{aligned} \tag{34}$$

Let $\gamma_{(r)} = (2-r)\gamma_1 + (r-1)\gamma_2,\ r \in [1,2]$, which is a homotopy between γ_1 and γ_2 and $\gamma_{(1)} = \gamma_1$, and $\gamma_{(2)} = \gamma_2$. Let $g(t,r)$ denote the solution of (24) with $\gamma = \gamma_{(r)}$ and initial data $g(0,r) = g_0$. Similar to the derivation of (30) we find

$$\begin{aligned}
\int_1^2 \left\|\frac{\partial}{\partial r}g(t,r)\right\| dr &\le \int_1^2\int_0^t \left\|\frac{\partial}{\partial r}\frac{\partial}{\partial s}g(s,r)\right\| dsdr \\
&\le \int_0^t\int_1^2 \left\|\frac{\partial}{\partial r}Q_{\epsilon,1}(g(s,r),\gamma_{(r)}(g(s,r)))\right\| drds \\
&\le \int_0^t\int_1^2 \Big\|DQ_{\epsilon,1}\Big(\frac{\partial}{\partial r}g(s,r), \frac{\partial}{\partial r}((2-r)\gamma_1(g(s,r)) \\
&\qquad + (r-1)\gamma_2(g(s,r)))\Big)\Big\| drds \\
&\le \int_0^t\int_1^2 (\epsilon\bar{M} + L(\delta)\delta)\left\|\frac{\partial}{\partial r}g(s,r)\right\| + (\epsilon\bar{M} + L(\delta)) \\
&\qquad \left(\frac{\delta}{4M}\left\|\frac{\partial}{\partial r}g(s,r)\right\| + \|\gamma_1(g(s,r)) - \gamma_2(g(s,r))\|\right) drds \\
&\le \int_0^t\int_1^2 (\epsilon\bar{M} + L(\delta))|\gamma_1 - \gamma_2| + C_3(\epsilon,\delta)\left\|\frac{\partial}{\partial r}g(s,r)\right\| drds \\
&\le (\epsilon\bar{M} + L(\delta))|\gamma_1 - \gamma_2|t + \int_0^t\int_1^2 C_3(\epsilon,\delta)\left\|\frac{\partial}{\partial r}g(s,r)\right\| drds.
\end{aligned}$$

Therefore,

$$\int_1^2 \left\|\frac{\partial}{\partial r}g(t,r)\right\| dr \le \frac{(\epsilon\bar{M} + L(\delta))e^{C_3(\epsilon,\delta)t}}{C_3(\epsilon,\delta)}\,|\gamma_1 - \gamma_2|. \tag{35}$$

Returning to (34),

$$\begin{aligned}\|E^+\gamma_1(g_0)-E^+\gamma_2(g_0)\| &\le \int_0^{+\infty} e^{-\alpha s}(C_4(\epsilon,\delta)(\epsilon\bar{M}+L(\delta))\frac{e^{C_3(\epsilon,\delta)s}}{C_3(\epsilon,\delta)} \\ &\quad +(\epsilon\bar{M}+L(\delta)))|\gamma_1-\gamma_2|ds \\ &\le (\epsilon\bar{M}+L(\delta))|\gamma_1-\gamma_2|\left(\frac{1}{\alpha}+\frac{C_4(\epsilon,\delta)}{C_3(\epsilon,\delta)}\frac{1}{\alpha-C_3(\epsilon,\delta)}\right). \end{aligned} \tag{36}$$

Similarly

$$\begin{aligned}&\|E^-\gamma_1(g_0)-E^-\gamma_2(g_0)\| \\ &\quad\le (\epsilon\bar{M}+L(\delta))|\gamma_1-\gamma_2|\left(\frac{1}{\alpha}+\frac{C_4(\epsilon,\delta)}{C_3(\epsilon,\delta)}\frac{1}{\alpha-C_3(\epsilon,\delta)}\right). \end{aligned} \tag{37}$$

Therefore, E is a contraction if condition (C3) holds.

Therefore there exists a unique fixed point $\gamma_0\in Z^+\times Z^-$. Next, we prove

Lemma 5. *$\{(g,\gamma_0(g))|\, g\in G\}$ is an invariant set of system (15), (17), (18).*

Proof. For $g_0\in G$, let $g(t)$ be the solution of (24) with$\gamma=\gamma_0$. Writing $\gamma_0(g)=(\gamma_0^+(g)+\gamma_0^-(g))\in X_u\oplus X_s$, we have

$$\gamma_0^+(g_0)=-\int_0^{+\infty} T_u(-s)Q^+_{\epsilon,2}(g(s))ds,$$

which implies, for $t_0>0$,

$$\begin{aligned}T_u(t_0)\gamma_0^+(g_0) &= -\int_0^{+\infty} T_u(t_0-s)Q^+_{\epsilon,2}(g(s))ds \\ &= -\int_0^{t_0} T_u(t_0-s)Q^+_{\epsilon,2}(g(s))ds-\int_0^{+\infty} T_u(-s)Q^+_{\epsilon,2}(g(s+t_0))ds \\ &= -\int_0^{t_0} T_u(t_0-s)Q^+_{\epsilon,2}(g(s))ds+\gamma_0^+(g(t_0)).\end{aligned}$$

In the same way, we have

$$\gamma_0^-(g_0)=T_s(t_0)\gamma_0^-(g_0)+\int_0^{t_0} T_s(t_0-s)Q^-_{\epsilon,2}(g(s))ds.$$

Therefore, $(g(t),\gamma_0(g(t)))$ is a solution of that system, which implies that $(g,\gamma_0(g))$ is an invariant manifold.

Finally, we consider the condition C1, C2, C3.

$$\text{(C1)}\ :\ \frac{\epsilon\bar{M}+L(\delta)\delta}{\alpha}<\frac{\delta}{8},$$

$$\text{(C2)}\ :\ \frac{\delta}{8M}\ge\frac{C_4(\epsilon,\delta)}{\alpha-C_3(\epsilon,\delta)}=\frac{\epsilon\bar{M}+\epsilon\delta\frac{\bar{M}}{4M}+\frac{L(\delta)\delta}{4M}}{\alpha-\left(\epsilon\bar{M}+L(\delta)\delta+\left(\epsilon\bar{M}+L(\delta)\right)\frac{\delta}{4M}\right)},$$

$$\begin{aligned}\text{(C3)}\ :\ \frac{1}{\alpha} &> \epsilon\bar{M}+L(\delta)\left(\frac{1}{\alpha}+\frac{C_4(\epsilon,\delta)}{C_3(\epsilon,\delta)}\ \frac{1}{\alpha-C_3(\epsilon,\delta)}\right) \\ &= \epsilon\bar{M}+L(\delta)\left(\frac{1}{\alpha}+\frac{\epsilon\bar{M}+\epsilon\delta\frac{\bar{M}}{4M}+\frac{L(\delta)\delta}{4M}}{\epsilon\bar{M}+L(\delta)\delta+\epsilon\delta\frac{\bar{M}}{4M}+\frac{L(\delta)\delta}{4M}}\ \frac{1}{\alpha-C_3(\epsilon,\delta)}\right).\end{aligned}$$

Note that $L(\delta) \to 0$ as $\delta \to 0$. It is easy to see that all these conditions are satisfied if $\epsilon \ll \delta \ll 1$. In $Y_{\delta/2}$, the equation is equivalent to (3) in X, therefore, when $\epsilon \ll \delta \ll 1$ there is an invariant manifold near $\phi_0(G)$, which is given by $\{g\gamma_0(g) \,|\, g \in G\}$. ■

Acknowledgment. I would like to thank the referee for carefully reading the manuscript and making very useful suggestions.

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