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Multiplicity of solutions for quasilinear elliptic boundary-value problems *

Idris Addou

Abstract

This paper is concerned with the existence of multiple solutions to the boundary-value problem

$$-(\varphi_p(u'))' = \lambda\varphi_q(u) + f(u) \quad \text{in } (0,1)\,, \quad u(0) = u(1) = 0\,,$$

where $p, q > 1$, $\varphi_x(y) = |y|^{x-2}y$, λ is a real parameter, and f is a function which may be sublinear, superlinear, or asymmetric. We use the time map method for showing the existence of solutions.

1 Introduction

We study the existence and multiplicity of solutions for the boundary-value problem

$$\begin{gathered} -(\varphi_p(u'))' = \lambda\varphi_q(u) + f(u) \quad \text{in } (0,1) \\ u(0) = u(1) = 0\,, \end{gathered} \tag{1}$$

where $p, q > 1$, $\varphi_x(y) = |y|^{x-2}y$, $\lambda \in \mathbb{R}$, and f is a continuous function such that

$$a_\pm = \lim_{s\to\pm\infty} f(s)/\varphi_q(s) \quad \text{and} \quad a_0 = \lim_{s\to 0} f(s)/\varphi_q(s)$$

exist as real numbers. Also, we assume that $m_\pm := \inf_{\pm s\geq 0} f(s)/\varphi_q(s)$ exists in $\mathbb{R}$, and define

$$m := \inf_{u\in\mathbb{R}} f(s)/\varphi_q(s) = \min(m_+, m_-) \quad \text{and} \quad m_k^\pm = \begin{cases} m_\pm & \text{if } k = 1 \\ m & \text{if } k \geq 2. \end{cases}$$

We shall consider the following three cases: Superlinear case, $a_0 < \min(a_-, a_+)$; Sublinear case, $a_0 > \max(a_-, a_+)$; and Asymmetric case, $a_- < a_0 < a_+$.

In the special case when $p = 2$, several authors have been interested in Problem (1), including higher dimensions under various assumptions. See, for instance, Amann and Zehnder [5], Castro and Lazer [18], Hess [41], Struwe [60],

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and particularly Esteban [29]. For the general case $p > 1$, we mention a recent paper by J. Wang [73] where positive solutions are studied.

Now we define some sets that will be used in the statement of the main results. For $k \geq 1$, let

$$S_k^+ = \left\{ \begin{array}{c} u \in C^1([0,1]) : u \text{ admits exactly } (k-1) \text{ zeros in } (0,1), \\ \text{all are simple, } u(0) = u(1) = 0 \text{ and } u'(0) > 0 \end{array} \right\},$$

then $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. If $u \in C([0,1])$ is a real-valued function vanishing at x_1 and x_2 and not between them, (with $x_1 < x_2$) we call its restriction to the open interval (x_1, x_2) a hump of u. So, each function in S_k^+ has exactly k humps such that the first one is positive, the second is negative, and so on with alternations.

Let A_k^+ ($k \geq 1$) be the subset of S_k^+ consisting of the functions u satisfying:

- Every hump of u is symmetrical about the center of the interval of its definition.
- Every positive (resp. negative) hump of u can be obtained by translating the first positive (resp. negative) hump.
- The derivative of each hump of u vanishes once and only once.

Let $A_k^- = -A_k^+$ and $A_k = A_k^+ \cup A_k^-$. Denote by $(\lambda_k)_{k\geq 1}$ the eigenvalues of the one dimensional p-Laplacian operator with Dirichlet boundary conditions,

$$\begin{gathered} -(\varphi_p(u'))' = \lambda \varphi_p(u) \quad \text{in } (0,1) \\ u(0) = u(1) = 0\,. \end{gathered} \tag{2}$$

Then for each integer $k \geq 1$ and $p > 1$, $\lambda_k = k^p \lambda_1$ and

$$\lambda_1 = (p-1)(2\int_0^1 (1-t^p)^{-1/p} dt)^p = (p-1)(\frac{2\pi}{p\sin(\pi/p)})^p.$$

For fixed real constants a_- and a_+, consider the boundary value problem

$$\begin{gathered} -(\varphi_p(u'))' = \lambda \varphi_p(u) + a_+ \varphi_p(u^+) - a_- \varphi_p(u^-), \text{in } (0,1) \\ u(0) = u(1) = 0\,, \end{gathered} \tag{3}$$

where λ is a real parameter. If λ is such that problem (3) admits a nontrivial solution u_λ, then λ is called a half-eigenvalue of (3). In the particular case where $p = 2$, this definition goes back to Berestycki [11]. For any integer $k \geq 1$, let

$$b_k^\pm = \begin{cases} a_\pm & \text{if } k = 1 \\ \min(a_-, a_+) & \text{if } k \geq 2 \end{cases} \quad \text{and} \quad c_k^\pm = \begin{cases} a_\pm & \text{if } k = 1 \\ \max(a_-, a_+) & \text{if } k \geq 2\,. \end{cases}$$

Proposition 1 *For fixed real constants a_- and a_+, the set of half-eigenvalues of problem (3) consists of two increasing sequences $(\lambda_k^+)_{k\geq 1}$ and $(\lambda_k^-)_{k\geq 1}$, satisfying $h_k^\pm(\lambda_k^\pm)=1$, for all $k\geq 1$, where*

$$h_{2n}^\pm(\lambda):=\frac{\lambda_n^{1/p}}{(a_\pm+\lambda)^{1/p}}+\frac{\lambda_n^{1/p}}{(a_\mp+\lambda)^{1/p}},\ \textit{for all } \lambda>-b_{2n}^\pm,\ n\geq 1,$$
$$h_{2n+1}^\pm(\lambda):=\frac{\lambda_{n+1}^{1/p}}{(a_\pm+\lambda)^{1/p}}+\frac{\lambda_n^{1/p}}{(a_\mp+\lambda)^{1/p}},\ \textit{for all } \lambda>-b_{2n+1}^\pm,\ n\geq 0\,,$$

with the convention $\lambda_0=0$. Moreover, if $a_\mp<a_\pm$ then

$$\lambda_{2n-1}^\pm<\lambda_{2n-1}^\mp<\lambda_{2n}^\pm<\lambda_{2n+1}^\pm<\lambda_{2n+1}^\mp, \forall n\geq 1\,,$$

and

$$\lambda_k-a_\pm<\lambda_k^\pm<\lambda_k-a_\mp,\quad \forall k\geq 1.$$

If $a_-=a_+$ then $\lambda_k^\pm=\lambda_k-a_\pm$, for all $k\geq 1$.

The first result reads as follows.

Theorem 2 *Assume that $q=p$. For each integer $k\geq 1$,*

1. *If $a_0<b_k^\pm$ and $\max(-m_k^\pm,\lambda_k^\pm)<\lambda_k-a_0$, problem (1) admits at least a solution in $A_k^\pm$ for all λ satisfying $\max(-m_k^\pm,\lambda_k^\pm)<\lambda<\lambda_k-a_0$.*
2. *If $a_0>c_k^\pm$ and $\max(-m_k^\pm,\lambda_k-a_0)<\lambda_k^\pm$, problem (1) admits at least a solution in $A_k^\pm$ for all λ satisfying $\max(-m_k^\pm,\lambda_k-a_0)<\lambda<\lambda_k^\pm$.*

The next result deals with the asymmetric case. For any integer $n\geq 1$, let

$$a_{2n+1}^\pm=\frac{na_\mp+(n+1)a_\pm}{2n+1}\quad\text{and}\quad a_{2n}^\pm=\frac{na_-+na_+}{2n}=\frac{a_-+a_+}{2}\,.$$

Theorem 3 *Assume that $q=p$ and $a_-<a_0<a_+$. Then*

1. *If $\max(-m_+,\lambda_1-a_+)<\lambda_1-a_0$, problem (1) admits at least a solution in A_1^+ for all λ satisfying $\max(-m_+,\lambda_1-a_+)<\lambda<\lambda_1-a_0$.*
2. *If $\max(-m_-,\lambda_1-a_0)<\lambda_1-a_-$, problem (1) admits at least a solution in A_1^- for all λ satisfying $\max(-m_-,\lambda_1-a_0)<\lambda<\lambda_1-a_-$.*
3. *For each integer $k\geq 2$, there exists $\tilde a_k^\pm: a_-<\tilde a_k^\pm<a_k^\pm<a_+$ such that,*

   (i) *if $a_-<a_0<\tilde a_k^\pm$ and $\max(-m,\lambda_k^\pm)<\lambda_k-a_0$, problem (1) admits at least a solution in $A_k^\pm$ for all λ satisfying $\max(-m,\lambda_k^\pm)<\lambda<\lambda_k-a_0$,*

   (ii) *if $\tilde a_k^\pm<a_0<a_+$ and $\max(-m,\lambda_k-a_0)<\lambda_k^\pm$, problem (1) admits at least a solution in $A_k^\pm$ for all λ satisfying $\max(-m,\lambda_k-a_0)<\lambda<\lambda_k^\pm$.*

Theorem 4 *Assume that $q \neq p$. Then for each integer $k \geq 1$, problem (1) admits at least a solution in $A_k^{\pm}$ for all $\lambda > -m_k^{\pm}$. In particular, (1) admits infinitely many solutions for all $\lambda > -m$.*

The paper is organized as follows. The next section is dedicated to the proof of Proposition 1. In Section 3, we present the method used for proving the main results of this paper. In Section 4, we present some preliminary lemmas. In Section 5, we prove the main results, present some remarks. The paper ends with an appendix which contains a brief historical overview on time maps.

2 Proof of Proposition 1

Notice that for all $u \in \mathbb{R}, \varphi_p(u) = \varphi_p(u^+) - \varphi_p(u^-)$; so, problem (3) may be written as

$$\begin{gathered}-(\varphi_p(u'))' = \alpha_+\varphi_p(u^+) - \alpha_-\varphi_p(u^-), \quad \text{in } (0,1)\,, \\ u(0) = u(1) = 0\,,\end{gathered} \tag{4}$$

with $\alpha_\pm = \lambda + a_\pm$. The set of $(\alpha_+, \alpha_-) \in \mathbb{R}^2$ such that problem (4) has a nontrivial solution is known as the Fučik spectrum for the operator $(\varphi_p(u'))'$ under Dirichlet boundary conditions. See for instance, Drábek [28], Boccardo et al. [12], Huang and Metzen [42]. We denote this spectrum by $\sum_p$.

One has $\sum_p = \bigcup_{k\geq 0} C_k^{\pm}$, where

$$\begin{aligned} C_0^{\pm} &= \left\{(\alpha_+,\alpha_-) \in \mathbb{R}^2 : \alpha_\pm = \lambda_1\right\}, \\ C_{2n}^{\pm} &= \left\{(\alpha_+,\alpha_-) \in (\mathbb{R}^+)^2 : (\lambda_n^{1/p}/\alpha_\pm^{1/p}) + (\lambda_n^{1/p}/\alpha_\mp^{1/p}) = 1\right\}, n \geq 1, \\ C_{2n+1}^{\pm} &= \left\{(\alpha_+,\alpha_-) \in (\mathbb{R}^+)^2 : (\lambda_{n+1}^{1/p}/\alpha_\pm^{1/p}) + (\lambda_n^{1/p}/\alpha_\mp^{1/p}) = 1\right\}, n \geq 0\,. \end{aligned}$$

So, λ is a half-eigenvalue of (3) if and only if $(\lambda + a_+, \lambda + a_-) \in \sum_p$, that is, $h_k^{\pm}(\lambda) = 1$, for some $k \geq 1$. Since the function $\lambda \mapsto h_k^{\pm}(\lambda), k \geq 1$, is strictly decreasing on $(-b_k^{\pm}, +\infty)$ and its limits at $-b_k^{\pm}$ and $+\infty$ are $+\infty$ and 0 respectively, it follows that the equation $h_k^{\pm}(\lambda) = 1$ admits a unique solution $\lambda = \lambda_k^{\pm}$.

It is easy to check that for $a_- < a_+$ and for all $\lambda > -a_-$ one has

$$h_{2n-1}^{+}(\lambda) < h_{2n-1}^{-}(\lambda) < h_{2n}^{\pm}(\lambda) < h_{2n+1}^{+}(\lambda) < h_{2n+1}^{-}(\lambda), \quad \forall n \geq 1.$$

Due to the fact that the function $\lambda \mapsto h_k^{\pm}(\lambda)$, $k \geq 1$, is strictly decreasing on $(-a_-, +\infty)$, it follows that

$$\lambda_{2n-1}^{+} < \lambda_{2n-1}^{-} < \lambda_{2n}^{\pm} < \lambda_{2n+1}^{+} < \lambda_{2n+1}^{-}, \forall n \geq 1\,.$$

On the other hand, if $a_- < a_+$, it follows that for all $\lambda > -a_-$, $k \geq 1$,

$$\frac{\lambda_k^{1/p}}{(a_+ + \lambda)^{1/p}} < \frac{\lambda_k^{1/p}}{(a_- + \lambda)^{1/p}}, \quad \text{so} \quad \frac{\lambda_{2k}^{1/p}}{(a_+ + \lambda)^{1/p}} < h_{2k}^{\pm}(\lambda) < \frac{\lambda_{2k}^{1/p}}{(a_- + \lambda)^{1/p}},$$

and then $\lambda_{2k} - a_+ < \lambda_{2k}^{\pm} < \lambda_{2k} - a_-$. The other cases may be handled the same way.

If $a_- = a_+$, problem (3) may be written as

$$-(\varphi_p(u'))' = \mu\varphi_p(u) \quad \text{in } (0,1)\,, \quad u(0) = u(1) = 0, \tag{5}$$

with $\mu = \lambda + a_\pm$. So, λ is a half-eigenvalue of (3) if and only if $\lambda + a_\pm$ is an eigenvalue of (2), that is, if and only if $\lambda = \lambda_k^{\pm} = \lambda_k - a_\pm$, which ends the proof of Proposition 1. $\diamondsuit$

3 The time map method

To prove the main results, we use the time mapping approach as used in [2], [3], [4]. To keep this paper self-contained, we describe it here.

Denote by g a nonlinearity and by p a real parameter, and we assume,

$$g \in C(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad 1 < p < +\infty\,. \tag{6}$$

Consider the boundary value problem

$$-(\varphi_p(u'))' = g(u) \quad \text{in } (0,1), u(0) = u(1) = 0, \tag{7}$$

where $\varphi_p(x) = |x|^{p-2}x$, $x \in \mathbb{R}$. Denote by $p' = p/(p-1)$ the conjugate exponent of p. Let $G(s) = \int_0^s g(t)dt$. For $E > 0$ and $\kappa = +, -$, let

$$X_\kappa(E) = \{s \in \mathbb{R} : \kappa s \geq 0 \quad \text{and} \quad E^p - p'G(\xi) > 0, \forall\, \xi, 0 \leq \kappa\xi < \kappa s\}$$

and

$$r_\kappa(E) = \begin{cases} 0 & \text{if } X_\kappa(E) = \emptyset \\ \kappa \sup(\kappa X_\kappa(E)) & \text{otherwise.} \end{cases}$$

Note that r_κ may be infinite. We shall also make use of the following sets,

$$D_\kappa = \{E > 0 : 0 < |r_\kappa(E)| < +\infty \text{ and } \kappa g(r_\kappa(E)) > 0\}$$

and $D = D_+ \cap D_-$. Also, let $D_k^\kappa := D$ if $k \geq 2$, and $D_1^\kappa := D_\kappa$. Define the following time-maps,

$$\begin{aligned} T_\kappa(E) &= \kappa \textstyle\int_0^{r_\kappa(E)} (E^p - p'G(t))^{-1/p} dt, \quad E \in D_\kappa\,, \\ T_{2n}^\kappa(E) &= n(T_+(E) + T_-(E)), \quad n \in \mathbb{N},\ E \in D\,, \\ T_{2n+1}^\kappa(E) &= T_{2n}^\kappa(E) + T_\kappa(E), \quad n \in \mathbb{N},\ E \in D. \end{aligned}$$

Theorem 5 (Quadrature method) *Assume that (6) holds. Let $E > 0$, $k \in \mathbb{N}^*$, $\kappa = +, -$. If $E \in D_k^\kappa$ and $T_k^\kappa(E) = 1/2$, problem (7) admits at least a solution $u_k^\kappa \in A_k^\kappa$ satisfying $(u_k^\kappa)'(0) = \kappa E$, and this solution is unique.*

This theorem is well-known, but we did not find a convenient reference to the precise statement used later. The paper by Guedda and Veron [40], seems to be the first one dealing with time maps approach when the differential operator is the one dimensional p-Laplacian. An easy adaptation of the ideas contained in [40] and in the paper by Del Pino and Manasevich [23], allows one to prove Theorem 5. Also, we mention the papers by Manasevich and Zanolin [44] and Manasevich et al. [45], where one can find time maps used when the differential operator is the one dimensional p-Laplacian.

Notice that time maps were also used when the differential operator generalizes the p-Laplacian; see for example Garcia-Huidobro et al. [33], [34], [35], [36], [37], Garcia-Huidobro and Ubilla [38], Huang and Metzen [42] and Ubilla [61]. A brief historical overview of time maps is given in the appendix.

For the sake of completeness, we dedicate the rest of this section to the proof of Theorem 5 for the case where $k = 1$ and $\kappa = +$. The adaptation of the other cases may be handled similarly.

We shall say that u is a solution of (7) if u and $\varphi_p(u')$ belong to $C^1([0,1])$ and u satisfies

$$-(\varphi_p(u'(x)))' = g(u(x)) \ \forall x \in (0,1)\,, \quad \text{with} \quad u(0) = u(1) = 0\,.$$

Let us assign to each function $u \in C^1([0,1])$ the set

$$Z(u') = \{x \in [0,1] : u'(x) = 0\}\,.$$

Lemma 6 *Assume that (6) holds and u is a solution of (7). Then $u \in C^2([0,1])$ if $1 < p \le 2$, and $u \in C^2([0,1]\backslash Z(u'))$ if $p > 2$.*

Proof. The identity $t = \varphi_{p'} o \varphi_p(t)$ for all $t \in \mathbb{R}$ implies that

$$u'(x) = \varphi_{p'} o(\varphi_p(u'))(x) \ \text{ for all } x \in [0,1].$$

Thus, if $1 < p \le 2$, it follows that $p' \ge 2$ and therefore $\varphi_{p'} \in C^1(\mathbb{R})$. By $\varphi_p(u') \in C^1([0,1])$, (since u is a solution of (7)), it follows that $u' \in C^1([0,1])$, that is $u \in C^2([0,1])$. If $p > 2$, it follows that $1 < p' < 2$ and therefore $\varphi_{p'} \in C^1(\mathbb{R}^*)$ and $\varphi'_{p'}(0) = +\infty$. Hence, u' is C^1 at each point x where $\varphi_p(u'(x)) \ne 0$. But $\varphi_p(u'(x)) = 0$ if and only if $u'(x) = 0$, that is, if and only if $x \in Z(u')$. Therefore, Lemma 6 is proved. ♢

By Lemma 6, if u is a solution of problem (7), then

$$-(p-1)|u'(x)|^{p-2}u''(x) = g(u(x)), \text{ for all } x \in [0,1]\backslash Z(u').$$

Multiplying both sides by $u'(x)$ one obtains for the left hand side

$$-(p-1)|u'(x)|^{p-2}u''(x)u'(x) = -(1/p')(|u'|^p)'(x),$$

for all $x \in [0,1]\backslash Z(u')$, and for the right hand side

$$g(u(x))u'(x) = (G(u(x)))', \ \text{ for all } x \in [0,1]\backslash Z(u').$$

Thus,

$$(|u'|^p(x) + p'G(u(x)))' = 0, \tag{8}$$

for all $x \in [0,1]\backslash Z(u')$. Let us prove that (8) holds even for $x \in Z(u')$.

Lemma 7 (Energy relation) *Let $p > 1$ and assume that u is a solution of problem (7). Then*

$$(|u'|^p(x) + p'G(u(x)))' = 0, \quad \textit{for all } x \in [0,1].$$

Proof. Let $x_0 \in Z(u')$. One has

$$(G(u))'(x_0) = g(u(x_0))u'(x_0) = 0\,.$$

Let us prove that

$$(|u'|^p)'(x_0) = 0\,. \tag{9}$$

One has for all $x \neq x_0$,

$$\frac{|u'(x)|^p - |u'(x_0)|^p}{x - x_0} = u'(x) \times \frac{\varphi_p(u'(x)) - \varphi_p(u'(x_0))}{x - x_0}.$$

Thus, $\lim_{x\to x_0} u'(x) = u'(x_0) = 0$, and

$$\lim_{x\to x_0} \frac{\varphi_p(u'(x)) - \varphi_p(u'(x_0))}{x - x_0} = (\varphi_p(u'))'(x_0) = -g(u(x_0)) \in \mathbb{R}.$$

Therefore, (9) is proved. Regarding (8), Lemma 7 follows. $\diamondsuit$

Remark. In Lemmas 6 and 7, the solutions, u, are arbitrary and are not necessarily in A_1^+.

Now, assume that u is a solution of problem (7) belonging to A_1^+. Thus,

$$u' > 0 \text{ in } [0,(1/2)) \;\text{ and }\; u'(1/2). \tag{10}$$

It follows that

$$\sup\{x \in [0,1) : u'(t) > 0, \forall t \in [0,x)\} = 1/2,$$

and by the energy relation one gets

$$u'(x) = \{(u'(0))^p - p'G(u(x))\}^{1/p}, \text{ for all } x \in [0,1]. \tag{11}$$

Thus,

$$\sup\{x \in [0,1) : (u'(0))^p - p'G(u(t)) > 0, \forall t \in [0,x)\} = 1/2,$$

or equivalently

$$\sup\{x \in [0,1) : (u'(0))^p - p'G(z) > 0, \forall z \in [0,u(x))\} = 1/2,$$

which implies that

$$\sup\{s \geq 0 : (u'(0))^p - p'G(\xi) > 0, \forall \xi \in [0,s)\} = u(1/2).$$

Also, by (11) it follows that

$$x = \int_0^{u(x)} \{(u'(0))^p - p'G(\xi)\}^{-1/p}\, d\xi, \quad \text{for all } x \in [0,(1/2)]. \tag{12}$$

Thus, the improper integral in (12) is convergent for all $x \in [0,(1/2)]$ and in particular, $u'(0)$ is such that the improper integral

$$\int_0^{r_+(u'(0))} \{(u'(0))^p - p'G(\xi)\}^{-1/p}\, d\xi$$

converges and is equal to $(1/2)$, where for all $E > 0$

$$r_+(E) = \sup X_+(E) \text{ if } X_+(E) \neq \emptyset \quad \text{and} \quad r_+(E) = 0 \text{ if } X_+(E) = \emptyset$$

and

$$X_+(E) = \{s \geq 0 : E^p - p'G(\xi) > 0, \forall \xi \in [0,s)\}\,.$$

It follows that if u is a solution of problem (7) belonging to A_1^+, then there exists $E_* \in \tilde{D}_+$ with

$$\tilde{D}_+ = \Big\{E > 0 : 0 < r_+(E) < +\infty \text{ and } \int_0^{r_+(E)} \{E^p - p'G(\xi)\}^{-1/p}\, d\xi < +\infty\Big\}\,,$$

such that

$$u'(0) = E_*\,, \quad u(1/2) = r_+(E_*)\,, \quad \text{and} \quad T_+(E_*) = 1/2\,,$$

where

$$T_+(E) = \int_0^{r_+(E)} \{E^p - p'G(\xi)\}^{-1/p}\, d\xi, \quad \text{for all } E \in \tilde{D}_+.$$

Conversely, it is possible to assign to each root E_* of the equation $T_+(E) = 1/2$ in the variable $E \in \tilde{D}_+$ a unique solution u of the problem (7) belonging to A_1^+ and satisfying $u'(0) = E_*$, $\max_{[0,1]} u = u(1/2) = r_+(E_*)$.

In fact, if $E_* \in \tilde{D}_+$ is such that $T_+(E_*) = 1/2$, define the function h_+ on $[0, r_+(E_*)]$ by $h_+(u) = \int_0^u \{E_*^p - p'G(\xi)\}^{-1/p}\, d\xi$. Notice that $h_+(r_+(E_*)) = T_+(E_*) = 1/2$ and

$$0 \leq h_+(u) \leq T_+(E_*), \text{ for all } u \in [0, r_+(E_*)].$$

Thus, h_+ is well defined on $[0, r_+(E_*)]$. Moreover, it is an increasing diffeomorphism from $(0, r_+(E_*))$ onto $(0, T_+(E_*))$,

$$h_+'(u) = \{E_*^p - p'G(u)\}^{-1/p} > 0 \text{ for all } u \in (0, r_+(E_*)).$$

Let u_+ be the inverse of h_+ defined by

$$u_+(x) = h_+^{-1}(x) \in [0, r_+(E_*)], \text{ for all } x \in [0, (1/2)]\,,$$

and let u be defined on $[0, 1]$ by

$$u(x) = \begin{cases} u_+(x) \text{ if } x \in [0, (1/2)] \\ u_+(1-x) \text{ if } x \in [(1/2), 1]. \end{cases}$$

It easy to show that this function u is a solution of problem (7) belonging to A_1^+ and satisfying $u'(0) = E_*$, $\max_{[0,1]} u = u(1/2) = r_+(E_*)$. Let us prove its uniqueness. Assume that v is also a solution of problem (7) belonging to A_1^+ and satisfies

$$v'(0) = E_*, \max_{[0,1]} v = v(1/2) = v_+(E_*).$$

By (12) it follows that

$$x = \int_0^{u(x)} \{E_*^p - p'G(\xi)\}^{-1/p}\, d\xi = \int_0^{v(x)} \{E_*^p - p'G(\xi)\}^{-1/p}\, d\xi,$$

for all $x \in [0, 1/2]$. Thus,

$$\int_{u(x)}^{v(x)} \{E_*^p - p'G(\xi)\}^{-1/p}\, d\xi = 0, \quad \text{for all } x \in [0, 1/2].$$

Thus, $u = v$ on $[0, 1/2]$, and by symmetry it follows that $u = v$ on $[0, 1]$. Therefore, because $D_+ \subset \tilde{D}_+$, Theorem 5 is proved for the case $k = 1$ and $\kappa = +$. $\diamondsuit$

4 Preliminary Lemmas

In order to define the time maps we need the following.

Lemma 8 *For $s \in \mathbb{R}$, consider the equation*

$$E^p - p'(F(s) + \lambda\Phi_q(s)) = 0, \tag{13}$$

where $p > 1$, $E \geq 0$ and $\lambda \in \mathbb{R}$ are real parameters, $F(s) = \int_0^s f(t)dt$ and $\Phi_q(s) = \int_0^s \varphi_q(t)dt$. If $\lambda > -m_\pm$, then for any $E > 0$, equation (13) admits a unique positive zero $s_+ = s_+(\lambda, E)$ (resp. a unique negative zero $s_- = s_-(\lambda, E)$) and if $E = 0$ it admits no positive (resp. negative) zero beside the trivial one $s_\pm = 0$. Moreover, for all $\lambda > -m_\pm, p > 1$,

(i) *The function $E \longmapsto s_\pm(\lambda, E)$ is C^1 on $(0, +\infty)$, and*

$$\pm\frac{\partial s_\pm}{\partial E}(\lambda, E) = \frac{\pm(p-1)E^{p-1}}{f(s_\pm(\lambda, E)) + \lambda\varphi_q(s_\pm(\lambda, E))} > 0, \ \forall E > 0\,.$$

(ii) $\lim_{E\to 0^+} s_\pm(\lambda, E) = 0$, $\lim_{E\to +\infty} s_\pm(\lambda, E) = \pm\infty$.

(iii) $\lim_{E\to 0} \frac{E^p}{|s_\pm(\lambda,E)|^q} = \frac{p'}{q}(a_0 + \lambda)$, $\lim_{E\to +\infty} \frac{E^p}{|s_\pm(\lambda,E)|^q} = \frac{p'}{q}(a_\pm + \lambda)$.

(iv) $\lim_{E\to 0} \frac{E}{|s_\pm(\lambda,E)|} = \begin{cases} 0 & \text{if } q-p>0 \\ ((a_0+\lambda)/(p-1))^{1/p} & \text{if } q-p=0 \\ +\infty & \text{if } q-p<0, \end{cases}$

$\lim_{E\to +\infty} \frac{E}{|s_\pm(\lambda,E)|} = \begin{cases} +\infty & \text{if } q-p>0 \\ ((a_\pm+\lambda)/(p-1))^{1/p} & \text{if } q-p=0 \\ 0 & \text{if } q-p<0. \end{cases}$

(v) $\lim_{E\to 0} \frac{F(s_\pm(\lambda,E)t)}{E^p} = \frac{t^p}{p'} \frac{a_0}{a_0+\lambda}, \forall t>0$, $\lim_{E\to +\infty} \frac{F(s_\pm(\lambda,E)t)}{E^p} = \frac{t^p}{p'} \frac{a_\pm}{a_\pm+\lambda}, \forall t>0$.

Proof. For $p > 1$ and $E \geq 0$ fixed, consider the function

$$s \longmapsto G_\pm(\lambda, E, s) := E^p - p'(F(s) + \lambda\Phi_q(s)),$$

defined in $\mathbb{R}^\pm$ and strictly decreasing on $(0, +\infty)$ (resp. strictly increasing on $(-\infty, 0)$), because

$$\frac{dG_\pm}{ds}(\lambda, E, s) = -p'\varphi_q(s)(\frac{f(s)}{\varphi_q(s)} + \lambda) \quad \text{and} \quad m_\pm + \lambda > 0\,.$$

One has $G_\pm(\lambda, E, 0) = E^p \geq 0$, and via l'Hospital's rule,

$$\begin{aligned} \lim_{s\to+\infty} G_\pm(\lambda, E, s) &= \lim_{s\to+\infty} E^p - p'\Phi_q(s)(\frac{F(s)}{\Phi_q(s)} + \lambda) \\ &= E^p - p' \lim_{s\to+\infty} \Phi_q(s)(\lim_{s\to+\infty} \frac{f(s)}{\varphi_q(s)} + \lambda) \\ &= -\infty. \end{aligned}$$

So, it is clear that in the case $m_+ + \lambda > 0$ (resp. $m_- + \lambda > 0$) for any $E > 0$, (13) admits a unique positive zero, $s_+ = s_+(\lambda, E)$, (resp. a unique negative zero, $s_- = s_-(\lambda, E)$), and if $E = 0$, it admits no positive (resp. negative) zero beside the trivial one $s = 0$.

Now, for any $p > 1$ and $\lambda > -m_\pm$, consider the real valued function,

$$(E, s) \longmapsto G_\pm(E, s) := E^p - p'(F(s) + \lambda\Phi_q(s)),$$

defined on $\Omega_+ = (0, +\infty)^2$ (resp. $\Omega_- = (0, +\infty) \times (-\infty, 0)$). One has $G_\pm \in C^1(\Omega_\pm)$ and,

$$\frac{\partial G_\pm}{\partial s}(E, s) = -p'\varphi_q(s)(\frac{f(s)}{\varphi_q(s)} + \lambda) \quad \text{in } \Omega_\pm\,;$$

hence, because $m_\pm + \lambda > 0$, it follows that

$$\pm\frac{\partial G_\pm}{\partial s}(E, s) < 0, \quad \text{in} \quad \Omega_\pm,$$

and one may observe that $s_\pm(\lambda,E)$ belongs to the open interval $(0,+\infty)$ (resp. $(-\infty,0)$) and from its definition satisfies

$$G_\pm(E,s_\pm(\lambda,E))=0\,. \tag{14}$$

So, one can make use of the implicit function theorem to show that the function $E\mapsto s_\pm(\lambda,E)$ is $C^1((0,+\infty),\mathbb{R})$ and to obtain the expression of $\frac{\partial s_\pm}{\partial E}(\lambda,E)$ given in **(i)**. Hence, by $m_\pm+\lambda>0$, it follows that for any fixed $p>1$ and $\lambda>-m_\pm$, the function defined on $(0,+\infty)$ by $E\mapsto s_\pm(\lambda,E)$ is strictly increasing (resp. strictly decreasing) and bounded from below by 0 (resp. by $-\infty$) and from above by $+\infty$ (resp. by 0). Then the limit $\lim_{E\to0^+}s_\pm(\lambda,E)=\ell_0^\pm$ exists as real number and the limit $\lim_{E\to+\infty}s_\pm(\lambda,E)=\ell_{+\infty}^\pm$ exists and belongs to $(0,+\infty]$ (resp. $[-\infty,0)$). Moreover,

$$-\infty\le\ell_{+\infty}^-<\ell_0^-\le0\le\ell_0^+<\ell_{+\infty}^+\le+\infty.$$

Let us observe that, for any fixed $p>1$ and $\lambda>-m_\pm$, the function $(E,s)\mapsto G_\pm(E,s)$ is continuous on $[0,+\infty)^2$ (resp. $[0,+\infty)\times(-\infty,0]$) and the function $E\longmapsto s_\pm(\lambda,E)$ is continuous on $(0,+\infty)$ and satisfies (14). So, by passing to the limit in (14) as E tends to 0^+ one obtains

$$0=\lim_{E\to0^+}G_\pm(E,s_\pm(\lambda,E))=G_\pm(0,\ell_0^+)\,.$$

Hence, $\ell_0^\pm$ is a zero, belonging to $[0,+\infty)$ (resp. $(-\infty,0]$), to the equation in s:

$$G_\pm(0,s)=0.$$

By solving this equation one gets: $\ell_0^\pm=0$.

Assume that $\ell_{+\infty}^\pm$ is finite, then by passing to the limit in (14) as E tends to $+\infty$ one gets,

$$+\infty=p'(F(\ell_{+\infty}^\pm)+\lambda\Phi_q(\ell_{+\infty}^\pm))<+\infty,$$

which is impossible. So, $\ell_{+\infty}^\pm=\pm\infty$.

Proof of (iii). Dividing equation (14) by $|s_\pm(\lambda,E)|^q$ one gets,

$$\frac{E^p}{|s_\pm(\lambda,E)|^q}=p'(\frac{F(s_\pm(\lambda,E))}{|s_\pm(\lambda,E)|^q}+\frac{\lambda}{q}),$$

and by passing to the limit as E tends to 0^+, (using l'Hospital's rule),

$$\lim_{E\to0^+}\frac{E^p}{|s_\pm(\lambda,E)|^q}=\frac{p'}{q}(\lim_{E\to0^+}\frac{f(s_\pm(\lambda,E))}{\varphi_q(s_\pm(\lambda,E))}+\lambda)=\frac{p'}{q}(a_0+\lambda).$$

The second limit is obtained by the same way.

Proof of (iv). Dividing equation (14) by $|s_\pm(\lambda,E)|^p$ one gets,

$$\begin{aligned}\lim_{E\to 0^+}\frac{E^p}{|s_\pm(\lambda,E)|^p} &= \lim_{E\to 0^+} p'\big(\frac{F(s_\pm(\lambda,E))}{|s_\pm(\lambda,E)|^p}+\frac{\lambda}{q}|s_\pm(\lambda,E)|^{q-p}\big)\\ &= \lim_{E\to 0^+} p'|s_\pm(\lambda,E)|^{q-p}\big(\frac{F(s_\pm(\lambda,E))}{|s_\pm(\lambda,E)|^q}+\frac{\lambda}{q}\big)\\ &= \frac{p'}{q}(a_0+\lambda)\lim_{E\to 0^+}|s_\pm(\lambda,E)|^{q-p}.\end{aligned}$$

Therefore, the first limit follows. The second one is obtained by the same way.

Proof of (v). Using l'Hospital's rule one gets, for any $t>0$,

$$\begin{aligned}\lim_{E\to 0^+}\frac{F(s_\pm(\lambda,E)t)}{E^p} &= \lim_{E\to 0^+}\frac{t\frac{ds_\pm}{dE}(\lambda,E)f(s_\pm(\lambda,E)t)}{pE^{p-1}}\\ &= \lim_{E\to 0^+}\frac{t(p-1)E^{p-1}f(s_\pm(\lambda,E)t)}{(f(s_\pm(\lambda,E))+\lambda\varphi_q(s_\pm(\lambda,E)))pE^{p-1}}\\ &= \frac{t}{p'}\lim_{E\to 0^+}\frac{\frac{f(s_\pm(\lambda,E)t)}{\varphi_q(s_\pm(\lambda,E)t)}}{\big(\frac{f(s_\pm(\lambda,E))}{\varphi_q(s_\pm(\lambda,E))}\frac{1}{\varphi_q(t)}+\frac{\lambda}{\varphi_q(t)}\big)}\\ &= \frac{t^q}{p'}\big(\frac{a_0}{a_0+\lambda}\big).\end{aligned}$$

The second limit may be computed by the same way, which completes the proof of Lemma 8. ♢

Now, for any $p>1$, $\lambda>-m_\pm$ and $E>0$ we compute $X_\pm(\lambda,E)$ as defined in Section 3. In fact, for all $E>0$,

$$X_+(\lambda,E)=(0,s_+(\lambda,E)),X_-(\lambda,E)=(s_-(\lambda,E),0),$$

where $s_\pm(\lambda,E)$ is defined in Lemma 8. Then

$$r_\pm(\lambda,E)=s_\pm(\lambda,E)\quad\text{for all } E>0.$$

Hence, for any $p>1,\lambda>-m_\pm$, $0<|s_\pm(\lambda,E)|<+\infty$ if and only if $E>0$. And for all $E>0$,

$$\pm(f(r_\pm(\lambda,E))+\lambda\varphi_q(r_\pm(\lambda,E)))=\pm\varphi_q(r_\pm(\lambda,E))\big(\frac{f((r_\pm(\lambda,E)))}{\varphi_q(r_\pm(\lambda,E))}+\lambda\big)>0.$$

So, $D_\pm(\lambda)=(0,+\infty)$ for all $\lambda>-m_\pm$ and

$$D(\lambda)=D_+(\lambda)\cap D_-(\lambda)=(0,+\infty),\forall\lambda>-m.$$

Before going further in the investigation, from Lemma 8, we deduce that for any fixed $p>1$ and $\lambda>-m_\pm$,

$$\pm\frac{\partial r_\pm}{\partial E}(\lambda,E)=\frac{\pm(p-1)E^{p-1}}{f(r_\pm(\lambda,E))+\lambda\varphi_q(r_\pm(\lambda,E))}>0,\forall E>0. \tag{15}$$

$$\lim_{E\to 0^+} r_\pm(\lambda,E)=0 \quad \text{and} \quad \lim_{E\to+\infty} r_\pm(\lambda,E)=\pm\infty \tag{16}$$

$$\lim_{E\to 0}\frac{E^p}{|r_\pm(\lambda,E)|^q}=\frac{p'}{q}(a_0+\lambda),\ \lim_{E\to+\infty}\frac{E^p}{|r_\pm(\lambda,E)|^q}=\frac{p'}{q}(a_\pm+\lambda). \tag{17}$$

$$\lim_{E\to 0}\frac{E}{|r_\pm(\lambda,E)|}=\begin{cases} 0 & \text{if} \quad q-p>0\\ (\frac{a_0+\lambda}{p-1})^{\frac{1}{p}} & \text{if} \quad q-p=0\\ +\infty & \text{if} \quad q-p<0\end{cases} \tag{18}$$

$$\lim_{E\to +\infty}\frac{E}{|r_\pm(\lambda,E)|}=\begin{cases} +\infty & \text{if} \quad q-p>0\\ (\frac{a_\pm+\lambda}{p-1})^{\frac{1}{p}} & \text{if} \quad q-p=0\\ 0 & \text{if} \quad q-p<0\end{cases} \tag{19}$$

$$\lim_{E\to 0}\frac{F(r_\pm(\lambda,E)t)}{E^p}=\frac{t^p}{p'}\frac{a_0}{a_0+\lambda},\forall t>0 \tag{20}$$

$$\lim_{E\to +\infty}\frac{F(r_\pm(\lambda,E)t)}{E^p}=\frac{t^p}{p'}\frac{a_\pm}{a_\pm+\lambda},\forall t>0. \tag{21}$$

At present we define, for any $p>1$, $\lambda>-m_\pm$, and $E>0$, the time map,

$$T_\pm(\lambda,E):=\pm\int_0^{r_\pm(\lambda,E)}\{E^p-p'(F(\xi)+\lambda\Phi_q(\xi))\}^{-\frac{1}{p}}\,d\xi, E>0,$$

and a simple change of variables shows that,

$$T_\pm(\lambda,E)=|r_\pm(\lambda,E)|\int_0^1\{E^p-p'(F(r_\pm(\lambda,E)\xi)+(\lambda/q)|r_\pm(\lambda,E)\xi|^q)\}^{-\frac{1}{p}}\,d\xi, \tag{22}$$

which may be written as,

$$\begin{aligned} T_\pm(\lambda,E) \quad = \quad (|r_\pm(\lambda,E)|/E)\int_0^1 &\big\{1-p'(F(r_\pm(\lambda,E)\xi)/E^p \\ &+(\lambda\xi^q/q)(|r_\pm(\lambda,E)|^q/E^p))\big\}^{-1/p}d\xi\,. \end{aligned} \tag{23}$$

Also, we define, the time maps

$$\begin{aligned} T^\pm_{2n}(\lambda,E)&:=n(T_+(\lambda,E)+T_-(\lambda,E)),E>0,\lambda>-m^\pm_{2n},n\geq 1\,,\\ T^\pm_{2n+1}(\lambda,E)&:=T^\pm_{2n}(\lambda,E)+T_\pm(\lambda,E),E>0,\lambda>-m^\pm_{2n+1},n\geq 0\,. \end{aligned}$$

To prove Theorems 2, 3, 4, it suffices to compute the limits of these time maps as E tends to 0^+ and $+\infty$, and then apply the intermediate value theorem. Recall that we have defined in Proposition 1 the functions $h_k^\pm$ and let us now define,

$$g_k(\lambda)=\frac{\lambda_k^{1/p}}{(a_0+\lambda)^{1/p}},\ \text{for all } \lambda>-a_0,\ k\geq 1\,.$$

Lemma 9 *Assume that $p,q>1$, then for all $k\geq 1$, and all $\lambda>-m_k^\pm$,*

1. $\lim_{E\to 0^+} T_k^{\pm}(\lambda,E)=+\infty$ *and* $\lim_{E\to +\infty} T_k^{\pm}(\lambda,E)=0$, *if* $q>p$,

2. $\lim_{E\to 0^+} T_k^{\pm}(\lambda,E)=0$ *and* $\lim_{E\to +\infty} T_k^{\pm}(\lambda,E)=+\infty$, *if* $q<p$,

3. $\lim_{E\to 0^+} T_k^{\pm}(\lambda,E)=\frac{1}{2}g_k(\lambda)$ *and* $\lim_{E\to +\infty} T_k^{\pm}(\lambda,E)=\frac{1}{2}h_k^{\pm}(\lambda)$, *if* $q=p$.

Proof. The limits of $T_{\pm}(\lambda,E)$ as E tends to 0^+ or $+\infty$ follow by making use of (23) and (17)–(21), together with the fact that

$$\lambda_k^{1/p}=2k(p-1)^{1/p}\int_0^1(1-t^p)^{-1/p}dt.$$

The limits of $T_{2n}^{\pm}(\lambda,E)$ and $T_{2n+1}^{\pm}(\lambda,E)$ as E tends to 0 or $+\infty$ follow immediately from the definition of these maps and the limits of $T_{\pm}(\lambda,E)$ as E tends to 0 or $+\infty$ respectively. The proof is complete. ♢

To apply the intermediate value theorem we need to know, for each integer $k\geq 1$, which one of the limits,

$$\lim_{E\to 0^+} T_k^{\pm}(\lambda,E) \quad \text{or} \quad \lim_{E\to +\infty} T_k^{\pm}(\lambda,E)$$

is greater than the other. If $q-p\neq 0$, the answer is evident from Lemma 9, but if $q-p=0$, a deep study is required. Let, for any integer $n\geq 1$,

$$\Lambda_{2n+1}^{\pm}=\frac{a_-a_+-a_0a_{2n+1}^{\mp}}{a_0-a_{2n+1}^{\pm}} \quad \text{and} \quad \Lambda_{2n}^{\pm}=\frac{a_-a_+-a_0a_{2n}^{\pm}}{a_0-a_{2n}^{\pm}}.$$

Lemma 10 *Let* $k\geq 1$ *be an integer,*

(i) *If* $a_0<b_k^{\pm}$, *then* $g_k(\lambda)>h_k^{\pm}(\lambda)$ *for all* $\lambda>-a_0$.

(ii) *If* $a_0>c_k^{\pm}$, *then* $g_k(\lambda)<h_k^{\pm}(\lambda)$ *for all* $\lambda>-b_k^{\pm}$.

(iii) *If* $a_-<a_k^{\pm}\leq a_0<a_+$, $(k\geq 2)$ *then* $g_k(\lambda)<h_k^{\pm}(\lambda)$ *for all* $\lambda>-a_-$.

(iv) *If* $a_-<a_0<a_k^{\pm}$, $(k\geq 2)$ *then there exists a unique* $\tilde\lambda_k^{\pm}=\tilde\lambda_k^{\pm}(a_0)\in(-a_-,\Lambda_k^{\pm})$ *such that,*

$$\begin{aligned}&g_k(\lambda)<h_k^{\pm}(\lambda) \text{ for all } \lambda\in(-a_-,\tilde\lambda_k^{\pm}),\\ &g_k(\tilde\lambda_k^{\pm})=h_k^{\pm}(\tilde\lambda_k^{\pm}),\\ &g_k(\lambda)>h_k^{\pm}(\lambda) \text{ for all } \lambda\in(\tilde\lambda_k^{\pm},+\infty).\end{aligned}$$

Moreover, the function $a_0\mapsto\tilde\lambda_k^{\pm}(a_0)$ *is strictly increasing on* $(a_-,a_k^{\pm})$ *and*

$$\lim_{a_0\to a_-}\tilde\lambda_k^{\pm}(a_0)=-a_-, \textit{ and } \lim_{a_0\to a_k^{\pm}}\tilde\lambda_k^{\pm}(a_0)=+\infty.$$

Proof. It is immediate to prove Assertions **(i)** and **(ii)** of this lemma since the function $a \mapsto (a+\lambda)^{-1/p}$ is strictly decreasing. Let us prove assertions **(iii)** and **(iv)**.

We are concerned by the case $k = 2n+1$ for some $n \geq 1$ and the superscript is $+ : (k = 2n+1, +)$. All the remaining cases may be handled similarly.

Let y^+_{2n+1} be the function defined on $(-a_-, +\infty)$ by

$$y^+_{2n+1}(\lambda) = (n+1)(a_+ + \lambda)^{-1/p} + n(a_- + \lambda)^{-1/p} - (2n+1)(a_0 + \lambda)^{-1/p}.$$

One has, $y^+_{2n+1}(\lambda) < 0$ if and only if

$$((\frac{n+1}{2n+1})(a_+ + \lambda)^{-1/p} + (\frac{n}{2n+1})(a_- + \lambda)^{-1/p})^p < (a_0 + \lambda)^{-1}$$

and $(y^+_{2n+1})'(\lambda) < 0$ if and only if

$$((\frac{n+1}{2n+1})(a_+ + \lambda)^{-1-1/p} + (\frac{n}{2n+1})(a_- + \lambda)^{-1-1/p})^{p/(p+1)} > (a_0 + \lambda)^{-1}.$$

Since the function $t \mapsto t^p$ (resp. $t \mapsto -t^{p/(p+1)}$) is convex on $(-a_-, +\infty)$,

$$\begin{aligned}((\frac{n+1}{2n+1})(a_+ + \lambda)^{-1/p} &+ (\frac{n}{2n+1})(a_- + \lambda)^{-1/p})^p \\ &< (\frac{n+1}{2n+1})(a_+ + \lambda)^{-1} + (\frac{n}{2n+1})(a_- + \lambda)^{-1}\end{aligned}$$

(resp.

$$\begin{aligned}((\frac{n+1}{2n+1})(a_+ + \lambda)^{-1-1/p} &+ (\frac{n}{2n+1})(a_- + \lambda)^{-1-1/p})^{p/(p+1)} \\ &> (\frac{n+1}{2n+1})(a_+ + \lambda)^{-1} + (\frac{n}{2n+1})(a_- + \lambda)^{-1}).\end{aligned}$$

So, if we define on $(-a_-, +\infty)$ the function x^+_{2n+1} by

$$x^+_{2n+1}(\lambda) = (\frac{n+1}{2n+1})(a_+ + \lambda)^{-1} + (\frac{n}{2n+1})(a_- + \lambda)^{-1} - (a_0 + \lambda)^{-1},$$

it follows that for all $\lambda > -a_-$,

$$x^+_{2n+1}(\lambda) \leq 0 \Longrightarrow y^+_{2n+1}(\lambda) < 0 \quad \text{and} \quad x^+_{2n+1}(\lambda) \geq 0 \Longrightarrow (y^+_{2n+1})'(\lambda) < 0.$$

Some simple computations show that for all $\lambda > -a_-$ and $\kappa = +, -$, one has,

$$\kappa x^+_{2n+1}(\lambda) > 0 \Longleftrightarrow \kappa(\lambda(a_0 - a^+_{2n+1}) - (a_- a_+ - a_0 a^-_{2n+1})) > 0.$$

Also, for all $\lambda \in \mathbb{R}$ and $\kappa = +, -$, one has in the case where $a_0 > a^+_{2n+1}$,

$$\kappa(\lambda(a_0 - a^+_{2n+1}) - (a_- a_+ - a_0 a^-_{2n+1})) > 0 \Longleftrightarrow \kappa(\lambda - \Lambda^+_{2n+1}) > 0,$$

and in the case where $a_0 < a^+_{2n+1}$,

$$\kappa(\lambda(a_0 - a^+_{2n+1}) - (a_-a_+ - a_0a^-_{2n+1})) > 0 \Longleftrightarrow \kappa(\lambda - \Lambda^+_{2n+1}) < 0.$$

On the other hand one has,

$$a_0 < a^+_{2n+1} \Longrightarrow \Lambda^+_{2n+1} > -a_- \quad \text{and} \quad a_0 > a^+_{2n+1} \Longrightarrow \Lambda^+_{2n+1} < -a_-.$$

Hence, an easy compilation of the above assertions shows that if $a_0 < a^+_{2n+1}$, then

$$\begin{aligned} \lambda \geq \Lambda^+_{2n+1}(> -a_-) &\Longrightarrow x^+_{2n+1}(\lambda) \leq 0 \\ -a_- < \lambda < \Lambda^+_{2n+1} &\Longrightarrow x^+_{2n+1}(\lambda) < 0 \end{aligned}$$

and if $a_0 > a^+_{2n+1}$, then $\lambda > -a_-$ implies $x^+_{2n+1}(\lambda) > 0$.

It remains to study the particular case $a_0 = a^+_{2n+1}$. For $\lambda > -a_-$, define

$$\psi_\lambda(t) = (\lambda + t)^{-1}, \text{ for all } t > -\lambda.$$

Let us observe that $x^+_{2n+1}(\lambda) > 0$ if and only if

$$\psi_\lambda(a_0) < (\frac{n}{2n+1})\psi_\lambda(a_-) + (\frac{n+1}{2n+1})\psi_\lambda(a_+).$$

But since ψ_λ is strictly convex, one has

$$\begin{aligned} \psi_\lambda(a_0) = \psi_\lambda(a^+_{2n+1}) &= \psi_\lambda((\frac{n}{2n+1})a_- + (\frac{n+1}{2n+1})a_+) \\ &< (\frac{n}{2n+1})\psi_\lambda(a_-) + (\frac{n+1}{2n+1})\psi_\lambda(a_+), \end{aligned}$$

that is, if $a_0 = a^+_{2n+1}$, then

$$\lambda > -a_- \Longrightarrow x^+_{2n+1}(\lambda) > 0.$$

Thus, in the case where $a_0 \geq a^+_{2n+1}$, y^+_{2n+1} is strictly decreasing on $(-a_-, +\infty)$ and by $\lim_{\lambda\to+\infty} y^+_{2n+1}(\lambda) = 0$ it follows that y^+_{2n+1} is strictly positive on $(-a_-, +\infty)$. Thus, Assertion **(iii)** is proved.

In the case where $a_0 < a^+_{2n+1}$, y^+_{2n+1} is strictly negative on $[\Lambda^+_{2n+1}, +\infty)$ and strictly decreasing on $(-a_-, \Lambda^+_{2n+1})$. By $\lim\limits_{\lambda\to -a_-} y^+_{2n+1}(\lambda) = +\infty$, it follows that there exists $\tilde\lambda^+_{2n+1} = \tilde\lambda^+_{2n+1}(a_0) \in (-a_-, \Lambda^+_{2n+1})$ such that y^+_{2n+1} is strictly positive on $(-a_-, \tilde\lambda^+_{2n+1}), y^+_{2n+1}(\tilde\lambda^+_{2n+1}) = 0$, and y^+_{2n+1} is strictly negative on $(\tilde\lambda^+_{2n+1}, +\infty)$.

One has $y^+_{2n+1}(a_0, \tilde\lambda^+_{2n+1}(a_0)) = 0$. So, the implicit function theorem yields

$$\frac{\partial\tilde\lambda^+_{2n+1}}{\partial a_0}(a_0) = -\frac{\partial y^+_{2n+1}}{\partial a_0}(a_0, \tilde\lambda^+_{2n+1}(a_0))/\frac{\partial y^+_{2n+1}}{\partial\lambda}(a_0, \tilde\lambda^+_{2n+1}(a_0)).$$

One has $\frac{\partial y^+_{2n+1}}{\partial\lambda}(a_0,\tilde\lambda^+_{2n+1}(a_0)) < 0$ since $x^+_{2n+1}(\tilde\lambda^+_{2n+1}(a_0)) > 0$, and

$$\frac{\partial y^+_{2n+1}}{\partial a_0}(a_0,\tilde\lambda^+_{2n+1}(a_0)) = \frac{2n+1}{p}(a_0+\tilde\lambda^+_{2n+1}(a_0))^{-1-1/p} > 0.$$

So, the function $a_0 \mapsto \tilde\lambda^+_{2n+1}(a_0)$ is strictly increasing on (a_-,a^+_{2n+1}). On the other hand, one has

$$-a_- < \tilde\lambda^+_{2n+1}(a_0) < \Lambda^+_{2n+1}(a_0); \forall a_0 \in (a_-,a^+_{2n+1}),$$

and $\lim_{a_0\mapsto a_-}\Lambda^+_{2n+1}(a_0) = -a_-$, which is easy to check. Thus, $\lim_{a_0\to a_-}\tilde\lambda^+_{2n+1}(a_0) = -a_-$.

Assume that the function $a_0 \mapsto \tilde\lambda^+_{2n+1}(a_0)$ is bounded as a_0 tends to a^+_{2n+1}. Denote by λ_* its limit, that is, $\lim_{a_0\to a^+_{2n+1}}\tilde\lambda^+_{2n+1}(a_0) = \lambda_*$. Since

$$y^+_{2n+1}(a_0,\tilde\lambda^+_{2n+1}(a_0)) = 0, \forall a_0 \in (a_-,a^+_{2n+1}),$$

it follows that $y^+_{2n+1}(a^+_{2n+1},\lambda_*) = 0$, that is,

$$\begin{aligned}
&((\frac{n+1}{2n+1})(a_+ + \lambda_*) + (\frac{n}{2n+1})(a_- + \lambda_*))^{-1/p} \\
&= (\frac{n+1}{2n+1})(a_+ + \lambda_*)^{-1/p} + (\frac{n}{2n+1})(a_- + \lambda_*)^{-1/p}\,.
\end{aligned}$$

By the strict convexity of the function $t \to t^{-1/p}$ on $(0,+\infty)$, it follows that $a_+ + \lambda_* = a_- + \lambda_*$, that is $a_- = a_+$, which contradicts the hypothesis $a_- < a_+$. So, the function $a_0 \mapsto \tilde\lambda^+_{2n+1}(a_0)$ is unbounded, and

$$\lim_{a_0\to a^+_{2n+1}}\tilde\lambda^+_{2n+1}(a_0) = +\infty\,.$$

Therefore, Assertion **(iv)** is proved for the case $(k = 2n+1,+)$.

In the case $(k = 2n+1,-)$ (resp. $(k = 2n,\pm)$) one considers y^-_{2n+1} and x^-_{2n+1} (resp. $y^\pm_{2n}$ and $x^\pm_{2n}$) defined by

$$\begin{aligned}
y^-_{2n+1}(\lambda) &= n(a_+ + \lambda)^{-1/p} + (n+1)(a_- + \lambda)^{-1/p} - (2n+1)(a_0+\lambda)^{-1/p} \\
x^-_{2n+1}(\lambda) &= (\frac{n}{2n+1})(a_+ + \lambda)^{-1} + (\frac{n+1}{2n+1})(a_- + \lambda)^{-1} - (a_0+\lambda)^{-1}
\end{aligned}$$

(resp.

$$\begin{aligned}
y^\pm_{2n}(\lambda) &= (a_+ + \lambda)^{-1/p} + (a_- + \lambda)^{-1/p} - 2(a_0+\lambda)^{-1/p} \\
x^\pm_{2n}(\lambda) &= \frac12(a_+ + \lambda)^{-1} + \frac12(a_- + \lambda)^{-1} - (a_0+\lambda)^{-1}).
\end{aligned}$$

The same reasoning as above also works here; therefore, Lemma 10 is proved. $\diamondsuit$

5 Proof of the main results

Proof of Theorem 2. If $a_0 < b_k^{\pm}$, then for all $k \geq 1$ and $\lambda > -m_k^{\pm}$

$$\lim_{E\to 0^+} T_k^{\pm}(\lambda, E) = \frac{1}{2} g_k(\lambda) \quad \text{and} \quad \lim_{E\to +\infty} T_k^{\pm}(\lambda, E) = \frac{1}{2} h_k^{\pm}(\lambda).$$

On the other hand, for all $\lambda > -a_0 : g_k(\lambda) > h_k^{\pm}(\lambda)$. So, since the function $\lambda \mapsto g_k(\lambda)$ (resp. $\lambda \mapsto h_k^{\pm}(\lambda)$) is strictly decreasing on $(-a_0, +\infty)$ (resp. on $(-b_k^{\pm}, +\infty)$), and

$$g_k(\lambda_k - a_0) = 1 \text{ (resp. } h_k^{\pm}(\lambda_k^{\pm}) = 1)\,,$$

then

$$g_k(\lambda) > 1 \quad \text{if and only if} \quad -a_0 < \lambda < \lambda_k - a_0$$

(resp.

$$h_k^{\pm}(\lambda) < 1 \quad \text{if and only if} \quad \lambda_k^{\pm} < \lambda).$$

Hence, one has,

$$g_k(\lambda) > 1 > h_k^{\pm}(\lambda) \quad \text{if and only if} \quad \max(-a_0, \lambda_k^{\pm}) < \lambda < \lambda_k - a_0.$$

Then the equation in $E > 0 : T_k^{\pm}(\lambda, E) = 1/2$ admits at least a solution for all λ satisfying

$$\lambda > -m_k^{\pm} \quad \text{and} \quad \max(-a_0, \lambda_k^{\pm}) < \lambda < \lambda_k - a_0,$$

that is, for all λ satisfying

$$\max(-m_k^{\pm}, \lambda_1^{\pm}) < \lambda < \lambda_k - a_0,$$

since $\max(-a_0, -m_k^{\pm}) = -m_k^{\pm}$.

If $a_0 > c_k^{\pm}$ then for all $k \geq 1$ and $\lambda > -m_k^{\pm}$,

$$\lim_{E\to 0^+} T_k^{\pm}(\lambda, E) = \frac{1}{2} g_k(\lambda) \quad \text{and} \quad \lim_{E\to +\infty} T_k^{\pm}(\lambda, E) = \frac{1}{2} h_k^{\pm}(\lambda).$$

On the other hand, for all $\lambda \in (-b_k^{\pm}, +\infty)$, $g_k(\lambda) < h_k^{\pm}(\lambda)$. So, since the function $\lambda \mapsto g_k(\lambda)$ (resp. $\lambda \mapsto h_k^{\pm}(\lambda)$) is strictly decreasing on $(-a_0, +\infty)$ (resp. on $(-b_k^{\pm}, +\infty)$) and

$$g_k(\lambda_k - a_0) = 1, \text{ (resp. } h_k^{\pm}(\lambda_k^{\pm}) = 1)\,,$$

it follows that

$$g_k(\lambda) < 1 \quad \text{if and only if} \quad \lambda_k - a_0 < \lambda$$

(resp.

$$h_k^{\pm}(\lambda) > 1 \quad \text{if and only if} \quad -b_k^{\pm} < \lambda < \lambda_k^{\pm}).$$

Hence, one has,

$$g_k(\lambda) < 1 < h_k^{\pm}(\lambda) \quad \text{if and only if} \quad \max(-b_k^{\pm}, \lambda_k - a_0) < \lambda < \lambda_k^{\pm}.$$

Then the equation in $E > 0 : T_k^{\pm}(\lambda, E) = 1/2$ admits at least a solution for all λ satisfying

$$\lambda > -m_k^{\pm} \quad \text{and} \quad \max(-b_k^{\pm}, \lambda_k - a_0) < \lambda < \lambda_k^{\pm},$$

that is, for all λ satisfying

$$\max(-m_k^{\pm}, \lambda_k - a_0) < \lambda < \lambda_k^{\pm}$$

since $\max(-b_k^{\pm}, -m_k^{\pm}) = -m_k^{\pm}$. The proof of Theorem 2 is complete. $\diamondsuit$

Proof of Theorem 3. If $a_- < a_0 < a_+$ then for all $\lambda > -m_{\pm}$,

$$\lim_{E\to 0^+} T_1^{\pm}(\lambda, E) = \frac{1}{2} g_1(\lambda) \quad \text{and} \quad \lim_{E\to+\infty} T_1^{\pm}(\lambda, E) = \frac{1}{2} h_1^{\pm}(\lambda).$$

On the other hand, one has,

$$\text{for all } \lambda > -a_0, g_1(\lambda) > h_1^+(\lambda) \quad (\text{resp. for all } \lambda > -a_-, g_1(\lambda) < h_1^-(\lambda)).$$

So, the same reasoning as in the superlinear (resp. sublinear) case leads to the proof of the first assertion (resp. the second assertion) of Theorem 3.

If $a_- < a_k^{\pm} \le a_0 < a_+$ and $k \ge 2$, for all $\lambda > -m_k^{\pm} = -m$,

$$\lim_{E\to 0^+} T_k^{\pm}(\lambda, E) = \frac{1}{2} g_k(\lambda) \quad \text{and} \quad \lim_{E\to+\infty} T_k^{\pm}(\lambda, E) = \frac{1}{2} h_k^{\pm}(\lambda).$$

On the other hand, one has for all $\lambda > -a_-, g_k(\lambda) < h_k^{\pm}(\lambda)$. So, the same reasoning as in the sublinear case leads to the fact that the equation in $E > 0 :$ $T_k^{\pm}(\lambda, E) = 1/2$ admits at least a solution for all λ satisfying

$$\max(-m, \lambda_k - a_0) < \lambda < \lambda_k^{\pm}.$$

If $a_- < a_0 < a_k^{\pm} < a_+$ and $k \ge 2$, for all $\lambda > -m$,

$$\lim_{E\to 0^+} T_k^{\pm}(\lambda, E) = \frac{1}{2} g_k(\lambda) \quad \text{and} \quad \lim_{E\to+\infty} T_k^{\pm}(\lambda, E) = \frac{1}{2} h_k^{\pm}(\lambda).$$

Since the function $a_0 \mapsto \tilde{\lambda}_k^{\pm}(a_0) \in (-a_-, \Lambda_k^{\pm})$ is strictly increasing on the interval $(a_-, a_k^{\pm})$ then the function $a_0 \mapsto h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0))$ is strictly decreasing on $(a_-, a_k^{\pm})$. Also, one has,

$$\lim_{a_0\to a_-} h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) = \lim_{x\to -a_-} h_k^{\pm}(x) = +\infty,$$

and

$$\lim_{a_0\to a_k^{\pm}} h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) = \lim_{x\to+\infty} h_k^{\pm}(x) = 0\,.$$

Then there exists a unique $\tilde{a}_k^{\pm} \in (a_-, a_k^{\pm})$ such that for all $a_0 \in (a_-, a_k^{\pm})$ one gets,

$$\begin{aligned} h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) > 1 &\iff a_- < a_0 < \tilde{a}_k^{\pm}, \\ h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) = 1 &\iff a_0 = \tilde{a}_k^{\pm}, \\ h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) < 1 &\iff \tilde{a}_k^{\pm} < a_0 < a_k^{\pm}. \end{aligned}$$

So, one has to distinguish two cases:

Case where $a_- < a_0 < \tilde{a}_k^{\pm}$. Then for all $\lambda \in (-a_-, \tilde{\lambda}_k^{\pm}(a_0))$,

$$1 < h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) = g_k(\tilde{\lambda}_k^{\pm}(a_0)) < g_k(\lambda) < h_k^{\pm}(\lambda),$$

and for all $\lambda > \tilde{\lambda}_k^{\pm}(a_0)$,

$$h_k^{\pm}(\lambda) < 1 < g_k(\lambda) \quad \text{if and only if} \quad \lambda_k^{\pm} < \lambda < \lambda_k - a_0\,,$$

hence, the equation in $E > 0 : T_k^{\pm}(\lambda, E) = 1/2$ admits at least a solution for all λ satisfying

$$\lambda > -m \quad \text{and} \quad \lambda_k^{\pm} < \lambda < \lambda_k - a_0\,,$$

that is, for all λ satisfying, $\max(-m, \lambda_k^{\pm}) < \lambda < \lambda_k - a_0$.

Case where $\tilde{a}_k^{\pm} < a_0 < a_k^{\pm}$. Then for all $\lambda > \tilde{\lambda}_k^{\pm}(a_0)$,

$$h_k^{\pm}(\lambda) < g_k(\lambda) < g_k(\tilde{\lambda}_k^{\pm}(a_0)) = h_k^{\pm}(\tilde{\lambda}_k^{\pm}(a_0)) < 1,$$

and for all $\lambda \in (-a_-, \tilde{\lambda}_k^{\pm}(a_0))$,

$$g_k(\lambda) < 1 < h_k^{\pm}(\lambda) \quad \text{if and only if} \quad \max(-a_-, \lambda_k - a_0) < \lambda < \lambda_k^{\pm}\,;$$

hence, the equation in $E > 0 : T_k^{\pm}(\lambda, E) = 1/2$ admits at least a solution for all λ satisfying

$$\lambda > -m \quad \text{and} \quad \max(-a_-, \lambda_k - a_0) < \lambda < \lambda_k^{\pm},$$

that is, for all λ satisfying $\max(-m, \lambda_k - a_0) < \lambda < \lambda_k^{\pm}$, this is so because $\max(-m, -a_-) = -m$. The proof of Theorem 3 is complete. $\diamondsuit$

Proof of Theorem 4. If $q - p > 0$ (resp. $q - p < 0$), by Lemma 9, for all $\lambda > -m_k^{\pm}, k \geq 1$,

$$\lim_{E \to 0^+} T_k^{\pm}(\lambda, E) = +\infty \text{ (resp. } = 0) \text{ and } \lim_{E \to 0^+} T_k^{\pm}(\lambda, E) = 0 \text{ (resp. } = +\infty).$$

So, the intermediate value theorem implies that the equation in $E > 0$: $T_k^{\pm}(\lambda, E) = 1/2$ admits at least a solution for all $\lambda > -m_k^{\pm}$. Theorem 4 is proved. $\diamondsuit$

Remark 1. Some easy computations show that

$$\frac{\partial T_+}{\partial E}(\lambda, E) = \frac{(p')^{-\frac{1}{p}}}{p}(\frac{\partial r_+}{\partial E}(\lambda, E)) \int_0^1 \frac{(H(\lambda, r_+(\lambda, E)) - H(\lambda, r_+(\lambda, E)\xi))}{(F(\lambda, r_+(\lambda, E)) - F(\lambda, r_+(\lambda, E)\xi))^{1+\frac{1}{p}}} d\xi,$$

where $H(\lambda, x) := p\tilde{F}(\lambda, x) - x\tilde{f}(\lambda, x), \tilde{F}(\lambda, x) = \lambda\Phi_q(x) + F(x)$ and $\tilde{f}(\lambda, x) = \lambda\varphi_q(x) + f(x)$. So, if $q = p$, one gets,

$$\frac{d}{dx}(\frac{f(x)}{\varphi_p(x)}) = \frac{-1}{x\varphi_p(x)}\frac{\partial H}{\partial x}(\lambda, x), x > 0.$$

Hence, in the particular case where the function $x \mapsto f(x)/\varphi_p(x), (q = p)$ is strictly decreasing on $(0, +\infty)$ the function $x \mapsto H(\lambda, x)$ is strictly increasing on $(0, +\infty)$ and then the time map $E \mapsto T_+(\lambda, E)$ is strictly increasing on $(0, +\infty)$. So, uniqueness (if existence) of the solution of the equation in $E > 0 :$ $T_+(\lambda, E) = 1/2$ is guaranteed. The exact number of positive solution(s) in A_1^+ should be obtained with this additional condition. The same remark works for the result of Guedda and Veron [40]. That is, their Theorem 2.1 holds without there condition (2.7).

Remark 2. If f is an odd function, $a_- = a_+$. The statements of Theorems 2, 3, and 4 may be simplified in this particular case.

Remark 3. Several corollaries may be deduced from Theorems 2, 3, and 4 and the above remarks. In fact, one may draw some bifurcation diagrams and compute a lower bound on the number of solutions of problem (1) in some cases and the exact number of positive solutions in others. This would require more space and patience, and is left to the diligent, patient reader. However, a qualitative feature of the variations of the bifurcation branches as a_0 varies is known. In fact, if $a_- < a_+$, then the bifurcation branches are trend towards a same direction for all $a_0 < \tilde{a}_k^{\pm}$ and towards the opposite for all $a_0 > \tilde{a}_k^{\pm}$. The case where $a_0 = \tilde{a}_k^{\pm}$ remains an open question. If $a_- > a_+$, one may study the asymmetric case, $a_- > a_0 > a_+$, as in Theorem 3.

Remark 4. One may observe that a common feature in Theorems 2, 3, and 4 is that the parameter λ is taken, in particular, such that the function $x \mapsto \lambda + f(x)/\varphi_q(x)$ is strictly positive on $(-\infty, 0)$ and/or $(0, +\infty)$. Some cases where this function changes sign once are studied by Guedda and Veron [40] and by Boucherif, Bouguima and Derhab [13], both for the particular case where f is odd. So, it is reasonable to ask the question of what happens if f is not necessarily odd.

6 Appendix: Historical overview on time maps

At the beginning of time maps' history, the authors used them with the one dimensional Laplacian operator, that is, with the one dimensional p-Laplacian and $p = 2$. From the 1960's we can mention Opial [46], [47], Urabe [62], [63], [64], [65], [66], Pimbley [48], [49], and Gavalas [39].

In the early 1970's, Laetsch [43] used time maps to study positive solutions to a class of boundary-value problems with Dirichlet boundary data. Since then many authors have referred to his work. We also want to mention Chafee and Infante [21], and Chafee [22]. Brown and Budin [14], [15] used the time map approach to study positive solutions to some boundary-value problems. Independently and about the same time, De-Mottoni and Tesei [24] studied positive solutions of some other class of boundary-value problems by means of the same method.

In the early 1980's Smoller and Wasserman [55] introduced a technique that, in some circumstances, can be used to prove uniqueness of the critical point of time maps. Their technique has been used subsequently by many authors; see for instance, Ammar Khodja [6], Ramaswamy [50], S. H. Wang and Kazarinoff, [67], [68], S. H. Wang and F. P. Lee [69], S. H. Wang [70], [71], [72], and recently by Addou and Benmezaï [4].

The study of sign-changing solutions by means of time maps was initiated by De-Mottoni and Tesei [25], and independently, some years after, by Shivaji [53].

During the last two decades, time maps have been used in many publications. Besides the above mentioned papers, we want to add the following ones: Addou and Ammar Khodja [1], Anuradha, Shivaji and Zhu [7], [8], Anuradha and Shivaji [9], Anuradha, C. Brown and Shivaji [10], Brown, Ibrahim and Shivaji [16], Brunovsky and Chow [17], Castro and Shivaji [19], [20], Ding and Zanolin [26], [27], Fernandes [30], Fonda and Zanolin [31], Fonda, Gossez and Zanolin [32], Schaaf [51], Shivaji [52], [53], [54], Smoller, Tromba and Wasserman [59], Smoller and Wasserman [56], [57], [58]. Notice that this list is in alphabetical order, and is not complete by any means. The differential operator in the equations studied in these papers is the p-Laplacian with $p = 2$. For more general differential operators, see references listed in Section 3 of this paper.

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IDRIS ADDOU
USTHB, Institut de Mathématiques
El-Alia, B.P. no. 32, Bab-Ezzouar
16111, Alger, Algérie
E-mail address: idrisaddou@hotmail.com