Electronic Journal of Differential Equations, Vol. **1999**(1999), No. 30, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

Radial minimizers of a Ginzburg-Landau functional *

Yutian Lei, Zhuoqun Wu, & Hongjun Yuan

Abstract

We consider the functional

$$E_\varepsilon(u,G)=\frac{1}{p}\int_G|\nabla u|^p+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2$$

with $p>2$ and $d>0$, on the class of functions $W=\{u(x)=f(r)e^{id\theta}\in W^{1,p}(B,C);f(1)=1,f(r)\geq 0\}$. The location of the zeroes of the minimizer and its convergence as ε approaches zero are established.

1 Introduction

Let $G\subset R^2$ be a bounded and simply connected domain with smooth boundary ∂G and g be a smooth map from ∂G into $S^1=\{x\in C;|x|=1\}$. Consider the functional of Ginzburg-Landau type

$$E_\varepsilon(u,G)=\frac{1}{p}\int_G|\nabla u|^p+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2,\quad(\varepsilon>0)\tag{1.1}$$

which has been well-studied in [1] for $p=2$, $d=\deg(g,\partial G)=0$ and in [2] for $p=2$, $\deg(g,\partial G)\neq 0$. Here $d=\deg(g,\partial G)$ denotes the Brouwer degree of the map g. For other related papers, we refer to [3],[5]–[13].

The first two authors of this paper studied the general case $p>1$, especially the case $p>2$ under the restriction $d=\deg(g,\partial G)=0$. In [9][10] some results on the asymptotic behaviour of the minimizer u_ε of $E_\varepsilon(u,G)$ are presented, in particular, if $p>2$, then for some $\alpha\in(0,1)$, the regularizable minimizer $\tilde{u}_\varepsilon$ of $E_\varepsilon(u,G)$ converges in $C^{1,\alpha}_{\rm loc}(G,C)$ as $\varepsilon\to 0$. By the regularizable minimizer of $E_\varepsilon(u,G)$, we mean a minimizer of $E_\varepsilon(u,G)$ which is the limit of a subsequence $u_\varepsilon^{\tau_k}$ of minimizers u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u,G)=\frac{1}{p}\int_G(|\nabla u|^2+\tau)^{p/2}+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2,\quad(\tau>0)\tag{1.2}$$

---

*1991 *Mathematics Subject Classifications:* 35J70.
Key words and phrases: Ginzburg-Landau functional,
radial functional, zeros of a minimizer.
©1999 Southwest Texas State University and University of North Texas.
Submitted June 8, 1999. Published September 9, 1999.

in $W^{1,p}(G,C)$ as $\tau_k \to 0$.

In this paper we assume that $d = \deg(g, \partial G) \neq 0$. Under this condition, if $1 < p < 2$, then, since $W^{1,p}_g(G,S^1)$ is nonempty, the existence of the p-harmonic u_p on G with given boundary value g and the convergence to u_p for a subsequence u_{ε_k} of u_ε in $W^{1,p}(G,C)$ as $\varepsilon_k \to 0$ can be proved similar to [9].

However if $p > 2$, then, since $d \neq 0, W^{1,p}_g(G,S^1)$ must be empty. In this case unlike the case $d = 0$ or $1 < p < 2$, it is impossible to have some subsequence of u_ε converging to a p-harmonic map on G. Under the condition $d \neq 0, p > 2$, the asymptotic analysis of the minimizers of $E_\varepsilon(u,G)$ seems to be a very difficult problem. In this paper, we assume that $G = B = \{x \in R^2; |x| < 1\}, g(x) = e^{id\theta}$, $x = (\cos\theta, \sin\theta)$ on $\partial B = S^1$ and consider the minimization of $E_\varepsilon(u,B)$ in the class of radial functions

$$u(x) = f(r)e^{id\theta} \in W^{1,p}_g(B,C), r = |x|$$

Such minimizers will be called radial minimizers.

Obviously, $u(x) = f(r)e^{id\theta} \in W^{1,p}_g(B,C)$ implies $f(1) = 1$. Notice that if $u(x) = f(r)e^{id\theta} \in W^{1,p}_g(B,C)$, then $|f(r)|e^{id\theta} \in W^{1,p}_g(B,C)$ and $E_\varepsilon(|f(r)|e^{id\theta}, B) = E_\varepsilon(f(r)e^{id\theta}, B)$. So, without loss of generality, we may choose the class of admissible functions as

$$W = \{u(x) = f(r)e^{id\theta} \in W^{1,p}(B,C); f(1) = 1, f(r) \geq 0\}.$$

In polar coordinates, for $u(x) = f(r)e^{id\theta}$ we have

$$|\nabla u| = (f_r^2 + d^2r^{-2}f^2)^{1/2},$$
$$\int_B |u|^p = 2\pi \int_0^1 r|f|^p\,dr,$$
$$\int_B |\nabla u|^p = 2\pi \int_0^1 r(f_r^2 + d^2r^{-2}f^2)^{p/2}\,dr.$$

It is easily seen that $f(r)e^{id\theta} \in W^{1,p}(B,C)$ implies $f(r)r^{\frac{1}{p}-1}, f_r(r)r^{\frac{1}{p}} \in L^p(0,1)$. Conversely, if $f(r) \in W^{1,p}_{\rm loc}(0,1], f(r)r^{\frac{1}{p}-1}, f_r(r)r^{\frac{1}{p}} \in L^p(0,1)$, then $f(r)e^{id\theta} \in W^{1,p}(B,C)$. Thus if we denote

$$\begin{aligned} V = \{f \in W^{1,p}_{\rm loc}(0,1]; \quad & r^{1/p}f_r \in L^p(0,1), r^{(1-p)/p}f \in L^p(0,1), \\ & f(1) = 1, f(r) \geq 0\} \end{aligned}$$

then $V = \{f(r); u(x) = f(r)e^{id\theta} \in W\}$.

Proposition 1.1 *The set V defined above is a subset of $\{f \in C[0,1]; f(0) = 0\}$.*

Proof. Let $f \in V, h(r) = f(r^{1+\frac{1}{p-2}})$.Then

$$\begin{aligned} \int_0^1 |h'(r)|^p\,dr &= (1 + \frac{1}{p-2})^p \int_0^1 |f'(r^{1+\frac{1}{p-2}})|^p r^{\frac{p}{p-2}}\,dr \\ &= (1 + \frac{1}{p-2})^p (1 - \frac{1}{p-1}) \int_0^1 s|f'(s)|^p\,ds < \infty \end{aligned}$$

which implies that $h(r) \in C[0,1]$ and hence $f(r) \in C[0,1]$.

Suppose $f(0) > 0$, then $f(r) \geq s > 0$ for $r \in [0,t)$ with $t > 0$ small enough. Since $p > 2$, we have

$$\int_0^1 r^{1-p} f^p \, dr \geq s^p \int_0^t r^{1-p} \, dr = \infty$$

which contradicts $r^{1/p-1} f \in L^p(0,1)$. Therefore $f(0) = 0$ and the proof is complete.

Substituting $u(x) = f(r)e^{id\theta} \in W$ into $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$, we obtain

$$E_\varepsilon(u,B) = 2\pi E_\varepsilon(f) \tag{1.3}$$

$$(E_\varepsilon^\tau(u,B) = 2\pi E_\varepsilon^\tau(f))$$

where

$$E_\varepsilon(f) = \int_0^1 [\frac{1}{p}(f_r^2 + d^2 r^{-2} f^2)^{p/2} + \frac{1}{4\varepsilon^p}(1-f^2)^2] r \, dr \tag{1.4}$$

$$(E_\varepsilon^\tau(f) = \int_0^1 [\frac{1}{p}(f_r^2 + d^2 r^{-2} f^2 + \tau)^{p/2} + \frac{1}{4\varepsilon^p}(1-f^2)^2] r \, dr)$$

This shows that $u = f(r)e^{id\theta} \in W$ is the minimizer of $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f)(E_\varepsilon^\tau(f))$.

Some basic properties of minimizers are given in §2. The main purpose of §3 is to prove that for any radial minimizer u_ε of $E_\varepsilon(u,B)$ and any given $\eta \in (0,1)$ there exists a constant $h(\eta) > 0$ such that

$$Z_\varepsilon = \{x \in B; |u_\varepsilon(x)| < 1-\eta\} \subset B(0,h\varepsilon) = \{x \in R^2; |x| < h\varepsilon\}.$$

(Theorem 3.5) which implies, in particular, that the zeroes of u_ε are contained in $B(0,h\varepsilon)$ and that

$$\lim_{\varepsilon \to 0} u_\varepsilon = e^{id\theta}, \quad in \ \ C_{\mathrm{loc}}(\overline{B} \setminus \{0\}, C)$$

In §4 the convergence rate for regularizable minimizers $\tilde{u}_\varepsilon$ is studied (Theorem 4.4). In §5 we prove the convergence of radial minimizers u_ε in $W_{\mathrm{loc}}^{1,p}(\overline{B} \setminus \{0\}, C)$ as $\varepsilon \to 0$ (Theorem 5.3) and the convergence of regularizable radial minimizers $\tilde{u}_\varepsilon$ in $C_{\mathrm{loc}}^{1,\alpha}(B \setminus \{0\}, C)$ as $\varepsilon \to 0$ (Theorem 5.4). Finally we indicate in §6 that our argument can be extended to the higher dimensional case.

2 Basic properties of minimizers

Proposition 2.1 *The functional $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$ achieves its minimum on W by a function $u_\varepsilon(x) = f_\varepsilon(r)e^{id\theta}(u_\varepsilon^\tau(x) = f_\varepsilon^\tau(r)e^{id\theta})$; $f_\varepsilon(r)(f_\varepsilon^\tau(r))$ is the minimizer of $E_\varepsilon(f)(E_\varepsilon^\tau(f))$.*

Proof. $W^{1,p}(B,C)$ is a reflexive Banach space. By a well-known result of Morrey (see for example [4]) $E_\varepsilon(u,B)$ is weakly lower-semi-continuous in $W^{1,p}_{\rm loc}(B,C)$. To prove the existence of the minimizers of $E_\varepsilon(u,B)$ in W, it suffices to verify that W is a weakly closed subset of $W^{1,p}(B,C)$. Clearly W is a convex subset of $W^{1,p}(B,C)$. Now we prove that W is a closed subset of $W^{1,p}(B,C)$.

Let $u_k = f_k(r)e^{id\theta} \in W$ and

$$\lim_{k\to\infty} u_k = u, \quad in \;\; W^{1,p}(B,C)$$

By the embedding theorem there exists a subsequence of u_k, supposed to be u_k itself, such that

$$\lim_{k\to\infty} u_k = u, \quad in \;\; C(\overline{B},C)$$

which implies

$$\lim_{k\to\infty} f_k = f, \quad in \;\; C[0,1]$$

and

$$u = f(r)e^{id\theta}$$

Combining this with $f_k(1) = 1, f_k(r) \geq 0$, we see that $f(1) = 1, f(r) \geq 0$. Thus $u \in W$. The existence of minimizers u_ε^τ of $E_\varepsilon^\tau(u,B)$ can be proved similarly.

Proposition 2.2 *The minimizer $f_\varepsilon(r)(f_\varepsilon^\tau(r))$ of the functional $E_\varepsilon(f)(E_\varepsilon^\tau(f))$ satisfies*

$$-(rAf')' + r^{-1}d^2Af = \frac{r}{\varepsilon^p}f(1-f^2), \quad A = (f_r^2 + r^{-2}d^2f^2)^{(p-2)/2} \tag{2.1}$$

in the following sense:

$$\begin{aligned} &\int_0^1 r(f_r^2 + r^{-2}d^2f^2)^{(p-2)/2}(f_r\phi_r + r^{-2}d^2f\phi)\,dr \\ &= \tfrac{1}{\varepsilon^p}\int_0^1 r(1-f^2)f\phi\,dr, \quad \forall\phi \in C_0^\infty(0,1) \end{aligned} \tag{2.2}$$

$$(-(rAf')' + r^{-1}d^2Af = \frac{r}{\varepsilon^p}f(1-f^2), \quad A = (f_r^2 + r^{-2}d^2f^2 + \tau)^{(p-2)/2} \tag{2.3}$$

in the classical sense).

By a limit process we see that the test function ϕ in (2.2) can be any member of

$$X = \{\phi(r) \in W^{1,p}_{\rm loc}(0,1]; \phi(0) = \phi(1) = 0, \phi(r) \geq 0, r^{\frac{1}{p}}\phi', r^{\frac{1}{p}-1}\phi \in L^p(0,1)\}$$

Proposition 2.3 *Let f_ε (f_ε^τ) be a nonnegative solution of (2.1)((2.3)) satisfying $f(0) = 0, f(1) = 1$ Then $f_\varepsilon \leq 1, (f_\varepsilon^\tau \leq 1)$ on [0,1].*

Proof. Denote $f = f_\varepsilon$ in (2.2) and set $\phi = f(f^2-1)_+$. Then

$$\int_0^1 r(f_r^2 + d^2r^{-2}f^2)^{(p-2)/2}[f_r^2(f^2-1)_+ + ff_r[(f^2-1)_+]_r + d^2r^{-2}f^2(f^2-1)_+]\,dr + \frac{1}{\varepsilon^p}\int_0^1 rf^2(f^2-1)_+^2\,dr = 0$$

from which it follows that

$$\frac{1}{\varepsilon^p}\int_0^1 rf^2(f^2-1)_+^2\,dr = 0$$

Thus $f=0$ or $(f^2-1)_+=0$ on $[0,1]$ and hence $f=f_\varepsilon \le 1$ on [0,1]. The proof of $f_\varepsilon^\tau \le 1$ is even easier.

Proposition 2.4 *Let $f_\varepsilon(f_\varepsilon^\tau)$ be a minimizer of $E_\varepsilon(f)(E_\varepsilon^\tau(f))$. Then*

$$E_\varepsilon(f_\varepsilon) \le C\varepsilon^{2-p}, (E_\varepsilon^\tau(f_\varepsilon^\tau) \le C\varepsilon^{2-p})$$

with a constant C independent of $\varepsilon \in (0,1)(\varepsilon, \tau \in (0,1))$.

Proof. Denote

$$I(\varepsilon,R) = Min\{\int_0^R [\frac{1}{p}(f_r^2 + \frac{d^2}{r^2}f^2)^{\frac{p}{2}} + \frac{1}{4\varepsilon^p}(1-f^2)^2]r\,dr; f \in V_R\}$$

where

$$V_R = \left\{f(r) \in W^{1,p}_{\rm loc}(0,R]; f(r) \ge 0, f(R) = 1, f(r)r^{\frac{1}{p}-1}, f'(r)r^{\frac{1}{p}} \in L^p(0,R)\right\}.$$

Then

$$\begin{aligned} I(\varepsilon,1) &= E_\varepsilon(f_\varepsilon) \\ &= \frac{1}{p}\int_0^1 r((f_\varepsilon)_r^2 + d^2r^{-2}f_\varepsilon^2)^{p/2}\,dr + \frac{1}{4\varepsilon^p}\int_0^1 r(1-f_\varepsilon^2)^2\,dr \\ &= \frac{1}{p}\int_0^{1/\varepsilon} \varepsilon^{2-p}s((f_\varepsilon)_s^2 + d^2s^{-2}f_\varepsilon^2)^{p/2}\,ds + \frac{1}{4\varepsilon^p}\int_0^{\varepsilon^{-1}} \varepsilon^2 s(1-f_\varepsilon^2)^2\,ds \\ &= \varepsilon^{2-p}I(1,\varepsilon^{-1}) \end{aligned}$$

Let f_1 be the minimizer for $I(1,1)$ and define

$$f_2 = f_1, 0 < s < 1; \quad f_2 = 1, 1 \le s \le \varepsilon^{-1}$$

We have

$$I(1,\varepsilon^{-1}) \le \frac{1}{p}\int_0^{\varepsilon^{-1}} s[(f_2')^2 + d^2s^{-2}f_2^2]^{p/2}\,ds + \frac{1}{4}\int_0^{\varepsilon^{-1}} s(1-f_2^2)\,ds$$

$$
\begin{aligned}
&\leq \frac{1}{p}\int_1^{\varepsilon^{-1}} s^{1-p}d^p\,ds + \frac{1}{p}\int_0^1 s((f_1')^2+d^2s^{-2}f_1^2)^{p/2}\,ds \\
&\quad +\frac{1}{4}\int_0^1 s(1-f_1^2)^2\,ds \\
&= \frac{d^p}{p(p-2)}(1-\varepsilon^{p-2})+I(1,1) \\
&\leq \frac{d^p}{p(p-2)}+I(1,1)=C
\end{aligned}
$$

Substituting into (2.4) follows the first conclusion of Proposition 2.4.

To prove another conclusion, note

$$
\begin{aligned}
E_\varepsilon^\tau(f_\varepsilon^\tau) &= \varepsilon^{2-p}[\frac{1}{p}\int_0^{1/\varepsilon} s((f_\varepsilon^\tau)_s^2+d^2s^{-2}(f_\varepsilon^\tau)^2+\varepsilon^2\tau)^{p/2}\,ds \\
&\quad +\frac{1}{4}\int_0^{\varepsilon^{-1}} s(1-(f_\varepsilon^\tau)^2)^2\,ds]
\end{aligned}
$$

Let f_1 be the minimizer for $I(1,1)$ and f_ε be the function defined above. Then

$$
\begin{aligned}
&E_\varepsilon^\tau(f_\varepsilon^\tau)\leq E_\varepsilon^\tau(f_\varepsilon) \\
&\leq \varepsilon^{2-p}[\tfrac{1}{p}\textstyle\int_0^{\varepsilon^{-1}} s[(f_2')^2+d^2s^{-2}f_2^2+\varepsilon^2\tau]^{p/2}\,ds+\tfrac{1}{4}\int_0^{\varepsilon^{-1}} s(1-f_2^2)^2\,ds] \\
&= \varepsilon^{2-p}[\tfrac{1}{p}\textstyle\int_1^{\varepsilon^{-1}} s[s^{-2}d^2+\varepsilon^2\tau]^{p/2}\,ds+\tfrac{1}{p}\int_0^1 s((f_1')^2+d^2s^{-2}f_1^2+\varepsilon^2\tau)^{p/2}\,ds \\
&\quad +\tfrac{1}{4}\textstyle\int_0^1 s(1-f_1^2)^2\,ds \\
&\leq \varepsilon^{2-p}[\tfrac{C}{p}\textstyle\int_1^{\varepsilon^{-1}} s[s^{-p}d^p+\varepsilon^p]^{p/2}\,ds+\tfrac{C}{p}\int_0^1 s[((f_1')^2+d^2s^{-2}f_1^2)^{p/2}+\varepsilon^p]\,ds \\
&\quad +\tfrac{1}{4}\textstyle\int_0^1 s(1-f_1^2)^2\,ds] \\
&\leq \varepsilon^{2-p}[CI(1,1)+C\varepsilon^p+C+C\varepsilon^{p-2}]\leq C\varepsilon^{2-p}
\end{aligned}
$$

The proof of Proposition 2.4 is complete.

3 Location of zeroes and C_{loc} convergence for minimizers

By the embedding theorem we first derive from Proposition 2.3 and Proposition 2.4 the following

Proposition 3.1 *Let $u_\varepsilon(u_\varepsilon^\tau)$ be a radial minimizer of $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$. Then there exists a constant C independent of $\varepsilon\in(0,1)(\varepsilon,\tau\in(0,1))$ such that*

$$|u_\varepsilon(x)-u_\varepsilon(x_0)|\leq C\varepsilon^{(2-p)/p}|x-x_0|^{1-2/p},\quad \forall x,x_0\in B$$

$$(|u_\varepsilon^\tau(x)-u_\varepsilon^\tau(x_0)|\leq C\varepsilon^{(2-p)/p}|x-x_0|^{1-2/p})\quad \forall x,x_0\in B$$

As a corollary of Proposition 2.4 we have

Proposition 3.2 *Let $u_\varepsilon(u_\varepsilon^\tau)$ be a radial minimizer of $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$. Then for some constant C independent of $\varepsilon(\varepsilon,\tau)\in(0,1]$*

$$\frac{1}{\varepsilon^2}\int_B(1-|u_\varepsilon|^2)^2\le C \tag{3.1}$$

$$(\frac{1}{\varepsilon^2}\int_B(1-|u_\varepsilon^\tau|^2)^2\le C)$$

Based on Proposition 3.1, we have the following interesting result:

Proposition 3.3 *Let $u_\varepsilon(u_\varepsilon^\tau)$ be a radial minimizer of $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$. Then for any $\eta\in(0,1)$, there exist positive constants λ,μ independent of $\varepsilon(\varepsilon,\tau)\in(0,1)$ such that if*

$$\frac{1}{\varepsilon^2}\int_{B\cap B^{2l\varepsilon}}(1-|u_\varepsilon|^2)^2\le\mu \tag{3.2}$$

$$(\frac{1}{\varepsilon^2}\int_{B\cap B^{2l\varepsilon}}(1-|u_\varepsilon^\tau|^2)^2\le\mu)$$

where $B^{2l\varepsilon}$ is some disc of radius $2l\varepsilon$ with $l\ge\lambda$, then

$$|u_\varepsilon(x)|\ge 1-\eta,\quad \forall x\in B\cap B^{l\varepsilon} \tag{3.3}$$

$$(|u_\varepsilon^\tau(x)|\ge 1-\eta,\quad \forall x\in B\cap B^{l\varepsilon})$$

Proof. First we observe that there exists a constant $\beta>0$ such that for any $x\in B$ and $0<\rho\le 1$,

$$mes(B\cap B(x,\rho))\ge\beta\rho^2$$

To prove the proposition, we choose

$$\lambda=(\frac{\eta}{2C})^{\frac{p}{p-2}},\quad \mu=\frac{\beta}{4}(\frac{1}{2C})^{\frac{2p}{p-2}}\eta^{2+\frac{2p}{p-2}}$$

where C is the constant in Proposition 3.1.

Suppose that there is a point $x_0\in B\cap B^{l\varepsilon}$ such that $|u_\varepsilon(x_0)|<1-\eta$. Then applying Proposition 3.1 we have

$$\begin{aligned}|u_\varepsilon(x)-u_\varepsilon(x_0)| &\le C\varepsilon^{(2-p)/p}|x-x_0|^{1-2/p}\le C\varepsilon^{(2-p)/p}(\lambda\varepsilon)^{1-2/p}\\ &= C\lambda^{1-2/p}=\frac{\eta}{2},\quad \forall x\in B(x_0,\lambda\varepsilon)\end{aligned}$$

Hence

$$(1-|u_\varepsilon(x)|^2)^2>\frac{\eta^2}{4},\quad \forall x\in B(x_0,\lambda\varepsilon)$$

$$\begin{aligned}&\textstyle\int_{B(x_0,\lambda\varepsilon)\cap B}(1-|u_\varepsilon|^2)^2>\frac{\eta^2}{4}mes(B\cap B(x_0,\lambda\varepsilon))\\ &\ge\beta\frac{\eta^2}{4}(\lambda\varepsilon)2=\beta\frac{\eta^2}{4}(\frac{\eta}{2C})^{\frac{2p}{p-2}}\varepsilon^2=\mu\varepsilon^2\end{aligned} \tag{3.4}$$

Since $x_0 \in B^{l\varepsilon} \cap B$, and $(B(x_0, \lambda\varepsilon) \cap B) \subset (B^{2l\varepsilon} \cap B)$, (3.4) implies

$$\int_{B^{2l\varepsilon} \cap B} (1 - |u_\varepsilon|^2)^2 > \mu\varepsilon^2$$

which contradicts (3.2) and thus the proposition is proved.

Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Given $\eta \in (0, 1)$. Let λ, μ be constants in Proposition 3.3 corresponding to η. If

$$\frac{1}{\varepsilon^2} \int_{B(x^\varepsilon, 2\lambda\varepsilon) \cap B} (1 - |u_\varepsilon|^2)^2 \le \mu \tag{3.5}$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called $\eta-$ good disc, or simply good disc. Otherwise $B(x^\varepsilon, \lambda\varepsilon)$ is called $\eta-$ bad disc or simply bad disc.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of discs satisfying

$$(i) : x_i^\varepsilon \in B, i \in I; \quad (ii) : B \subset \cup_{i \in I} B(x_i^\varepsilon, \lambda\varepsilon)$$
$$(iii) : B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \ne j \tag{3.6}$$

Denote

$$J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon) \;\; is \;\; a \;\; bad \;\; disc\}$$

Proposition 3.4 *There exists a positive integer N such that the number of bad discs* card $J_\varepsilon \le N$

Proof. Since (3.6) implies that every point in B can be covered by finite, say m (independent of ε) discs, from (3.2) and the definition of bad discs,we have

$$\begin{aligned} \mu\varepsilon^2 \operatorname{card} J_\varepsilon &\le \sum_{i \in J_\varepsilon} \int_{B(x_i^\varepsilon, 2\lambda\varepsilon) \cap B} (1 - |u_\varepsilon|^2)^2 \\ &\le m \int_{\cup_{i \in J_\varepsilon} B(x_i^\varepsilon, 2\lambda\varepsilon) \cap B} (1 - |u_\varepsilon|^2)^2 \\ &\le m \int_B (1 - |u_\varepsilon|^2)^2 \le mC\varepsilon^2 \end{aligned}$$

and hence card $J_\varepsilon \le \frac{mC}{\mu} \le N$.

Applying Theorem IV.1 in [2], we may modify the family of bad discs such that the new one, denoted by $\{B(x_i^\varepsilon, h\varepsilon); i \in J\}$, satisfies

$$\cup_{i \in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \cup_{i \in J} B(x_i^\varepsilon, h\varepsilon),$$
$$\lambda \le h; \quad \operatorname{card} J \le \operatorname{card} J_\varepsilon \tag{3.7}$$
$$|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, i, j \in J, i \ne j$$

The last condition implies that every two discs in the new family are Dis-intersected.

The argument on the good and bad discs can be applied to the radial minimizer u_ε^τ of $E_\varepsilon^\tau(u,B)$. In particular, we may obtain a family of discs $\{B(x_i^{\varepsilon,\tau},\lambda\varepsilon), i\in I\}$ such that the number of bad discs is bounded by a positive integer N independent of both $\varepsilon\in(0,1)$ and $\tau\in(0,1)$. The family of bad discs can be modified such that the new one satisfies the conditions corresponding to (3.7).

Now we prove our main result of this section.

Theorem 3.5 *Let $u_\varepsilon(u_\varepsilon^\tau)$ be a radial minimizer of $E_\varepsilon(u,B)(E_\varepsilon^\tau(u,B))$. Then for any $\eta\in(0,1)$, there exists a constant $h=h(\eta)$ independent of $\varepsilon(\varepsilon,\tau)\in(0,1)$ such that $Z_\varepsilon=\{x\in B;|u_\varepsilon(x)|<1-\eta\}\subset B(0,h\varepsilon)(Z_\varepsilon^\tau=\{x\in B;|u_\varepsilon^\tau(x)|<1-\eta\}\subset B(0,h\varepsilon))$.In particular the zeroes of $u_\varepsilon(u_\varepsilon^\tau)$ are contained in $B(0,h\varepsilon)$.*

Proof. Suppose there exists a point $x_0\in Z_\varepsilon$ such that $x_0\overline{\in}B(0,h\varepsilon)$. Then all points on the circle
$$S_0=\{x\in B;\ |x|=|x_0|\}$$
satisfy $|u_\varepsilon(x)|<1-\eta$ and hence by virtue of Proposition 3.3 all points on S_0 are contained in bad discs. However, since $|x_0|\geq h\varepsilon, S_0$ can not be covered by a single bad disc. S_0 can be covered by at least two bad discs. However this is impossible. The same is true for u_ε^τ.

Theorem 3.6 *Let $u_\varepsilon=f_\varepsilon(r)e^{id\theta}$ be a radial minimizer of $E_\varepsilon(u,B)$. Then*
$$\lim_{\varepsilon\to0}f_\varepsilon=1,\quad in\quad C_{\rm loc}((0,1],R)$$
$$\lim_{\varepsilon\to0}u_\varepsilon=e^{id\theta},\quad in\quad C_{\rm loc}(\overline{B}\setminus\{0\},C)$$

4 Convergence rate for minimizers

Proposition 4.1 *Let u_ε^τ be a radial minimizer of $E_\varepsilon^\tau(u,B)$. Then there exists a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ with $\tau_k\to0$ such that*
$$\lim_{\tau_k\to0}u_\varepsilon^{\tau_k}=\tilde u_\varepsilon,\quad in\quad W^{1,p}(B,C)\tag{4.1}$$
and $\tilde u_\varepsilon$ is a radial minimizer of $E_\varepsilon(u,B)$.

Proof. Since $u_\varepsilon\in W$ and u_ε^τ is a radial minimizer of $E_\varepsilon^\tau(u,B)$ in W, we have
$$E_\varepsilon^\tau(u_\varepsilon^\tau,B)\leq E_\varepsilon^\tau(u_\varepsilon,B)\leq C$$
with a constant C independent of $\tau\in(0,1)$. This and $|u_\varepsilon^\tau|\leq1$ on $\overline{B}$ imply the existence of a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ with $\tau_k\to0$ and a function $\tilde u_\varepsilon\in W^{1,p}(B,C)$ such that
$$\lim_{\tau_k\to0}u_\varepsilon^{\tau_k}=\tilde u_\varepsilon,\quad weakly\quad in\quad W^{1,p}(B,C)\tag{4.2}$$

$$\lim_{\tau_k\to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon, \quad in \ \ C(\overline{B}, C) \tag{4.3}$$

Thus, $\tilde{u}_\varepsilon \in W$ and we have

$$\liminf_{\tau_k\to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B) \le \limsup_{\tau_k\to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B) \le \lim_{\tau_k\to 0} E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon, B)$$

$$\lim_{\tau_k\to 0}\int_B (1-|u_\varepsilon^{\tau_k}|^2)^2 = \int_B (1-|\tilde{u}_\varepsilon|^2)^2$$

Hence

$$\begin{aligned} &\liminf\nolimits_{\tau_k\to 0} \textstyle\int_B (|\nabla u_\varepsilon^{\tau_k}|^2+\tau_k)^{p/2} \le \limsup\nolimits_{\tau_k\to 0} \textstyle\int_B (|\nabla u_\varepsilon^{\tau_k}|^2+\tau_k)^{p/2} \\ &\le \lim\nolimits_{\tau_k\to 0} \textstyle\int_B (|\nabla \tilde{u}_\varepsilon|^2+\tau_k)^{p/2} = \int_B |\nabla \tilde{u}_\varepsilon|^p \end{aligned} \tag{4.4}$$

On the other hand, (4.2) and the lower semicontinuity of $\int_B |\nabla v|^p$ imply

$$\int_B |\nabla \tilde{u}_\varepsilon|^p \le \liminf_{\tau_k\to 0} \int_B |\nabla u_\varepsilon^{\tau_k}|^p$$

¿From this and (4.4) we obtain

$$\lim_{\tau_k\to 0}\int_B |\nabla u_\varepsilon^{\tau_k}|^p = \int_B |\nabla \tilde{u}_\varepsilon|^p$$

which combined with (4.2) gives

$$\lim_{\tau_k\to 0}\int_B |\nabla (u_\varepsilon^{\tau_k} - \tilde{u}_\varepsilon)|^p = 0 \tag{4.5}$$

(4.1) follows from (4.3) and (4.5).

For any $v \in W$, we have

$$E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B) \le E_\varepsilon^{\tau_k}(v, B)$$

Letting $\tau_k \to 0$ and noticing that

$$\lim_{\tau_k\to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B) = E_\varepsilon(\tilde{u}_\varepsilon, B)$$

we are led to $E_\varepsilon(\tilde{u}_\varepsilon, B) \le E_\varepsilon(v, B)$ Thus $\tilde{u}_\varepsilon$ is a radial minimizer of $E_\varepsilon(u, B)$

Proposition 4.2 *Let f_ε^τ be a minimizer of the regularized functional $E_\varepsilon^\tau(f)$ in V. Then there exist a subsequence $f_\varepsilon^{\tau_k}$ of f_ε^τ with $\tau_k \to 0$ and a function $\tilde{f}_\varepsilon \in V$, such that*

$$\lim_{\tau_k\to 0}\int_0^1 r(f_\varepsilon^{\tau_k} - \tilde{f}_\varepsilon)_r^p \, dr = 0;$$

$\tilde{f}_\varepsilon$ is a minimizer of $E_\varepsilon(f)$ in V.

Now we prove the main result of this section.

Theorem 4.3 *Suppose $p > 4$. Let $\tilde{f}_\varepsilon$ be a regularizable minimizer of $E_\varepsilon(f)$. Then there exists a constant C independent of $\varepsilon \in (0,1)$ such that*

$$\|(\tilde{f}_\varepsilon)'\|_{L^2(r_0,r_1)} \le C(r_0, r_1)\varepsilon \tag{4.6}$$

where $[r_0, r_1]$ is an arbitrary closed interval of (0,1).

Proof. Substitute $f = f_\varepsilon^\tau$ into (2.3) and let $w = 1 - f$. Then w satisfies

$$w - \varepsilon^p(2-w)^{-1}(1-w)^{-1}[(Aw')' + Ar^{-1}w' + d^2r^{-2}A(1-w)] = 0$$

Differentiate with respect to r, multiply by $rw'\zeta^2$ with $\zeta \in C_0^\infty(0,1)$, such that $0 \le \zeta \le 1$ on $[0,1], \zeta = 1$ on $[t_1, t_2], \zeta = 0$ on $[0,1] - [t, t_3]$, where $0 < t < t_1 < t_2 < t_3 < 1, |\zeta'| \le C$, and integrate over $(0,1)$. Then we have

$$\begin{aligned}\int_t^1 r(w')^2\zeta^2\,dr + \varepsilon^p \int_t^1 (rw'\zeta^2)'(2-w)^{-1}(1-w)^{-1}\\ \cdot[(Aw')' + Ar^{-1}w' + d^2r^{-2}A(1-w)] = 0\end{aligned} \tag{4.7}$$

¿From Theorem 3.5, f has a positive uniform lower bound on $[t,1]$ for $\varepsilon > 0$ small enough. Hence

$$C^{-1} \le (2-w)^{-1}(1-w)^{-1} \le C$$

for some constant $C > 0$ independent of $\varepsilon \in (0,\eta), \tau \in (0,1)$. Substituting

$$A' = (p-2)A^{\frac{p-4}{p-2}} \cdot (w'w'' - d^2r^{-2}(1-w)w' - 2(1-w)^2d^2r^{-3})$$

into (4.7), we obtain

$$\begin{aligned}&\int_t^1 r(w')^2\zeta^2\,dr + \frac{\varepsilon^p}{C}\int_t^1 rA(w'')^2\zeta^2\,dr + \frac{p-2}{C}\varepsilon^p\int_t^1 r(w'w'')^2A^{\frac{p-4}{p-2}}\zeta^2\,dr\\ &\le C\varepsilon^p \textstyle\int_t^1 [Aw'w''\zeta^2 + d^2r^{-1}A(1-w)w''\zeta^2\\ &\quad + (w'\zeta^2 + 2\zeta\zeta'rw')(A'w' + Aw'' + r^{-1}Aw' + d^2r^{-2}A(1-w))\\ &\quad - (p-2)A^{\frac{p-4}{p-2}}rw'w''\zeta^2(d^2r^{-2}(1-w)w' - 2(1-w)^2d^2r^{-3})]\,dr\end{aligned}$$

and after putting in order

$$\begin{aligned}&\int_t^1 r(w')^2\zeta^2\,dr + \frac{\varepsilon^p}{Ct}\int_t^1 A(w'')^2\zeta^2\,dr + \frac{p-2}{Ct}\varepsilon^p\int_t^1 (w'w'')^2A^{\frac{p-4}{p-2}}\zeta^2\,dr\\ &\le C(t,d)\varepsilon^p \textstyle\int_t^1 [Aw'w''(\zeta^2 + \zeta\zeta') + Aw''\zeta^2\\ &\quad + A(w')^2(\zeta^2 + \zeta\zeta') + Aw'(\zeta^2 + \zeta\zeta')]\,dr\\ &\quad + C(t,d,p)\varepsilon^p \textstyle\int_t^1 A^{\frac{p-4}{p-2}}[(w')^3w''(\zeta^2 + \zeta\zeta') + (w')^3(\zeta^2 + \zeta\zeta')\\ &\quad + (w')^2(\zeta^2 + \zeta\zeta') + (w')^2w''\zeta^2 + w'w''\zeta^2]\,dr\\ &= C(t,d)\varepsilon^pJ_1 + C(t,d,p)\varepsilon^pJ_2\end{aligned} \tag{4.8}$$

Using the Young inequality we see that for any $\delta \in (0,1)$

$$J_1 \le \delta\int_t^1 A(w'')^2\zeta^2\,dr + C(\delta)\int_t^1 A[(w')^2 + 1]\,dr \tag{4.9}$$

Noticing that $p > 4$ and using the Young inequality again we have for any $\delta \in (0,1)$

$$\begin{aligned} J_2 &\le \delta \textstyle\int_t^1 A^{\frac{p-4}{p-2}}(w'w'')^2\zeta^2\,dr + C(\delta)\textstyle\int_t^1 A^{\frac{p-4}{p-2}}[(w')^4+1]\,dr \\ &\le \delta \textstyle\int_t^1 A^{\frac{p-4}{p-2}}(w'w'')^2\zeta^2\,dr + C(\delta)\textstyle\int_t^1 (A^{\frac{p}{p-2}}+1)\,dr \end{aligned} \tag{4.10}$$

Combining (4.8) with (4.9)(4.10) and choosing δ small enough we are led to

$$\int_t^1 r(w')^2\zeta^2\,dr + \varepsilon^p \int_t^1 A(w'')^2\zeta^2\,dr$$

$$+\varepsilon^p \int_t^1 (w'w'')^2 A^{\frac{p-4}{p-2}}\zeta^2\,dr \le C\varepsilon^p(1+\int_t^1 A^{\frac{p}{p-2}}\,dr)$$

In particular

$$\begin{aligned} &\textstyle\int_t^1 r(w')^2\zeta^2\,dr \le C\varepsilon^p(\textstyle\int_t^1 A^{\frac{p}{p-2}}\,dr+1) \\ &\le C\varepsilon^p(1+t^{-1}\textstyle\int_t^1 rA^{\frac{p}{p-2}}\,dr) \le C(t)\varepsilon^{2-p} \end{aligned}$$

Here Proposition 2.4 is applied. Thus we have

$$\int_{t_1}^{t_2} (w')^2 r\,dr \le C\varepsilon^2$$

namely

$$\int_{t_1}^{t_2} (f_\varepsilon^\tau)_r^2 r\,dr \le C\varepsilon^2 \tag{4.11}$$

As a regularizable minimizer of $E_\varepsilon(f)$, $\tilde f_\varepsilon$ is the limit of a subsequence $f_\varepsilon^{\tau_k}$ of f_ε^τ in the sense of Proposition 4.2. Therefore, taking $\tau = \tau_k$ in (4.11) and letting $\tau_k \to 0$, we finally obtain

$$\int_{t_1}^{t_2} (\tilde f_\varepsilon)_r^2\,dr \le Ct_1^{-1}\varepsilon^2$$

which is just (4.6).

It follows from Theorem 4.3 immediately

Theorem 4.4 *Suppose $p > 4$. Let $\tilde u_\varepsilon = \tilde f_\varepsilon e^{id\theta}$ be a regularizable radial minimizer of $E_\varepsilon(u,B)$. Then there exists a constant C independent of ε, such that*

$$\|1-\tilde f_\varepsilon\|_{H^1(r_0,r_1)} \le C(r_0,r_1)\varepsilon$$

$$\|\tilde u_\varepsilon - e^{id\theta}\|_{H^1(K,C)} \le C(K)\varepsilon$$

where $[r_0,r_1]$ is an arbitrary closed interval of (0,1) and K is an arbitrary compact subset of $B \setminus \{0\}$.

5 $W^{1,p}_{\rm loc}$ convergence and $C^{1,\alpha}_{\rm loc}$ convergence for minimizers

Let $u_\varepsilon(x) = f_\varepsilon(r)e^{id\theta}$ be a radial minimizer of $E_\varepsilon(u, B)$, namely f_ε be a minimizer of

$$E_\varepsilon(f) = \frac{1}{p}\int_0^1 (f_r^2 + d^2r^{-2}f^2)^{p/2} r\,dr + \frac{1}{4\varepsilon^p}\int_0^1 (1-f^2)^2 r\,dr$$

in V. From Proposition 2.4, we have

$$E_\varepsilon(f_\varepsilon) \le C\varepsilon^{2-p} \tag{5.1}$$

for some constant C independent of $\varepsilon \in (0,1)$.

In this section we further prove that for any $\eta \in (0,1)$, there exists a constant $C(\eta)$ such that

$$E_\varepsilon(f_\varepsilon;\eta) \le C(\eta) \tag{5.2}$$

for $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 > 0$ small be enough, where

$$E_\varepsilon(f_\varepsilon;\eta) = \frac{1}{p}\int_\eta^1 (f_r^2 + d^2r^{-2}f^2)^{p/2} r\,dr + \frac{1}{4\varepsilon^p}\int_\eta^1 (1-f^2)^2 r\,dr$$

In fact we can prove a more accurate estimate on $E_\varepsilon(f_\varepsilon;\eta)$ (see Proposition 5.2). Based on this estimate and Theorem 3.5, we may obtain better convergence for minimizers, namely the $W^{1,p}_{\rm loc}$ convergence and $C^{1,\alpha}_{\rm loc}$ convergence.

We first prove

Proposition 5.1 *Given $\eta \in (0,1)$. There exist constants*

$$\eta_j \in [\frac{(j-1)\eta}{N+1}, \frac{j\eta}{N+1}], (N = [p])$$

and C_j, such that

$$E_\varepsilon(f_\varepsilon, \eta_j) \le C_j\varepsilon^{j-p} \tag{5.3}$$

for $j = 2, ..., N$, where $\varepsilon \in (0, \varepsilon_0)$.

Proof. For $j = 2$, the inequality (5.3) is just the one in Proposition 2.4.

Suppose that (5.3) holds for all $j \le n$. Then we have, in particular

$$E_\varepsilon(f_\varepsilon;\eta_n) \le C_n\varepsilon^{n-p} \tag{5.4}$$

If $n = N$ then we are done. Suppose $n < N$. We want to prove (5.3) for $j = n+1$.

Obviously (5.4) implies

$$\begin{aligned}\tfrac{1}{p}\int_{\frac{n\eta}{N+1}}^{\frac{(n+1)\eta}{N+1}} [(f_\varepsilon)_r^2 + d^2r^{-2}f_\varepsilon^2]^{p/2} r\,dr \quad &+ \tfrac{1}{4\varepsilon^p}\int_{\frac{n\eta}{N+1}}^{\frac{(n+1)\eta}{N+1}} (1-f_\varepsilon^2)^2 r\,dr \\ &\le C_n\varepsilon^{n-p}\end{aligned}$$

from which we see by integral mean value theorem that there exists

$$\eta_{n+1} \in [\frac{n\eta}{N+1}, \frac{(n+1)\eta}{N+1}]$$

such that

$$[(f_\varepsilon)_r^2 + d^2 r^{-2} f_\varepsilon^2]_{r=\eta_{n+1}} \le C_n \varepsilon^{n-p} \tag{5.5}$$

$$[\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]_{r=\eta_{n+1}} \le C_n \varepsilon^{n-p} \tag{5.6}$$

Consider the functional

$$E(\rho, \eta_{n+1}) = \frac{1}{p}\int_{\eta_{n+1}}^1 (\rho_r^2+1)^{p/2}\,dr + \frac{1}{2\varepsilon^p}\int_{\eta_{n+1}}^1 (1-\rho)^2\,dr$$

It is easy to prove that the minimizer ρ_1 of $E(\rho, \eta_{n+1})$ on $W^{1,p}_{f_\varepsilon}((\eta_{n+1},1), R^+)$ exists and satisfies

$$-\varepsilon^p (v^{(p-2)/2}\rho_r)_r = 1-\rho, \quad in \ \ (\eta_{n+1}, 1) \tag{5.7}$$

$$\rho|_{r=\eta_{n+1}} = f_\varepsilon, \quad \rho|_{r=1} = f_\varepsilon(1) = 1 \tag{5.8}$$

where $v = \rho_r^2 + 1$.

Applying Theorem 3.5 and (5.4) we see easily that

$$E(\rho_1; \eta_{n+1}) \le E(f_\varepsilon; \eta_{n+1}) \le C_n E_\varepsilon(f_\varepsilon; \eta_{n+1}) \le C_n \varepsilon^{n-p} \tag{5.9}$$

for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ small enough.

Since $f_\varepsilon \le 1$, it follows from the maximum principle

$$\rho_1 \le 1 \tag{5.10}$$

Now choosing a smooth function $\zeta(r)$ such that $\zeta = 1$ on $(0,\eta), \zeta = 0$ near $r = 1$, multiplying (5.7) by $\zeta\rho_r(\rho = \rho_1)$ and integrating over $(\eta_{n+1}, 1)$ we obtain

$$\begin{aligned} v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}} \quad &+ \textstyle\int_{\eta_{n+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\,dr \\ &= \textstyle\frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 (1-\rho)\zeta\rho_r\,dr \end{aligned} \tag{5.11}$$

Using (5.9) we have

$$\begin{aligned} &|\textstyle\int_{\eta_{n+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\,dr| \\ &\le \textstyle\int_{\eta_{n+1}}^1 v^{(p-2)/2}|\zeta_r|\rho_r^2\,dr + \frac{1}{p}|\int_{\eta_{n+1}}^1 (v^{p/2}\zeta)_r\,dr - \int_{\eta_{n+1}}^1 v^{p/2}\zeta_r\,dr| \\ &\le C\textstyle\int_{\eta_{n+1}}^1 v^{p/2} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}} + \frac{C}{p}\int_{\eta_{n+1}}^1 v^{p/2} \\ &\le C\textstyle\int_{\eta_{n+1}}^1 v^{p/2} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}} \le C_n\varepsilon^{n-p} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}} \end{aligned} \tag{5.12}$$

and using (5.6)(5.9) we have

$$\begin{aligned} &|\tfrac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 (1-\rho)\zeta\rho_r\,dr| = \tfrac{1}{2\varepsilon^p}|\int_{\eta_{n+1}}^1((1-\rho)^2\zeta)_r\,dr - \int_{\eta_{n+1}}^1(1-\rho)^2\zeta_r\,dr| \\ &\le \tfrac{1}{2\varepsilon^p}(1-\rho)^2|_{r=\eta_{n+1}} + \tfrac{C}{2\varepsilon^p}\int_{\eta_{n+1}}^1(1-\rho)^2\,dr| \le C_n\varepsilon^{n-p} \end{aligned} \tag{5.13}$$

Combining (5.11) with (5.12)(5.13) yields

$$v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}} \le C_n\varepsilon^{n-p} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}$$

Hence

$$\begin{aligned} v^{p/2}|_{r=\eta_{n+1}} &= v^{(p-2)/2}(\rho_r^2+1)|_{r=\eta_{n+1}} = v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}} + v^{(p-2)/2}|_{r=\eta_{n+1}} \\ &\le C_n\varepsilon^{n-p} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}} + v^{(p-2)/2}|_{r=\eta_{n+1}} \\ &\le C_n\varepsilon^{n-p} + \frac{1}{p}v^{p/2}|_{r=\eta_{n+1}} + \delta v^{p/2}|_{r=\eta_{n+1}} + C(\delta) \\ &= C_n\varepsilon^{n-p} + (\frac{1}{p}+\delta)v^{p/2}|_{r=\eta_{n+1}} + C(\delta) \end{aligned}$$

from which it follows by choosing $\delta > 0$ small enough that

$$v^{p/2}|_{r=\eta_{n+1}} \le C_n\varepsilon^{n-p} \tag{5.14}$$

Now we multiply both sides of (5.7) by $\rho - 1$ and integrate. Then

$$-\varepsilon^p\int_{\eta_{n+1}}^1[v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr + \varepsilon^p\int_{\eta_{n+1}}^1 v^{(p-2)/2}\rho_r^2\,dr + \int_{\eta_{n+1}}^1(\rho-1)^2\,dr = 0$$

¿From this, using(5.8)(5.14)(5.6) and noticing that $n < p$, we obtain

$$\begin{aligned} &E(\rho_1;\eta_{n+1}) \le |\int_{\eta_{n+1}}^1[v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr| \\ &= v^{(p-2)/2}\rho_r|\rho-1|_{r=\eta_{n+1}} \le v^{(p-1)/2}|\rho-1|_{r=\eta_{n+1}} \\ &\le (C_n\varepsilon^{n-p})^{(p-1)/p}(C_n\varepsilon^n)^{1/2} \le C_{n+1}\varepsilon^{n+1-p+(n/2-n/p)} \end{aligned}$$

which implies

$$E(\rho_1;\eta_{n+1}) \le C_{n+1}\varepsilon^{n+1-p} \tag{5.15}$$

Define

$$w_\varepsilon = f_\varepsilon,\ \ for\ r \in (0,\eta_{n+1});\quad w_\varepsilon = \rho_1,\ \ for\ r\in[\eta_{n+1},1]$$

Since f_ε is a minimizer of $E_\varepsilon(f)$, we have

$$E_\varepsilon(f_\varepsilon) \le E_\varepsilon(w_\varepsilon)$$

namely

$$\begin{aligned} E_\varepsilon(f_\varepsilon;\eta_{n+1}) &\le \frac{1}{p}\int_{\eta_{n+1}}^1 (\rho_r^2+d^2r^{-2}\rho^2)^{p/2}r\,dr + \frac{1}{4e^p}\int_{\eta_{n+1}}^1 (1-\rho_r^2)^2 r\,dr \\ &\le \frac{C}{p}\int_{\eta_{n+1}}^1 (\rho_r^2+1)^{p/2}\,dr + \frac{C}{2\varepsilon^p}\int_{\eta_{n+1}}^1 (1-\rho_r)^2\,dr + C \\ &= CE(\rho_1;\eta_{n+1})+C \end{aligned}$$

Thus, using (5.15) yields

$$E_\varepsilon(f_\varepsilon;\eta_{n+1}) \le C_{n+1}\varepsilon^{n-p+1}$$

for $\varepsilon\in(0,\varepsilon_0)$. This is just (5.3) for $j=n+1$.

Proposition 5.2 *Given $\eta\in(0,1)$. There exist constants $\eta_{N+1}\in[\frac{N\eta}{N+1},\eta]$ and C_{N+1} such that*

$$E_\varepsilon(f_\varepsilon;\eta_{N+1}) \le C_{N+1}\varepsilon^{2(N-p+1)/p} + \frac{1}{p}\int_{\eta_{N+1}}^1 \frac{d^p}{r^{p-1}}\,dr \tag{5.16}$$

where $N=[p]$.

Proof. Similar to the derivation of (5.6) we may obtain from Proposition 5.1 for $j=N$ that there exists $\eta_{N+1}\in[\frac{N\eta}{N+1},\frac{(N+1)\eta}{N+1}]$, such that

$$\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2|_{r=\eta_{N+1}} \le C_N\varepsilon^{N-p} \tag{5.17}$$

Also similarly, consider the functional

$$E(\rho,\eta_{N+1}) = \frac{1}{p}\int_{\eta_{N+1}}^1 (\rho_r^2+1)^{p/2}\,dr + \frac{1}{2\varepsilon^p}\int_{\eta_{N+1}}^1 (1-\rho)^2\,dr$$

whose minimizer ρ_2 on $W^{1,p}_{f_\varepsilon}((\eta_{N+1},1),R^+)$ exists and satisfies

$$-\varepsilon^p(v^{(p-2)/2}\rho_r)_r = 1-\rho, \quad in \ \ (\eta_{N+1},1) \tag{5.18}$$

$$\rho|_{r=\eta_{N+1}} = f_\varepsilon, \quad \rho|_{r=1} = f_\varepsilon(1)=1$$

where $v=\rho_r^2+1$. By the maximum principle we have

$$\rho_2\le 1 \tag{5.19}$$

¿From (5.4) for $n=N$ it follows immediately that

$$E(\rho_2;\eta_{N+1}) \le E(f_\varepsilon;\eta_{N+1}) \le C_N E_\varepsilon(f_\varepsilon;\eta_{N+1}) \le C_N E_\varepsilon(f_\varepsilon;\eta_N) \le C_N\varepsilon^{N-p} \tag{5.20}$$

Similar to the proof of (5.14) and (5.15), we get from (5.17) that

$$v^{p/2}|_{r=\eta_{N+1}} \le C_N \varepsilon^{N-p}$$
$$E(\rho_2;\eta_{N+1}) \le C_{N+1}\varepsilon^{N+1-p} \tag{5.21}$$

Now we define

$$w_\varepsilon = f_\varepsilon,\ for\ r\in(0,\eta_{N+1});\quad w_\varepsilon=\rho_2,\ for\ r\in[\eta_{N+1},1]$$

and then we have

$$E_\varepsilon(f_\varepsilon) \le E_\varepsilon(w_\varepsilon)$$

Notice that

$$\begin{aligned}
&\int_{\eta_{N+1}}^1 (\rho_r^2+d^2r^{-2}\rho^2)^{p/2} r\,dr - \int_{\eta_{N+1}}^1 (d^2r^{-2})^{p/2}\,dr\\
&= \tfrac{p}{2}\int_{\eta_{N+1}}^1\int_0^1 [(\rho_r^2+d^2r^{-2}\rho^2)s+(d^2r^{-2}\rho^2)(1-s)]^{(p-2)/2}\,ds\rho_r^2 r\,dr\\
&\le C\int_{\eta_{N+1}}^1\int_0^1 [(\rho_r^2+d^2r^{-2}\rho^2)^{(p-2)/2}s^{(p-2)/2}\\
&\quad + (d^2r^{-2}\rho^2)^{(p-2)/2}(1-s)^{(p-2)/2}]\,ds\rho_r^2 r\,dr\\
&= C\int_{\eta_{N+1}}^1 (\rho_r^2+d^2r^{-2}\rho^2)^{(p-2)/2}\rho_r^2 r\,dr\int_0^1 s^{(p-2)/2}\,ds\\
&\quad + C\int_{\eta_{N+1}}^1 (d^2r^{-2}\rho^2)^{(p-2)/2}\rho_r^2 r\,dr\int_0^1 (1-s)^{(p-2)/2}\,ds\\
&\le C\Big(\int_{\eta_{N+1}}^1 \rho_r^p\,dr+\int_{\eta_{N+1}}^1\rho_r^2\,dr\Big)
\end{aligned}$$

Hence

$$\begin{aligned}
&E_\varepsilon(f_\varepsilon;\eta_{N+1})\\
&\le \frac1p\int_{\eta_{N+1}}^1((\rho_2)_r^2+d^2r^{-2}(\rho_2)^2)^{p/2}r\,dr+\frac{1}{4e^p}\int_{\eta_{N+1}}^1(1-(\rho_2)^2)^2 r\,dr\\
&\le \frac1p\int_{\eta_{N+1}}^1(d^2r^{-2})^{p/2}\,dr+\frac{1}{4\varepsilon^p}\int_{\eta_{N+1}}^1(1-(\rho_2)^2)^2\,dr\\
&\quad +C\Big(\int_{\eta_{N+1}}^1(\rho_2)_r^p\,dr+\int_{\eta_{N+1}}^1(\rho_2)_r^2\,dr\Big)
\end{aligned}$$

Using (5.21) we have

$$E_\varepsilon(f_\varepsilon;\eta_{N+1}) \le \frac1p\int_{\eta_{N+1}}^1 (d^2r^{-2})^{p/2}\,dr + C_{N+1}\varepsilon^{2(N-p+1)/p}.$$

Theorem 5.3 *Let $u_\varepsilon = f_\varepsilon(r)e^{id\theta}$ be a radial minimizer of $E_\varepsilon(u,B)$. Then*

$$\lim_{\varepsilon\to0} f_\varepsilon = 1,\quad in\ \ W^{1,p}((\eta,1],R) \tag{5.22}$$
$$\lim_{\varepsilon\to0} u_\varepsilon = e^{id\theta},\quad in\ \ W^{1,p}(K,C) \tag{5.23}$$

for any $\eta\in(0,1)$ and compact subset $K\subset\overline{B}\setminus\{0\}$.

Proof. It suffices to prove (5.23), since (5.23) implies (5.22). Without loss of generality, we may assume $K = B \setminus B(0, \eta_{N+1})$. ¿From Proposition 5.2, We have

$$E_\varepsilon(u_\varepsilon, K) = 2\pi E_\varepsilon(f_\varepsilon, \eta_{N+1}) \le C$$

where C is independent of ε, namely

$$\int_K |\nabla u_\varepsilon|^p \le C \tag{5.24}$$

$$\int_K (1-|u_\varepsilon|^2)^2 \le C\varepsilon^p \tag{5.25}$$

(5.24) and $|u_\varepsilon| \le 1$ imply the existence of a subsequence u_{ε_k} of u_ε and a function $u_* \in W^{1,p}(K, C)$, such that

$$\lim_{\varepsilon_k \to 0} u_{\varepsilon_k} = u_*, \quad \text{weakly in } W^{1,p}(K, C) \tag{5.26}$$

$$\lim_{\varepsilon_k \to 0} u_{\varepsilon_k} = u_*, \quad \text{in } C^\alpha(K, C), \alpha \in (0, 1 - \frac{2}{p}) \tag{5.27}$$

(5.27) implies $u_* = e^{id\theta}$. Noticing that any subsequence of u_ε has a convergence subsequence and the limit is always $e^{id\theta}$, we can assert

$$\lim_{\varepsilon \to 0} u_\varepsilon = e^{id\theta}, \quad \text{weakly in } W^{1,p}(K, C) \tag{5.28}$$

¿From this and the weakly lower semicontinuity of $\int_K |\nabla u|^p$, using Proposition 5.2, we have

$$\begin{aligned}\int_K |\nabla e^{id\theta}|^p &\le \liminf_{\varepsilon_k \to 0} \int_K |\nabla u_\varepsilon|^p \le \limsup_{\varepsilon_k \to 0} \int_K |\nabla u_\varepsilon|^p \\ &\le C \lim_{\varepsilon \to 0} \varepsilon^{2(N+1-p)/p} + 2\pi \int_{\eta_{N+1}}^1 (d^2 r^{-2})^{p/2} r\,dr\end{aligned}$$

and hence

$$\lim_{\varepsilon \to 0} \int_K |\nabla u_\varepsilon|^p = \int_K |\nabla e^{id\theta}|^p$$

since

$$\int_K |\nabla e^{id\theta}|^p = 2\pi \int_{\eta_{N+1}}^1 (d^2 r^{-2})^{p/2} r\,dr$$

Combining this with (5.28)(5.27) completes the proof of (5.23).

For the regularizable radial minimizer $\tilde{u}_\varepsilon = \tilde{f}_\varepsilon(r) e^{id\theta}$, we may prove

$$\begin{aligned}E_\varepsilon^\tau(f_\varepsilon^\tau; \eta) &= \frac{1}{p}\int_\eta^1 [(f_\varepsilon^\tau)_r^2 + d^2 r^{-2}(f_\varepsilon^\tau)^2 + \tau]^{p/2} r\,dr + \frac{1}{4\varepsilon^p}\int_\eta^1 (1-(f_\varepsilon^\tau)^2)^2 r\,dr \\ &\le C(\eta),\end{aligned}$$

where f_ε^τ is the regularized minimizer of $E_\varepsilon(f)$. On the basis of this fact and the conclusion for f_ε^τ similar to Theorem 3.5, we may obtain better convergence for the regularizable

minimizer $\tilde{f}_\varepsilon$ by means of the argument applied in [10]. Precisely we have

Theorem 5.4 *Let $\tilde{u}_\varepsilon = \tilde{f}_\varepsilon(r)e^{id\theta}$ be a regularizable radial minimizer of $E_\varepsilon(u, B)$. Then for some $\alpha \in (0, 1)$*

$$\lim_{\varepsilon\to 0} \tilde{f}_\varepsilon = 1 \ \textit{in}\ C^{1,\alpha}_{\rm loc}((0,1), R), \qquad \lim_{\varepsilon\to 0} \tilde{u}_\varepsilon = e^{id\theta} \ \textit{in}\ C^{1,\alpha}_{\rm loc}(B\setminus\{0\}, C)\,.$$

6 Generalization

Let $G \subset R^n$ be a bounded and simply connected domain with smooth boundary $\partial G, n > 2, g : \partial G \to S^{n-1} = \{x \in R^n; |x| = 1\}$ be a smooth map with $d = \deg(g, \partial G) \neq 0$. Consider the minimization of the functional

$$E_\varepsilon(u, G) = \frac{1}{p}\int_G |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_G (1 - |u|^2)^2$$

on $W = \{v \in W^{1,p}(G, R^n); v|_{\partial G} = g\}$. When $1 < p < n$, we have $W^{1,p}_g(G, S^{n-1}) \neq \emptyset$ and hence it is easy to prove that $\int_G |\nabla u|^p$ achieves its minimum on $W^{1,p}_g(G, S^{n-1})$ by a p-harmonic map with boundary value g. One can also prove that the minimizer u_ε of $E_\varepsilon(u, G)$ on W exists and for a subsequence u_{ε_k} of u_ε there holds,

$$\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_p \quad \text{in } W^{1,p}(G, R^n)$$

where u_p is a p-harmonic map with boundary value g.

In case $p = n$, M.C.Hong studied in [6] the asymptotic behavior of the regularizable minimizer of $E_\varepsilon(u, G)$. He proved that the minimizer u^τ_ε of

$$E^\tau_\varepsilon(u, G) = \frac{1}{n}\int_G (|\nabla u|^2 + \tau)^{n/2} + \frac{1}{4\varepsilon^n}\int_G (1 - |u|^2)^2$$

on $W^{1,n}_g(G, R^n)$ converges to a minimizer $\tilde{u_\varepsilon}$ (called regularizable minimizer) of $E_\varepsilon(u, G)$ on $W^{1,p}(G, R^n)$ as $\tau \to 0$ and that $\tilde{u_\varepsilon}$ contains a subsequence $\tilde{u}_{\varepsilon_k}$ such that

$$\lim_{\varepsilon_k\to 0} \tilde{u_{\varepsilon_k}} = u_n, \qquad \text{weakly in } W^{1,n}_{\rm loc}(G \setminus \cup_{j=1}^J \{a_j\}, R^n)$$

where $a_j (j = 1, 2, ..., J) \in G$ and u_n is an n-harmonic map on $G \setminus \cup_{j=1}^J \{a_j\}$. In case $G = B = \{x \in R^n; |x| < 1\}, g = x$, he proved that for a subsequence $\tilde{u}_{\varepsilon_k}$ of the regularizable radial minimizer $\tilde{u}_\varepsilon$

$$\lim_{\varepsilon_k\to 0} \tilde{u}_{\varepsilon_k} = \frac{x}{|x|}, quad \text{ weakly in } W^{1,n}_{\rm loc}(B \setminus \{0\}, R^n).$$

In this section we are concerned with the case $p > n$. Assume that $G = B$, and $g = x$ where B is the unit ball centered at the origin, and consider the minimizers of $E_\varepsilon(u, B)$ on the class of radial functions

$$W = \{u \in W^{1,p}_g(B, R^n); u(x) = f(r)x|x|^{-1}, f(r) \geq 0, r = |x|\}$$

we call them radial minimizers.

Denote as in §1

$$V = \{f(r) \in W^{1,p}_{\rm loc}(0,1]; r^{(1-p)/p}f, r^{1/p}f_r \in L^p(0,1), f(1)=1, f(r) \geq 0\}$$

Substituting $u = f(r)x|x|^{-1}$ into $E_\varepsilon(u,B)$ we obtain

$$E_\varepsilon(u,B) = \mathrm{meas}(S^{n-1})E_\varepsilon(f)$$

where

$$E_\varepsilon(f) = \int_0^1 r^{n-1}[\frac{1}{p}(f_r^2 + (n-1)r^{-2}f^2)^{p/2} + \frac{1}{4\varepsilon^p}(1-f^2)^2]\,dr$$

This means that $u_\varepsilon(x) = f_\varepsilon(r)x|x|^{-1}$ is the minimizer of $E_\varepsilon(u,B)$ on W if and only if $f_\varepsilon(r)$ is the minimizer of $E_\varepsilon(f)$ on V.

Parallel to the discussions in the previous sections we can obtain the corresponding results. In particular, we have the results on the location of zeroes of minimizers and on the convergence rate for minimizers. Also it can be proved that if $u_\varepsilon(x) = f_\varepsilon(r)x|x|^{-1}$ is a radial minimizer of $E_\varepsilon(u,B)$, then

$$\lim_{\varepsilon\to 0} f_\varepsilon = 1 \quad \text{in } W^{1,p}((\eta,1],R), \qquad \lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|} \quad \text{in } W^{1,p}(K,R^n)$$

for any $\eta \in (0,1)$ and any compact subset $K \subset \overline{B}\setminus\{0\}$. If $p > 2n-2$, then for the regularizable minimizer $\tilde{u}_\varepsilon(x) = \tilde{f}_\varepsilon(r)x|x|^{-1}$, we have

$$\lim_{\varepsilon\to 0} \tilde{f}_\varepsilon = 1 \quad \text{in } C^{1,\alpha}_{\rm loc}((0,1),R) \qquad \lim_{\varepsilon\to 0} \tilde{u}_\varepsilon = \frac{x}{|x|} \quad \text{in } C^{1,\alpha}_{\rm loc}(B\setminus\{0\},R^n)$$

with some constant $\alpha \in (0,1)$.

References

[1] F.Bethuel, H.Brezis, F.Helein: *Asymptotics for the minimization of a Ginzburg-Landau functional,* Calc. Var.PDE.,**1** (1993).123-148.

[2] F.Bethuel, H.Brezis, F.Helein: *Ginzburg-Landau Vortices,* Birkhauser. 1994.

[3] S.J.Ding, Z.H.Liu: *On the zeroes and asymptotic behaviour of minimizers to the Ginzburg-Landau functional with variable coefficient* J. Partial Diff. Eqs.,**10** No.1. (1997).45-64.

[4] M.Giaquinta: *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Ann. Math. Stud.,**105** 1989.

[5] R.Hardt, F.H.Lin: *Singularities for p-energy minimizing unit vector fields on planner domains,* Cal. Var. PDE.,**3** (1995), 311-341.

[6] M.C.Hong: *Asymptotic behavior for minimizers of a Ginzburg-Landau type functional in higher dimensions associated with n-harmonic maps,* Adv. in Diff. Equa.,**1** (1996), 611-634.

[7] M.C.Hong: *On a problem of Bethuel, Brezis and Helein concerning the Ginzburg-Landau functional,* C. R. Acad. Sic. Paris ,**320** (1995), 679-684.

[8] M.C.Hong: *Two estimates concerning asymptotics of the minimizations of a Ginzburg-Landau functional,* J. Austral. Math. Soc. Ser. A ,**62** No.1, (1997), 128-140.

[9] Y.T.Lei: *Asymptotic behavior of minimizers for functional* Acta. Scien. Natur. Jilin Univ. ,**1** (1996), 1-6.

[10] Y.T.Lei, Z.Q.Wu: $C^{1,\alpha}$ *convergence for minimizers of a Ginzburg-Landau-type functional* to appear.

[11] F.H.Lin: *Solutions of Ginzburg-Landau equations and critical points of the renormalized energy,* Ann. Inst. P. Poincare Anal. Nonlineare, **12** No.5, (1995), 599-622.

[12] M.Struwe: *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2-dimensions,* Diff. and Int. Equa.,**7** (1994), 1613-1624.

[13] M.Struwe: *An asymptotic estimate for the Ginzburg-Landau model,* C.R.Acad. Sci. Paris.,**317** (1993), 677-680.

Yutian Lei, Zhuoqun Wu, & Hongjun Yuan
Institute of Mathematics, Jilin University
130023 Changchun China
E-mail:wzq@mail.jlu.edu.cn