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A note on quasilinear elliptic eigenvalue problems *

Gianni Arioli

Abstract

We study an eigenvalue problem by a non-smooth critical point theory. Under general assumptions, we prove the existence of at least one solution as a minimum of a constrained energy functional. We apply some results on critical point theory with symmetry to provide a multiplicity result.

1 Introduction

In this note we consider the following problem which was addressed by Struwe in [17]. Let Ω be a bounded domain in $\mathbb{R}^n$, let $a : \Omega \times \mathbb{R}^m \to \mathbb{R}^{n^2}$ be a uniformly elliptic, symmetric and bounded matrix function, and for $u \in H_0^1(\Omega, \mathbb{R}^m)$ define the energy integral

$$E(u) = \frac{1}{2} \int_\Omega a_{ij}(x, u) \langle D_i u, D_j u \rangle \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^m$ and the sum over repeated indices is understood. Let $F : H_0^1(\Omega, \mathbb{R}^m) \to \mathbb{R}$ be a sufficiently regular functional and let

$$M = \{u \in H_0^1(\Omega, \mathbb{R}^m) : F(u) = 1\} \,.$$

Struwe studied the eigenvalue problem

$$\nabla E(u) = \mu \nabla F(u) \,, \quad u \in M \,, \quad \mu \in \mathbb{R} \,. \tag{1.1}$$

We refer the reader to [17] for more details. Here, we just wish to point out that the energy functional E is not Fréchet differentiable in $H_0^1(\Omega, \mathbb{R}^m)$ whenever $n \geq 2$ and the matrix $a_{ij}(x, u)$ depends on u: therefore, problem (1.1) cannot be addressed by a standard constrained minimization procedure. In fact, if the energy functional E is not differentiable, then (1.1) does not make sense. On the other hand, it is easy to check that the Fréchet derivative of E exists at least in smooth directions; so (1.1) can be understood at least in a distributional sense.

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In [17] the problem was addressed by replacing the Palais-Smale condition with a weaker condition called Criterion A* which was introduced by the same author in [16]. There, under suitable assumptions, existence and multiplicity results were proved.

Here we study a similar problem using the non-smooth critical point theory developed in [8, 13, 15]. According to such theory it is possible to define a critical point of a continuous functional in a generalized sense and it is possible to prove that, under additional assumptions, such a critical point is a weak solution of the corresponding Euler equation. By applying the non-smooth critical point theory, some compactness results proved in [2] and the equivariant critical point theory developed by Bartsch, Clapp and Puppe [3, 4, 12], we can extend the results from [17] in different directions: we set the problem in the whole space $\mathbb{R}^n$ (and in fact in any open and regular set, bounded or unbounded, although for simplicity we only deal with $\mathbb{R}^n$), we give different conditions on the matrix a_{ij} and on the "regular" term F and we give a multiplicity result in the case when the cat $M \geq 2$ or the system is invariant under the action of a compact Lie group G. Finally we provide sufficient conditions for the existence of infinitely many solutions.

2 Setting

We denote by $\mathcal{D}^{1,2}(\mathbb{R}^n, \mathbb{R}^m)$ the closure of $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ (the space of smooth vector functions with compact support in $\mathbb{R}^n$) with respect to the norm induced by the scalar product $(\psi, \phi) = \int_{\mathbb{R}^n} \langle D_i\psi, D_i\phi\rangle$. We denote by $\mathcal{D}^*$ the dual space.

The following condition is standard in this kind of problems. ($\nabla = \{\partial/\partial s_k\}$)

(A1) The matrix $[a_{ij}(x,s)]$ satisfies

$$\begin{gathered} a_{ij} \equiv a_{ji} \\ a_{ij}(x,s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}) \\ a_{ij}(x,\cdot) \in C^1(\mathbb{R}^m) \text{ for a.e. } x \in \mathbb{R}^n \\ \nabla a_{ij}(x,s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)\,. \end{gathered} \tag{2.2}$$

Also there exists $\nu > 0$ such that for a.e. $x \in \mathbb{R}^n$, all $s \in \mathbb{R}^m$ and all $\xi \in \mathbb{R}^n$

$$a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2 \text{ and } \langle s, \nabla a_{ij}(x,s)\rangle \xi_i\xi_j \geq 0\,. \tag{2.3}$$

The following assumption was introduced in [2]. It is a control required on the matrix $[a_{ij}(x,s)]$ in order to prove the compactness of bounded Palais-Smale sequences.

(A2) There exist $K > 0$ and a function $\psi : [0,+\infty) \to [0,+\infty)$ continuously differentiable and satisfying

$$\begin{aligned}
&\text{(i) } \psi(0)=0 \text{ and } \lim_{t\to+\infty}\psi(t)=K \\
&\text{(ii) } \psi'(t)\geq 0 \text{ for all } t\in[0,+\infty) \\
&\text{(iii) } \psi' \text{ is non-increasing} \\
&\text{(iv) } \sum_{k=1}^{m}\left|\tfrac{\partial}{\partial s_k}a_{ij}(x,s)\xi_i\xi_j\right| \leq 2e^{-4K}\psi'(|s|)a_{ij}(x,s)\xi_i\xi_j \\
&\qquad \text{for all } s\in\mathbb{R}^m, \text{ for all } \xi\in\mathbb{R}^n \text{ and for a.e. } x\in\mathbb{R}^n.
\end{aligned} \tag{2.4}$$

In some sense, ψ is a measure of the growth of ellipticity of the differential operator as the value of $|s|$ grows. We assume that such growth is “not too large”, see the appendix for some examples. In fact it is enough to require the continuous differentiability of ψ almost everywhere, provided $\lim_{t\to 0}\psi'(t)\neq+\infty$. In Struwe's paper it is required $|\nabla a_{ij}(x,s)|\leq\rho$, where $\rho>0$ is a constant depending on the critical level, instead of (A2). We point out that assumption (A2) is not comparable to the corresponding Struwe's assumption in the sense that neither is stronger than the other one. Our assumption looks somehow less natural, but on the other hand in this setting the Palais-Smale condition holds at all levels, so that under suitable assumptions it is possible to prove the existence of infinitely many solutions, see the following section.

On the regular term F we assume the following:

(F1) $F\in C^1(\mathcal{D},\mathbb{R})$ and $\nabla F:\mathcal{D}\to\mathcal{D}^*$ is weak-to-strong (sequentially) continuous; in other words, all sequences $\{u^h\}$ converging weakly to some u satisfy $\nabla F(u^h)\to\nabla F(u)$, up to a subsequence.

(F2) $M=F^{-1}(1)\neq\emptyset$, $0\notin F^{-1}(1)$, and $\nabla F(u)\neq 0$ for all $u\in F^{-1}(1)$.

These assumptions suffice to prove the existence of at least $\operatorname{cat} M$ weak solutions $(u,\mu)\in M\times\mathbb{R}$ of the eigenvalue problem $\nabla E(u)=\mu\nabla F(u)$. Such result comes as a by-product of our main theorem, see Corollary 3.2.

3 Results

We consider the case when the eigenvalue problem is invariant with respect to the action of a compact Lie group G, possibly the trivial group. First we recall some definitions and known results from representation theory, that can be found in [3, 6, 12]. In the following, G denotes a compact Lie group.

Definition A compact Lie group G is said to be *solvable* if there exists a sequence $G_0\subset G_1\subset\ldots\subset G_r=G$ of subgroups such that G_0 is a maximal torus of G, G_{i-1} is a normal subgroup of G_i and $G_i/G_{i-1}\cong\mathbb{Z}/p_i$, $1\leq i\leq r$. Here the p_i's are prime numbers.

Remark 3.1 If G is Abelian, then G is isomorphic to the product of a torus with a finite Abelian group [6, Corollary I.3.7]. In particular, all Abelian compact Lie groups are solvable.

Definition A G-space is a topological space E together with a continuous action

$$G \times E \to E \quad (g, x) \mapsto gx$$

satisfying $(gh)x = g(hx)$ and $ex = x$ for all $g, h \in G$ and all $x \in E$, where $e \in G$ denotes the unit element.

Definition Let E and $\widetilde{E}$ be two G-spaces. A subset B of E is said to be *invariant* if $gB \subset B$ for all $g \in G$. A functional $I : E \to \mathbb{R}$ is said to be *invariant* if $I(gx) = I(x)$ for all $g \in G$ and all $x \in E$. A map $f : E \to \widetilde{E}$ is said to be *equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and all $x \in E$. A continuous equivariant mapping between two G-spaces is called a G-map.

Definition Let E be a G-space. A homotopy $\eta : [0,1] \times E \to E$ is called a G-homotopy if $\eta(t, \cdot)$ is a G-map for all $t \in [0,1]$.

Definition Let E be a G-space equipped with a Hilbert structure under the scalar product $\langle \cdot, \cdot \rangle$. If the group action is linear and preserves the scalar product, i.e. $\langle gx, gy \rangle = \langle x, y \rangle$ for all $x, y \in E$ and all $g \in G$, then E is called a G-Hilbert space.

Definition The space $E^G := \{x \in E : gx = x \text{ for all } g \in G\}$ is called the *fixed point space* of (the representation of) G.

Definition The *orbit* of x is defined by $\mathcal{O}_G(x) := \{gx : g \in G\}$. We say that $x, y \in E$ are *geometrically distinct* if $y \notin \mathcal{O}_G(x)$.

Let $\mathcal{A}$ be a class of G-spaces.

Definition The $\mathcal{A}$-category of a G-space X, $\mathcal{A}$-cat(X), is the smallest number k such that X can be covered by G-invariant open subspaces $X_1, X_2, \dots, X_k$ with the following property: For each $1 \le i \le k$ there exist $A_i \in \mathcal{A}$ and G-maps $\alpha_i : X_i \to A_i$ and $\beta_i : A_i \to X$ such that the inclusion $X_i \hookrightarrow X$ is G-homotopic to the composition $\beta_i \circ \alpha_i$. If no such covering exists we define $\mathcal{A}$-cat$(X) := \infty$.

If G is the trivial group and $\mathcal{A}$ is a singleton, then $\mathcal{A}$-cat$(X) = \text{cat}(X)$ is just the Lusternik-Schnirelmann category of X. In general, a good choice for $\mathcal{A}$ is the set of all G-orbits of X, i.e. the set $\mathcal{G}$ of all homogeneous G-spaces G/G_x where $G_x := \{g \in G : gx = x\}$ is the isotropy subgroup of a point $x \in X$.

By exploiting the results in [12] we prove the following:

Theorem 3.1 *Assume (A1), (A2), (F1), (F2) and that the functionals E and F are invariant under some linear action of a compact Lie group G on $\mathcal{D}$. Let $\mathcal{G}$ be a set of homogeneous G-spaces which contains all critical G-orbits of M*

(up to G-homeomorphism). Then there exist at least $\mathcal{G}$-cat(M) *geometrically distinct weak solutions* $(u,\mu)\in M\times\mathbb{R}$ *of the eigenvalue problem*

$$\nabla E(u)=\mu\nabla F(u)\,. \tag{3.5}$$

Furthermore, if $\mathcal{G}$-cat$(M)=\infty$*, then* $E\mid_M$ *has an unbounded sequence of critical values.*

A particular case of this theorem is the following corollary, which considers the case when G is the trivial group, i.e. when there is no G-action.

Corollary 3.2 *Assume (A1), (A2), (F1) and (F2). Then there exist at least* cat(M) *weak solutions of the eigenvalue problem (3.5).*

The multiplicity result of Theorem 3.1 looks very abstract. In order to have a concrete multiplicity result, one has to provide conditions which allow the computability of $\mathcal{G}$-cat(M). This may be done by exploiting the theory in [3]. First we choose a representation of a compact Lie group G in $\mathcal{D}$: given a representation of G in $\mathbb{R}^m$ (i.e. a finite dimensional G-space which we identify with $\mathbb{R}^m$) the natural choice is $g(u)(x):=g(u(x))$. We have to restrict our choice of representations as follows:

Definition Let V be a finite-dimensional G-space. V is called *admissible* if for each open, bounded and invariant neighborhood $\mathcal{U}$ of 0 in V^k $(k\geq 1)$ and each equivariant map $f:\overline{\mathcal{U}}\to V^{k-1}$, $f^{-1}(0)\cap\partial\mathcal{U}\neq\emptyset$.

Remark 3.2 The admissibility of a representation space consists substantially in requiring that a generalized Borsuk-Ulam theorem holds. In [3, Theorem 3.7] it is proved that V is admissible if and only if there exist subgroups $K\subset H$ of G such that K is normal in H, H/K is solvable, $V^K\neq 0$ and $V^H=0$. It follows that, if G is solvable, then any finite-dimensional representation space V with $V^G=\{0\}$ is admissible. Furthermore, all admissible representations have a trivial fixed point space.

Definition Let E be a G-Hilbert space, $V\equiv\mathbb{R}^m$ an admissible representation for G and $\Sigma=\{A\subset E : A \text{ is closed and invariant}\}$. We define the index $\gamma:\Sigma\to\mathbb{N}\cup\{+\infty\}$ as follows: $\gamma(A)$ is the smallest integer k such that there exists an equivariant map $\Psi:A\to\{x\in V^k:|x|=1\}$ (the action of G on V^k is given by $gx=(gx_1,\dots,gx_k)$ for all $x=(x_1,\dots,x_k)\in V^k$).

We point out that the index γ defined above corresponds to the $\mathcal{A}$-genus defined in [3] with $\mathcal{A}=\{SV\}$, $SV=\{x\in V:|x|=1\}$, and therefore it satisfies the usual properties of indices, see Proposition 2.15 in [3].

One can easily see that $\mathcal{G}$- genus$(M)\leq\mathcal{G}$- cat(M) for all sets $\mathcal{G}$ of homogeneous G-spaces (see Section 2 in [12]). Furthermore, if a set of G-spaces $\mathcal{A}$ has the property that every G/H in $\mathcal{G}$ can be mapped G-equivariantly into some A in $\mathcal{A}$, then $\mathcal{A}$- genus$(M)\leq\mathcal{G}$- genus(M); but the equality does not hold in general.

By the definitions admissible space and of index it follows that the index of the unit sphere of an infinite dimensional G-Hilbert space is infinity. The monotonicity property of the index yields the following.

Remark 3.3 If V is admissible and there exists a G-map from the unit sphere in an infinite dimensional G-Hilbert space into M (see e.g. Examples 6.4 and 6.5), then $\gamma(M) = \mathcal{G}\text{-}\operatorname{cat}(M) = \infty$, where $\mathcal{G}$ is the set of all G-orbits of M.

These observations lead to the following corollary of Theorem 3.1.

Corollary 3.3 *Assume (A1), (A2), (F1), (F2), and assume that the functionals E and F are invariant under the G-action on $\mathcal{D}$ induced by some admissible representation of a compact Lie group G on $\mathbb{R}^n$. Then there exist at least $\gamma(M)$ geometrically distinct weak solutions $(u, \mu) \in M \times \mathbb{R}$ of the eigenvalue problem (3.5). Furthermore, if $\gamma(M) = \infty$, then $E \mid_M$ has an unbounded sequence of critical values.*

Another possibility for computing $\mathcal{G}$-cat(M) is provided by Corollary 3.3 in [3], which states that for an arbitrary fixed point free G-action, $\mathcal{G}\text{-}\operatorname{genus}(M) = \mathcal{G}\text{-}\operatorname{cat}(M) = \infty$ if M is (non equivariantly) contractible and there exists a prime p which divides the Euler characteristic of every G-orbit of M. Further computations of $\mathcal{G}\text{-}\operatorname{genus}(M)$ can be found in [3], [4] and [12].

4 Non-smooth critical point theory

The non-smooth critical point theory was introduced in [13, 15]. We refer to those papers and to [2] for details. We recall here the alternative definition of the weak slope given in [8], which is equivalent to the original definition.

Definition Let (X, d) be a metric space, $I : X \to \mathbb{R} \cup \{+\infty\}$, and $x \in X$ satisfy $I(x) \in \mathbb{R}$. We denote by $|dI|(x)$ the supremum of the $\sigma \in [0, +\infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : (B_\delta(x, I(x)) \cap \operatorname{epi}(I)) \times [0, \delta] \longrightarrow X$$

such that for all $(y, k) \in (B_\delta(x, I(x)) \cap \operatorname{epi}(I))$ and for all $t \in [0, \delta]$ we have

$$d(\mathcal{H}((y, k), t), y) \leq t \text{ and } I(\mathcal{H}((y, k), t)) \leq k - \sigma t.$$

We call the extended real number $|dI|(x)$ the *weak slope* of I at x.

Definition Let (X, d) be a metric space and $I : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional. A point $x \in X$ is said to be critical for I if $|dI|(x) = 0$. A real number c is said to be a critical value for I if there exists $x \in X$ such that $I(x) = c$ and $|dI|(x) = 0$.

Definition Let (X,d) be a metric space and $I : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous functional. We say that I satisfies the Palais-Smale condition if every sequence $\{x_m\} \subset X$ such that $I(x_m)$ is bounded and $|dI|(x_m) \to 0$ admits a converging subsequence.

The following is the natural extension of the weak slope to the setting of constrained functionals.

Definition Let X be a Banach space, $I \in C(X,\mathbb{R})$ and let $M \subset X$ be a C^1 manifold defined by $M = \{u \in X : F(u) = 1\}$, where $F \in C^1(X,\mathbb{R})$ satisfies $\nabla F(u) \neq 0$ for all $u \in M$. The manifold M is a metric space when endowed with the metric of X; therefore the weak slope $|I|_M|(x)$ of $I|_M : M \to \mathbb{R}$ at x is well defined for all $x \in M$, where $I|_M$ denotes the restriction of I to M. We say that $\{u^h\} \subset M$ is a constrained Palais-Smale sequence for I if $I(u^h)$ is bounded and $|dI|_M|(u^h) \to 0$.

The following theorem gives the connection between the weak slope and the problem of finding weak solutions of (3.5) and it is a generalization of the Lagrange Multipliers Theorem.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ be regular and open (bounded or unbounded). Let $J : \mathcal{D} = \mathcal{D}^{1,2}(\Omega,\mathbb{R}^m) \to \mathbb{R}$ be a functional of the type*

$$J(u) = \int_\Omega L(x,u,\nabla u)\,dx\,,$$

where $L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}$ satisfies the following assumptions:

$$\begin{gathered} L(x,s,\xi) \text{ is measurable with respect to } x \text{ for all } (s,\xi) \in \mathbb{R}^m \times \mathbb{R}^{mn} \\ L(x,s,\xi) \text{ is of class } C^1 \text{ with respect to } (s,\xi) \text{ for a.e. } x \in \Omega \end{gathered}$$

and there exist $h_1 \in L^1(\Omega,\mathbb{R})$, $h_2 \in L^1_{loc}(\Omega,\mathbb{R})$, $h_3 \in L^\infty_{loc}(\Omega,\mathbb{R})$ and $c > 0$ such that for all $(s,\xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$, and a.e. $x \in \Omega$ the following inequalities hold:

$$\begin{aligned} |L(x,s,\xi)| &\leq h_1(x) + c(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ |\frac{\partial L}{\partial s}(x,s,\xi)| &\leq h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ |\frac{\partial L}{\partial \xi}(x,s,\xi)| &\leq h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2). \end{aligned}$$

Let $M \subset \mathcal{D}$ be the C^1 manifold $M = \{u \in \mathcal{D} : F(u) = 1\}$ where $F \in C^1(\mathcal{D},\mathbb{R})$ satisfies $\nabla F(u) \neq 0$ for all $u \in M$. Then J is continuous, differentiable in the smooth directions and there exists $\mu \in \mathbb{R}$ such that

$$|dJ|_M|(u) \geq \sup_{\varphi \in C_c^\infty, ||\varphi||=1} \left(\nabla J(u)[\varphi] - \mu \nabla F(u)[\varphi]\right)\,.$$

Proof Under the growth conditions which we assume on L, it is easy to check that the functional is continuous and differentiable in the smooth directions. Let $\widetilde{J} : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ be defined by $\widetilde{J}(u) = J(u) + I_M$, where I_M is the indicator function of M. More precisely

$$\widetilde{J}(u) = \begin{cases} J(u) & \text{if } u \in M \\ +\infty & \text{if } u \notin M. \end{cases}$$

We can assume that $|dJ|_M|(u) < +\infty$. It follows directly from the definition of weak slope that $|d\widetilde{J}|(u) = |dJ|_M|(u)$, hence by Theorem 4.13 in [8] $\partial\widetilde{J}(u) \neq \emptyset$ and there exists $\alpha \in \partial\widetilde{J}(u) \subset \mathcal{D}^*$ such that $|d\widetilde{J}|(u) \geq ||\alpha||_{\mathcal{D}^*}$, where $\partial\widetilde{J}$ is the subdifferential of $\widetilde{J}$ as defined in Definition 4.1 in [8]. Since J is differentiable in the smooth directions and smooth functions are dense in $\mathcal{D}$, then the assumptions of Corollary 5.10 in [8] hold. Hence $\partial\widetilde{J}(u) \subset \partial J(u) + \mathbb{R}\nabla F(u)$. By Theorem 6.1 in [8] we have $\partial J(u) = \{\beta\}$, where $\beta \in \mathcal{D}^*$ satisfies

$$\langle \beta, \varphi \rangle = \int_{\mathbb{R}^n} \left\langle \frac{\partial L(x,u,\nabla u)}{\partial s}, \varphi \right\rangle + \left\langle \frac{\partial L(x,u,\nabla u)}{\partial \xi}, \nabla\varphi \right\rangle$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$, hence the thesis. $\diamondsuit$

To prove our results we use of the following corollary to the previous theorem.

Corollary 4.2 *Let J and M be as above and let $\{u^h\} \subset M$ be a Palais-Smale sequence for $J|_M$. Then there exists a sequence $\{\mu^h\} \subset \mathbb{R}$ such that*

$$\sup_{\varphi \in C_c^\infty, ||\varphi||=1} \left(\nabla J(u^h)[\varphi] - \mu^h \nabla F(u^h)[\varphi] \right) \to 0.$$

5 Proofs

The following lemma is an adaptation of Lemma 6.1 in [2] to the case of a constrained critical point.

Lemma 5.1 *Assume (A1) and (A2), let $\{u^h\} \subset \mathcal{D}$ be a bounded sequence, let $\{\mu^h\} \subset \mathbb{R}$, and assume that $\{w^h\} \subset \mathcal{D}^*$ satisfies*

$$\mu^h w^h = -D_j(a_{ij}(x,u^h)D_i u^h) + \frac{1}{2}\nabla a_{ij}(x,u^h)\langle D_i u^h, D_j u^h \rangle\,.$$

If $\{w^h\}$ is strongly convergent to some $w \neq 0$, then there exists $(u,\mu) \in \mathcal{D} \times \mathbb{R}$ such that $u^h \to u$ and $\mu^h \to \mu$ (up to a subsequence).

Proof Since $\{u^h\}$ is bounded, then $u^h \rightharpoonup u$ for some u, up to a subsequence. Since $w \neq 0$ and smooth functions are dense in $\mathcal{D}$, there exists $\bar\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ such that $\lim w^h[\bar\varphi] = w[\bar\varphi] > 0$. It is easy to check that the sequence

$$\left\{ \int_{\mathbb{R}^n} a_{ij}(x,u^h)\langle D_i u^h, D_j \bar\varphi \rangle + \frac{1}{2}\langle \nabla a_{ij}(x,u^h), \bar\varphi \rangle \langle D_i u^h, D_j u^h \rangle \right\}$$

is bounded, hence the sequence $\{\mu^h\}$ is bounded and it converges to some $\mu \in \mathbb{R}$ up to a subsequence. Then $\mu^h w^h \to \mu w$ and the result follows from Lemma 6.1 in [2]. $\diamondsuit$

With the previous lemma we can prove the compactness of constrained Palais-Smale sequences.

Lemma 5.2 *Assume (A1), (A2), (F1), (F2); let $\{u^h\}$ be a Palais-Smale sequence for the functional E constrained to M. Then there exists $(u, \mu) \in \mathcal{D} \times \mathbb{R}$ such that $u^h \to u$ and $\mu^h \to \mu$ (up to a subsequence), $\nabla E(u)[\varphi] = \mu \nabla F(u)[\varphi]$ for all $\varphi \in \mathcal{D}$, $\lim E(u^h) = E(u)$ and $F(u) = 1$.*

Proof Since by (2.3) $E(u)$ is coercive, then $\{u^h\}$ is bounded and $u^h \rightharpoonup u$ for some u, up to a subsequence. By (F1) we have $\nabla F(u^h) \to \nabla F(u)$ in $\mathcal{D}^*$, possibly on a sub-subsequence; furthermore $F(u^h) \to F(u)$ by Lagrange's theorem, hence $F(u) = 1$, i.e. $u \in M$ and $\nabla F(u) \neq 0$. By Corollary 4.2 there exists a sequence $\{\mu^h\} \subset \mathbb{R}$ such that

$$\sup_{\varphi \in C^\infty, ||\varphi||=1} \left(\nabla E(u^h)[\varphi] - \mu^h \nabla F(u^h)[\varphi] \right) \to 0 \,.$$

Let $w^h = \nabla F(u^h)$; then the sequences $\{u^h\} \subset \mathcal{D}$, $\{\mu^h\} \subset \mathbb{R}$ and $\{w^h\} \subset \mathcal{D}^*$ satisfy the assumptions of Lemma 5.1 and there exists $(u, \mu) \in \mathcal{D} \times \mathbb{R}$ such that $u^h \to u$ and $\mu^h \to \mu$ (up to a subsequence). The statement follows by the continuity of the functionals E and F and the density of smooth functions in $\mathcal{D}$. $\diamondsuit$

Proof of Theorem 3.1. The result follows from Theorem 1.1 in [12]. By Lemma 5.2 and Theorem 1.2.5 in [10] the strong G-deformation property required by the assumptions of Theorem 1.1 in [12] holds in M (see [12] for a definition of the strong G-deformation property and [10] for details about building an equivariant deformation in the non-smooth setting when the Palais-Smale condition holds).

6 Examples

The following examples of matrices satisfying assumptions (A1) and (A2) were given in [2]; we refer the reader to that paper for more details.

Example 6.1 Assume that there exists $M > 0$ such that

$$\nabla a_{ij}(x,s) \equiv 0 \quad \text{if } |s| \geq M$$

and, for all $\xi \in \mathbb{R}^n$,

$$\sum_k |\partial_k a_{ij}(x,s)\xi_i\xi_j| \leq \frac{1}{2eM} a_{ij}(x,s)\xi_i\xi_j \quad \text{if } |s| \leq M \,.$$

Example 6.2 Take $a_{ij}(x,s) = a_{ij}(s) = (\nu + \arctan(|s|^2))\delta_{ij}$ with $\nu \geq e\sqrt{m}(\sqrt{3} + \pi)$.

Example 6.3 More generally, consider the (continuous) function $\beta : [0,+\infty) \to [0,+\infty)$ defined by

$$\beta(t) = \frac{1}{2\nu}\sum_{k=1}^{m}\left(\operatorname*{esssup}_{x\in\mathbb{R}^n}\max_{|u|=t}\max_{|\xi|=1}|\partial_k a_{ij}(x,u)\xi_i\xi_j|\right);$$

clearly, $\beta(0) = 0$, β admits a global maximum point and $\lim_{t\to+\infty}\beta(t) = 0$. Next, define the function $\alpha : [0,+\infty) \to [0,+\infty)$ by

$$\alpha(t) = \max_{\tau\geq t}\beta(\tau) \tag{6.6}$$

If

$$\int_0^{+\infty}\alpha(\tau)d\tau \leq \frac{1}{4e}\,, \tag{6.7}$$

then a_{ij} satisfies (A1) and (A2).

If the equation is invariant under a representation of $O(m)$, then it is easy to see that it admits a radial solution which may be obtained by considering a single equation and extending it by symmetry. The following example we give a matrix a_{ij} and a functional F which satisfy the assumption of Corollary 3.3, but raise a nontrivial problem.

Example 6.4 Let $m = 4$, $G = SO(2)$ and consider the action $\mathbb{R}^4 \times SO(2) \to \mathbb{R}^4$ defined by

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \vartheta\right) \mapsto \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 & 0 \\ \sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & \cos\vartheta & -\sin\vartheta \\ 0 & 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Let $[a_{ij}(x,s)]$ be any matrix satisfying (A1), (A2) and $[a_{ij}(x,s)] = [a_{ij}(x,|s|)]$. Let $F : \mathcal{D} \to \mathbb{R}$ be defined by

$$F(u) = \tanh\left(||fu_1||_2^2 + ||fu_2||_2^2\right) + \tanh\left(||fu_3||_2^2 + ||fu_4||_2^2\right)\,,$$

where $f \in L^\infty(\mathbb{R}^n)$, $f(x) \neq 0$ a.e. Since the representation has only the origin as fixed point and $SO(2)$ is an Abelian group, then the representation is admissible by Remark 3.2. It is easy to check that the functional F satisfies (F1) and (F2) and furthermore $\gamma(M) = +\infty$, hence Corollary 3.3 applies and $E|_M$ has an unbounded sequence of critical values. To prove that $\gamma(M) = +\infty$, note that there exists a $SO(2)$-map $\varphi : S \to M$, where S is the unit sphere in $\mathcal{D}^{1,2}(\mathbb{R}^n,\mathbb{R}^2)$. To build the map note that for all $u = (u_1,u_2) \in S$ there exists a unique positive number $\lambda(u)$ such that $|\lambda(u)|^2\left(||fu_1||_2^2 + ||fu_2||_2^2\right) = \tanh^{-1}(1/2)$, hence the map φ defined by $u \mapsto (\lambda(u)u, \lambda(u)u)$ has the desired properties and Remark 3.3 applies.

Example 6.5 In general, given an admissible representation of a compact Lie group G on $\mathbb{R}^m$, consider a functional F defined by $F(u) = \int_{\mathbb{R}^n} H(x,u)$, where $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies suitable growth and regularity conditions which ensure that F is well defined on $\mathcal{D}$ and satisfies (F1), $H(x,\cdot)$ is invariant, $H(x,0) = 0$, $\langle \nabla H(x,s), s\rangle > 0$ for all $s \neq 0$ and $\langle \nabla H(x,s), s\rangle \geq \delta > 0$ for large s. Then F satisfies (F2) and $\gamma(M) = +\infty$. Indeed M is invariant and there exists a G-map, φ from the unit sphere S in $\mathcal{D}$ onto M. To prove the latter statement, take $u \in S$ and note that the function $\xi : [0,+\infty) \to [0,+\infty)$ defined by $\lambda \mapsto \xi(\lambda) = F(\lambda u)$ satisfies $\xi(0) = 0$, $\xi'(\lambda) = \int_{\mathbb{R}^n} \langle \nabla H(x, \lambda u), u\rangle > 0$ and $\lim_{\lambda\to+\infty} \xi(\lambda) = +\infty$, hence there exists a unique λ satisfying $\xi(\lambda) = 1$. Take $\varphi(u) = \lambda(u)u$. Finally $M$is nonempty, $F(0) = 0$ and $\nabla F(u)[u] = \int_{\mathbb{R}^n} \langle \nabla H(x,u), u\rangle > 0$, hence F satisfies (F2).

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Gianni Arioli
Dipartimento di Scienze e T.A.
Università del Piemonte Orientale
C.so Borsalino 54
15100 Alessandria Italy
e-mail: gianni@mfn.unipmn.it
http://www.mfn.unipmn.it/~gianni