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LIOUVILLIAN FIRST INTEGRALS OF SECOND ORDER POLYNOMIAL DIFFERENTIAL EQUATIONS

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Abstract

We consider polynomial differential systems in the plane with Liouvillian first integrals. It is shown that all such systems have Darbouxian integrating factors, and that the search for such integrals can be reduced to a search for the invariant algebraic curves of the system and their 'degenerate' counterparts.

1. Introduction

The purpose of this paper is to provide a more satisfactory conclusion to the work of Singer on the existence of Liouvillian first integrals of second order polynomial differential equations. In his paper ([S] Theorem 1 and its corollary), the following result is obtained:

Theorem 1. *If the second order polynomial differential equation*

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1.1}$$

has a local Liouvillian first integral, then there is a Liouvillian first integral of the form

$$\int_{(x_0,y_0)}^{(x,y)} RQ\,dx - RP\,dy,$$

where

$$R = \exp\left\{ \int_{(x_0,y_0)}^{(x,y)} U\,dx + V\,dy \right\},$$

with U and V rational functions in x and y such that

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}.$$

Our aim here is to prove the following theorem, which reduces the classification to a single quadrature.

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Theorem 2. *If the system* (1.1) *has an integrating factor of the form*

$$\exp\left\{\int U\,dx + V\,dy\right\}, \qquad U_y = V_x, \tag{1.2}$$

where U *and* V *are rational function of* x *and* y*, then there exists a integrating factor of the system* (1.1) *of the form*

$$exp(D/E)\prod {C_i}^{l_i},$$

where D*,* E *and the* C_i *are polynomials in* x *and* y*.*

The significance of this theorem is that the Darbouxian integrating factor above defines a collection of invariant algebraic curves of the system, $C_i(x, y) = 0$, satisfying the equation

$$\frac{d}{dt}C_i(x, y) = C_i(x, y)L_i(x, y), \tag{1.3}$$

for some polynomial $L_i(x, y)$ of smaller degree than the system. The other term in the product, $\exp(D/E)$, can be considered as a degenerate counterpart to these curves which satisfies a similar equation. In this way the search for Liouvillian integrating factors can be reduced to the search for invariant algebraic curves and 'degenerate algebraic curves' or exponential factors of the system satisfying the equation (1.3).

As an example, consider the Lokta-Volterra equations.

$$\dot{x} = x(a + bx + cy), \qquad \dot{y} = y(d + ex + fy).$$

If $af(e - b) = bd(c - f)$ then we can find an integrating factor of the form $x^\alpha y^\beta$.

Another example is given by the Kukles' system

$$\dot{x} = y, \qquad \dot{y} = -x + x^2 - 2y^2 - \tfrac{1}{3}x^3/3 - k^{-1}x^2y + \tfrac{1}{3}k^{-1}y^3,$$

with $k = \pm 2^{1/2}$, examined in [Ch&L]. This has an integrating factor

$$(x(y + kx) + 3k(1 - x))^{-3}\exp(x(1 - \tfrac{1}{2}x)).$$

It may be enquired whether the other integration is also unnecessary. That is, whether there is an elementary first integral whenever there is a Darbouxian integrating factor. However, this is not always the case. Generically, the Lokta-Volterra system has a first integral of the form

$$x^{\alpha+1}y^{\alpha+1}L(x, y),$$

with L a degree polynomial of degree one, but, in the second example the first integral is

$$\begin{aligned}&(y^2(x + 1) + 2kyx(x - 2) + 6(3x5))(x(y + kx) + 3k(1 - x))^{-2}\exp(x(1 - \tfrac{1}{2}x))\\ &+ \int_0^x \exp(u(1 - \tfrac{1}{2}u))\,du,\end{aligned}$$

which is not elementary. However, it is known that for 'generic' classes of Darbouxian integrating factors there also exists a Darbouxian first integral [Ch1].

In the general Żołądek has highlighted two classes of systems with Darbouxian integrating factors whose first integrals are not Darbouxian: those with so-called *Darboux-Schwatz-Christoffel* and *Darboux-Hyperelliptic* first integrals. Each of these can be distinguished from the Darbouxian case by their holonomy groups. The Darboux-Hyperelliptic case is also elementary (an example of which can be found in the counter-example of Prelle and Singer [P&S] p216), whilst the Darboux-Schwatz-Christoffel is non-elementary. It is not known whether these comprise all the possible first integrals.

In conclusion, the search for integrating factors by the use of algebraic invariant curves, which has been in use since the time of Darboux captures all first integrals (1.1) which can be expressed in closed form with quadratures. It is interesting to consider what place other families of functions play in the study of integrable systems. Żołądek [Z], for example, has found new classes of integrals which can be defined using hypergeometric functions.

Similar results are known in the local analytic case. The following proof has the advantage that it can be adapted easily to the case of algebraic differential equations. I would like to thank the referee for helpful comments on the first draft of the paper.

2. Proof of the Theorem

The proof of the theorem follows directly from evaluating the integral (1.2). In fact, the system (1.1) plays no role here at all.

Let K be an algebraic extension of $\mathbf{C}(y)$ which is a splitting field for the numerators and denominators of U and V considered as polynomials in x over $\mathbf{C}(y)$. We can thus rewrite U and V in their partial fraction expansions

$$U = \sum_{\substack{i=1,\ldots,r \\ j=1,\ldots,n_i}} \frac{\alpha_{i,j}}{(x-\beta_i)^j} + \sum_{i=0}^{N} \gamma_i x^i, \qquad V = \sum_{\substack{i=1,\ldots,\bar{r} \\ j=1,\ldots,\bar{n}_i}} \frac{\bar{\alpha}_{i,j}}{(x-\beta_i)^j} + \sum_{i=0}^{\bar{N}} \bar{\gamma}_i x^i,$$

where the $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, β_i and γ_i are elements of K. By taking $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, and γ_i to be zero outside their defined values, we can neglect the explicit mention of the summation limits without confusion.

We now apply the condition $U_y = V_x$ to the above expressions. Gathering terms and using the uniqueness of the partial fraction expansion, we see that

$$\gamma_i' = \bar{\gamma}_{i+1}(i+1), \qquad \alpha_{i,j+1}' + j\beta_i'\alpha_{i,j} + j\bar{\alpha}_{i,j} = 0. \tag{2.2}$$

In particular we have $\alpha_{i,1}' = 0$.

We now write down a function and show that it is indeed the integral of the equation (2.1). Let ϕ be given by

$$\phi = \sum \alpha_{i,1} \log(x-\beta_i) + \sum \frac{\alpha_{i,j}}{(x-\beta_i)^{j-1}} \frac{-1}{j-1} + \sum \frac{\gamma_i x^{i+1}}{i+1} + \int \bar{\gamma}_0 \, dy.$$

It is easy to verify that $\phi_x = U$ and $\phi_y = V$ using (2.2).

We now take the trace of ϕ divided by the number of terms in the summation of the trace. Call this function $\bar{\phi}$. It is clear that this function also satisfies $\phi_x = U$ and $\phi_y = V$. Futhermore we have

$$\bar{\phi} = \sum l_i \log(R_i(x,y)) + R(x,y) + \int S(y)\,dy,$$

where R_i, R and S are rational functions. We can now evalute the integral via the partial fraction expansion of S as a sum

$$\int S(y)\,dy = \sum \alpha_i \log(S_i(y)) + S_0(y),$$

Where the S_i are polynomials in y. Taking exponentials, the integrating factor obtained is of the form stated in the theorem.

3. Applications

We want now to apply the existence of a Darbouxian first integral to the system (1.1) to demonstrate how the theorem applies to the search for Liouvillian first integrals.

In the case where the integral is elementary, It follows from the work of Prelle and Singer [P&S] that there is an integrating factor in the form

$$\prod C_i^{l_i}, \tag{3.1}$$

for some polynomials C_i which satisfy the equation

$$\frac{d}{dt} C_i = C_{ix}P + C_{iy}Q = C_i L_i, \tag{3.2}$$

for some polynomial L_i, which we call the *cofactor* of C_i. If the degrees of P and Q are at most m, then the degree of the cofactor will be at most $m-1$. This is equivalent to the fact that the curves $C = 0$ are invariant algebraic curves of the complex system [Ch2].

The search for elementary solutions to the system is therefore equivalent to the search for invariant algebraic curves of the system. Furthermore, for a fixed system, there is either a finite number of irreducible algebraic curves or a rational first integral. Therefore for any system there exists a bound on the degree of its irreducible algebraic curves. However, there is not an effective way to compute this as yet.

The problem of integration in elementary terms is thus reduced to the following question (Prelle and Singer [P&S] p227):

Give an effective proceedure for computing, given a system of the form (1.1), a bound for the degree of any irreducible polynomials $C(x,y)$ which satisfy (3.2).

Substantial progress has been made recently in this direction by Campillo and Carnicer [C&C].

In the Liouvillian case, Theorem 1 is not powerful enough to give such a formulation. It was the lack of this information which motivated the work here. Now that we have a Darbouxian integrating factor, we are able to finish the characterisation problem. To do this we need only introduce the concept of a 'degenerate algebraic curve', or exponential factor, to cover the term $\exp(D/E)$ which does not appear in (3.1).

It is worth noting that the difference between the integrating factor (3.1) and the Darbouxian integrating factor we obtained in the previous section is more than just the term $\exp(D/E)$, because in the elementary case the l_i can also be chosen to be rational. For example, in the Lokta-Volterra system considered in the introduction we can also find an integrating factor of the form $(xyL)^{-1}$. In the Liouvillian case this is no longer true. For example, the system

$$\dot{x} = -x(1-x)s, \qquad \dot{y} = (1+rx+sy)((\alpha+1)(1-x)-(\beta+1)x)+rx(1-x)+1,$$

has an integrating factor $x^\alpha(1-x)^\beta$ and first integral

$$x^{\alpha+1}(1-x)^{\beta+1}(1+rx+sy)+\int_0^x t^\alpha(1-t)^\beta\,dt.$$

The Kukles' system considered in the introduction would give an example of a system which requires the exponential factor in its integrating factor.

From Theorem 2, we have a first integral

$$R = \exp(D/E)\prod C_i^{l_i},$$

which therefore satisfies the differential equation

$$R_xP + R_y + R(P_x+Q_y) = 0.$$

Without loss of generality, we can assume that D and E are coprime and that the C_i are distinct and irreducible. After some rearrangement, we then have

$$\begin{aligned}\Big((D_xP+D_yQ)E-(E_xP+E_yQ)D\Big)&\prod C_i\\ &+E^2\Big(\sum l_i(C_{ix}P+C_{iy}Q)\prod_{j\neq i}C_i+(P_x+Q_y)\prod C_i\Big)=0.\end{aligned} \tag{3.3}$$

First suppose that E has an irreducible factor F with multiplicity n. If we take $E=F^nG$, then it is clear from the equation above that

$$F^{2n}|\big((D_xP+D_yQ)E-(E_xP+E_yQ)D\big)\prod C_i.$$

since all the C_i are distinct, this implies that

$$F^{2n-1}|(D_xP+D_yQ)E-(E_xP+E_yQ)D$$

and since $2n-1\geq n$ for $n\geq 1$, we have

$$F^n|E_xP+E_yQ = n(F_xP+F_yQ)F^{n-1}+(G_xP+G_yQ)F^n,$$

and so

$$F | F_x P + F_y Q.$$

Thus $F = 0$ is an invariant algebraic curve. Suppose now that C_i is not one of the irreducible factors of E, then equation (3.3) implies that

$$C_i | C_{ix} P + C_{iy} Q,$$

And so all the C_i define invariant algebraic curves of the system.

It now remains to characterise the polynomial D. This is best done by considering the expression $\exp(D/E)$ as a whole. It is clear from equation (3.3), using equation (3.2) for the C_i, that we have:

$$\exp(D/E)_x P + \exp(D/E)_y Q = \exp(D/E) M, \tag{3.4}$$

where M is a polynomial of degree at most $m-1$. That the form of this equation is the same as (3.2) is no coincidence, and in fact the term $\exp(D/E)$ can be considered as a limit of coalescing curves—in the same way as the coalescence of two exponential solutions in a linear ODE give rise to the 'degenerate' solution $xe^{\lambda x}$ (see [Ch2] for a more detailed justification of this). Rewriting equation (3.4), we obtain

$$D_x P + D_y Q - DL = EM, \tag{3.5}$$

Where L is the cofactor of E.

Suppose we have more than $m(m+1)/2$ linearly independent rational functions D_i/E_i such that

$$\frac{d}{dt} \exp(D_i/E_i) = \exp(D_i/E_i) M_i.$$

with M_i of degree at most $m-1$, then we can choose constants l_i such that

$$\frac{d}{dt} \prod (\exp(D_i/E_i))^{l_i} = 0;$$

whence

$$\sum l_i (D_i/E_i)$$

is a rational first integral of the system. Thus, if (1.1) does not have a rational first integral, there exists a bound on the number of independent D_i/E_i which satisfy (3.5) and hence on the degree of the numerator and denominator of any rational function D/E which satisfies (3.4).

Hence the search for Liouvillian first integrals is reduced to the search for invariant algebraic curves, and exponential factors of the form $\exp(D/E)$. Of course to make the whole thing algorithmic, we need to be able to answer the following question in the spirit of Prelle and Singer:

(i) Give an effective proceedure for computing, given a system of the form (1.1), a bound for the degree of any irreducible polynomials $C(x, y)$ which satisfy (3.2).

(ii) Give an effective proceedure for computing, given a particular system (1.1) *without rational first integral, a bound on the degree of any polynomials D and E which satisfy equation (3.5), given that E also satifies* (3.2) *with cofactor L.*

The question of the existence of a rational first integral required in (ii) can be effectively decided from (i) and so the whole process of seeking Liouvillian first integrals would be algorithmic.

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