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Local existence and stability for a hyperbolic–elliptic system modeling two–phase reservoir flow *

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Abstract

A system arising in the modeling of oil–recovery processes is analyzed. It consists of a hyperbolic conservation law governing the saturation and an elliptic equation for the pressure. By an operator splitting approach, an approximate solution is constructed. For this approximation appropriate a–priori bounds are derived. Applying the Arzela–Ascoli theorem, local existence and uniqueness of a classical solution for the original hyperbolic–elliptic system is proved. Furthermore, convergence of the approximation generated by operator splitting towards the unique solution follows. It is also proved that the unique solution is stable with respect to perturbations of the initial data.

1 Introduction

The purpose of this paper is to study the system of partial differential equations

$$\begin{aligned} s_t + \nabla \cdot [f(s)v] &= g\,, \\ -\nabla \cdot [\lambda(s)\nabla p] &= q\,, \\ v &= -\lambda(s)\nabla p\,. \end{aligned} \tag{1}$$

This system is a prototypical model of incompressible two–phase flow in an oil reservoir. Here s denotes the water–saturation and $1-s$ is the oil–saturation. The function $f = f(s)$ is referred to as the fractional–flow function and $\lambda = \lambda(s)$ denotes the sum of the phase mobilities. The pressure and the total velocity are denoted by p and v respectively, and g, q denote source/sink terms of the model.

The system (1) is a vital part of virtually any reservoir simulator. The first equation is usually referred to as the *saturation equation* and the second is correspondingly referred to as the *pressure equation.* The saturation equation is

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of hyperbolic nature whereas the pressure equation is elliptic; thus we consider a coupled *hyperbolic–elliptic* system. The basic properties of the system are derived by e.g. Peaceman [18] and Ewing [8]. Variants of the system have been analyzed by a series of authors; Alt and DiBenedetto [2] and Kruzkov and Sukorjanskii [16] proved existence and uniqueness results for smooth solutions of this system in the presence of capillary forces. In this case there is an additional viscous term in the saturation equation. For miscible flow, Feng [9] and recently Chen and Ewing [4] obtained similar results. In all these cases the system is a coupled parabolic–elliptic system whereas the system considered here is a coupled hyperbolic–elliptic problem. The latter problem was analyzed by Frid [10] who used a regularization of the velocity field to obtain smooth solutions.

The question related to viscous fingering has been addressed by e.g. Chorin [5], Christie [6], Glimm, Marchesin and McBryan [13] and others. Also the questions related to numerical solution of this system have gained a lot of interest; cf. e.g. Glimm, Isaacson, Marchesin and McBryan [12] or Bratvedt, Bratvedt, Buchholz, Gimse, Holden, Holden, and Risebro [3].

In the present paper our aim is to analyze the hyperbolic–elliptic model above, i.e. the case of immiscible, incompressible two–phase flow in a porous medium. We will prove that this system, for a finite time, possesses a unique and stable smooth solution. The existence is proved by a constructive argument relying on an application of the Arzela–Ascoli theorem to obtain the limit of a family of approximate solutions. Uniform bounds, in the discretization parameter, on derivatives of the approximate solutions are derived in proper Hölder–norms. These bounds blow up in finite time as should be expected since the solutions are known to develop discontinuities as time evolves. Thus, our results are only valid for a limited time. After blow–up of the derivatives, the solutions may continue to exist in a weaker topology. However, an existence result of the system (1) allowing shocks in the saturations are not known to the authors. The main contribution of the present paper is that the system is analyzed without any regularization and without smoothing diffusion terms.

The outline of this paper is as follows. In the next section we state the precise assumptions on the model and present the main result. In Section 3 an outline of the proof is given. Section 4 is concerned with the properties of the pressure equation. The basic bounds needed for the velocities are derived through Schauder estimates carefully derived in Ladyzenskaya and Ural'tseva [17], cf. also Gilbarg and Trudinger [11]. In Section 5 we study linear advection problems and the results for these equations are applied to the linearized saturation equation in Section 6. As mentioned above, estimates on the spatial derivatives of the approximate solutions imply, by the Arzela–Ascoli theorem, the convergence of the family of approximate solutions. In Section 7 it is proved that the limit also is sufficiently smooth in time and the nonlinear saturation equation holds. Finally, we derive stability estimates for the initial value problem.

2 The main result

In this section we will give the precise assumptions on the mathematical model under consideration. But first, we have to introduce some notation. The basic properties of norms and function spaces used here are discussed in e.g. [1, 11, 17]. With $x \cdot y = \sum_j x_j y_j$ and $|x| = \sqrt{x \cdot x}$ we denote the Euklidian inner product and norm in $\mathbb{R}^n$. The usual sup–norm for bounded functions $f \in L^\infty$ is denoted by $\|\cdot\|_{L^\infty}$. Similarly $\|\cdot\|_{L^p}$ is the L^p–norm and $\|\cdot\|_{H^m}$ is the Sobolev norm $\|f\|_{H^m} = \sum_{|\beta| \le m} \|D^\beta f\|_{L^2}$, where $\beta \in \mathbb{N}_0^d$ is a multi-index, $|\beta| = \sum_{j=1}^d \beta_j$ and

$$D^\beta = \frac{\partial^{|\beta|}}{\partial_{x_1}^{\beta_1} \ldots \partial_{x_d}^{\beta_d}}.$$

For functions $f \in C^k(\Omega)$ we write $\|f\|_k = \sum_{|\beta| \le k} \|D^\beta f\|_{L^\infty}$. A function f is called α–Hölder continuous, if the quantity

$$|f|_\alpha = \sup_{x \ne y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \quad 0 < \alpha \le 1$$

is finite. Note that Hölder continuity with $\alpha = 1$ is the same as Lipschitz continuity. The Hölder spaces are defined as subspaces of $C^k(\Omega)$

$$C^{k,\alpha}(\Omega) := \{f \in C^k(\Omega) : \ \|f\|_{k,\alpha} < \infty\}, \quad k \in \mathbb{N}, \quad 0 < \alpha \le 1$$

with the associated norms $\|f\|_{k,\alpha} = \|f\|_k + \sum_{|\beta|=k} |D^\beta f|_\alpha$.

We consider incompressible and immiscible two–phase flow in a two– or three–dimensional reservoir $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. A derivation of the model can be found in the books by Ewing [8] and Peaceman [18]:

$$\begin{aligned} s_t + \nabla \cdot [f(s)v] &= g\,, \\ -\nabla \cdot [\lambda(s)\nabla p] &= q\,, \\ v = -\lambda(s)\nabla p\,,\ t \ge 0\,,\ x &\in \Omega \end{aligned} \tag{2}$$

Here, Ω is the domain defined by the reservoir. Since s denotes the saturation of the aqueous phase, the first equation is often referred to as the *saturation equation*. Similarly, the second equation is referred to as the *pressure equation*.

Furthermore, the system (2) is augmented with the following initial and boundary conditions

$$s(0, x) = s^0(x), \quad x \in \bar{\Omega}; \quad \frac{\partial p}{\partial n} = 0, \quad x \in \partial\Omega; \quad \int_\Omega p\, d\Omega = 0. \tag{3}$$

Here, n denotes the outward normal to the boundary $\partial\Omega$. We assume that the reservoir initially is filled with oil such that

$$0 \le s^0(x) < s^+ \ll 1, \quad x \in \bar{\Omega}.$$

In order to ensure that the saturation remains in the unit interval, we only seek solutions up to time

$$t \leq t_\star = \frac{1 - s^+}{\|q_o\|_{L^\infty} + \|q_w\|_{L^\infty}}. \tag{4}$$

More precisely, we will seek a solution for $t \leq t_\star$ if the solution stays sufficiently smooth in this time-interval. If smoothness breaks down at $T_\star < t_\star$ we will only prove existence up to $T_\star$. This point will be clarified below where we derive a–priori estimates of the appropriate Hölder norms of the solutions.

The functions involved are assumed to satisfy the following requirements

$$\begin{array}{rl}
(i) & f \in C^\infty([0,1],[0,1]), \\
(ii) & \lambda(s) \geq c_0 > 0, \quad \lambda \in C^\infty[0,1], \\
(iii) & s^0 \in C^{1,\alpha}(\bar\Omega), \quad 0 \leq s^0(x) \leq s^+ \ll 1, \quad x \in \bar\Omega, \\
(iv) & q, g \in C^\infty(\mathbb{R}^+ \times \bar\Omega), \quad q(t,\cdot), g(t,\cdot) \in C_0^\infty(\bar\Omega), \\
(v) & g \geq 0, \quad g - q \geq 0, \quad \displaystyle\int_\Omega q(t,\cdot)\, d\Omega = 0, \quad (t,x,y) \in \mathbb{R}^+ \times \Omega, \\
(vi) & \Omega \subset \mathbb{R}^d \text{ is a smoothly bounded domain } \partial\Omega \in C^{2,\alpha} \text{ and } d = 2 \text{ or } 3.
\end{array} \tag{5}$$

Now, we are in the position to state the main result of the present paper.

Theorem 2.1 *Assume the data satisfy the requirements listed in (5). Then, the two–phase reservoir model (2) with initial and boundary conditions (3) has locally in time a unique, classical solution. More precisely: There is a time* $T > 0$ *and a unique solution of (2) and (3) such that*

$$s(t,\cdot) \in C^{1,\alpha}(\bar\Omega), \quad s(\cdot,x) \in C^1([0,T]), \quad p(t,\cdot) \in C^{2,\alpha}(\bar\Omega), \quad t \in [0,T], \quad x \in \bar\Omega.$$

For any two solutions (s,p) *and* (S,P) *with initial data* s^0 *and* S^0 *satisfying (5) (iii) the following stability estimates hold*

$$\begin{array}{rcl}
\|(s-S)(t,\cdot)\|_{L^2} & \leq & M\|s^0 - S^0\|_{L^2}, \\
\|(p-P)(t,\cdot)\|_{H^1} & \leq & M\|s^0 - S^0\|_{L^2}, \quad t \in [0,T].
\end{array} \tag{6}$$

The constant M *depends on the smoothness of the solutions and the time* T.

Remark. The corresponding stability estimate for the flow speed $v = -\lambda(s)\nabla p$ follows easily from the above estimates for the saturation and the pressure gradient

$$\|(v - V)(t,\cdot)\|_{L^2} \leq M\|s^0 - S^0\|_{L^2}, \quad t \in [0,T].$$

The model considered in this paper relies on a series of assumptions. Some of these are reasonable from a physical point of view and some are not. The assumption of compact support for the source terms is natural, as injection and production usually takes place only in a few spots in the reservoir. Let us discuss out the non–physical assumptions more specifically.

1. The main disadvantage of the present analysis is, that we are only able to handle the case of smooth solutions. We prove that such solutions exist for a while, but we also know that they will subsequently develop discontinuities and thus probably exist in a weaker topology. Compared with other work in this field, we are able to avoid the introduction of diffusion terms and other types of regularization at the prize of getting existence locally in time.
2. We have assumed constant porosity ϕ and constant absolute permeability K. These assumptions can, without difficulties, be relaxed to cover the case of sufficiently smooth and uniformly positive functions. However, real porosity and absolute permeability can be discontinuous functions in which case the solutions cease to exist in the spaces considered here.
3. More general, smooth source terms can be included. Also more general smooth initial data can be allowed provided that these data and the associated source terms imply the proper invariant region for the saturations. The assumptions applied here are put up in order to reduce the technicalities.
4. The present analysis balances the smoothness of the saturation equation and the pressure equation such that subsequent solutions of these equations generate solutions of proper regularity. If discontinuities are allowed in either porosity, sources, sinks or absolute permeabilities, this balance is lost and our argument fails. In order to prove existence for such data, a theory covering discontinuous solutions must be developed. Typically, one would look for existence of a BV saturation and a H^1 pressure. But such a theory is not known for this system.

3 Outline of the argument

Our approach to prove local existence of a solution is based on a–priori bounds for a family of approximate solutions and the Arzela–Ascoli theorem. For the elliptic pressure equation so–called Schauder estimates are available in the literature. To make use of these estimates, we decompose the coupled system into the hyperbolic and the elliptic sub–problems by a fractional step approach.

Define an **approximate solution** (s_Δ, p_Δ) as follows:
Initially $n = 0$ and $s^0_\Delta = s^0 \in C^{1,\alpha}$ is given.
Then repeat the following three steps for $n = 0, 1, 2, \ldots, N = \min(t_\star, T_\star)/\Delta t$.

i) Define p^n_Δ as the solution of

$$\begin{aligned} -\nabla\cdot(\lambda(s^n_\Delta)\nabla p^n_\Delta) &= q^n\,,\ x\in\Omega\,,\\ \frac{\partial p^n_\Delta}{\partial n} &= 0\,,\ x\in\partial\Omega\,,\\ \int_\Omega p^n_\Delta\, d\Omega &= 0\,. \end{aligned} \tag{7}$$

ii) Set $v^n_\Delta = -\lambda(s^n_\Delta)\nabla p^n_\Delta$, $x \in \bar\Omega$.

iii) Solve the variable coefficient advection equation

$$\frac{\partial}{\partial t} s_\Delta + f'(s^n_\Delta) v^n_\Delta \cdot \nabla s_\Delta = g^n - f(s^n_\Delta) q^n, \quad (t,x) \in (t_n, t_{n+1}) \times \Omega, \quad (8)$$

with initial data $s_\Delta(t_n, \cdot) = s^n_\Delta$ for one time step $t_n \to t_{n+1} = (n+1)\Delta t$.

Note that we can switch between the conservative and the non–conservative form of the saturation equation. ¿From the pressure equation (7) it follows

$$\nabla \cdot v^n_\Delta = q^n.$$

Therefore, for smooth solutions the saturation equation is equivalent to

$$\frac{\partial}{\partial t} s_\Delta + f'(s_\Delta) v^n_\Delta \cdot \nabla s_\Delta = g - f(s_\Delta) q^n,$$

which is approximated by (8).

Clearly, the approximate solution (s_Δ, p_Δ) depends on the time step Δt. In the next section we shall derive bounds on the pressure p^n_Δ and the flow speed v^n_Δ in terms of the saturation s^n_Δ. Since the flow speed is frozen during one time step for the saturation equation (8), these bounds can be used to obtain bounds for the saturation s^{n+1}_Δ at the next time level. Up to some time $T > 0$ we obtain a family of uniformly smooth functions $(s_\Delta(t,\cdot), p_\Delta(t,\cdot))$ in space ie. we bound $\|s_\Delta(t,\cdot)\|_{1,\alpha}$ and hence $\|p_\Delta(t,\cdot)\|_{2,\alpha}$ independent of Δt. In Section 6 existence of a smooth limit as $\Delta t \to 0$ can be inferred from the Arzela–Ascoli theorem. By construction $s_\Delta(\cdot, x)$ is continuous but not differentiable with respect to time. Differentiability of the limit $s := \lim_{\Delta t \to 0} s_\Delta$ depends on the continuity of the speed $v_\Delta = -\lambda(s_\Delta)\nabla p_\Delta$ as a function in time. In Section 7 we show that v_Δ is continuous in time. Sending Δt to zero, we obtain a classical solution for the coupled reservoir system (2). The stability of this solution follows from the stability of the pressure equation with respect to the right hand side and an energy estimate for the saturation. In Section 10 we summarize the proof of Theorem 2.1.

Finally, the uniqueness of the limit implies that not only a subsequence —obtained by the Arzela–Ascoli theorem— but also the complete family of approximate solutions (s_Δ, p_Δ) converges as $\Delta t \to 0$. Since the above operator splitting approach is used in most of the available reservoir simulation software, we state this convergence as the second main result of the present paper. It will be proved in Section 9.

Theorem 3.1 *Under the general assumptions (5), the approximation (s_Δ, p_Δ) generated by the operator splitting approach above converges to the unique classical solution (s, p) of the reservoir model (2) with boundary conditions (3):*

$$\|s_\Delta(t,\cdot) - s(t,\cdot)\|_1 \to 0, \quad \|p_\Delta(t,\cdot) - p(t,\cdot)\|_2 \to 0, \quad t \in [0,T],$$

as $\Delta t \to 0$.

4 The pressure equation

As mentioned above, we have to solve an elliptic pressure equation at each time step. In this section we discuss the regularity of the solution of this equation.

Consider a given saturation function $s = s(x)$ and a source term $q = q(x)$ such that

$$s \in C^{1,\alpha}(\bar{\Omega}), \quad 0 \le s(x) \le 1, \quad q \in C^{\infty}(\bar{\Omega}), \quad \int_{\Omega} q \, d\Omega = 0.$$

The corresponding pressure is defined by

$$\begin{aligned} -\nabla \cdot (\lambda(s)\nabla p) &= q\,,\ x \in \Omega\,, \\ \frac{\partial p}{\partial n} &= 0,\ x \in \partial\Omega\,, \\ \int_{\Omega} p \, d\Omega &= 0\,. \end{aligned} \tag{9}$$

This variable-coefficient elliptic type boundary value problem has been carefully studied, cf. e.g. Gilbarg and Trudinger [11] or Ladyzenskaya and Ural'tseva [17]. It follows from [17] (Section 3.3 estimate (3.6)) that (9) has a unique solution $p \in C^{2,\alpha}(\bar{\Omega})$ satisfying the bound

$$\begin{aligned} \|p\|_{2,\alpha} \quad \le \quad & M\Big\{(1+|s|_{\alpha})\|q\|_{L^\infty} + |q|_{\alpha} + \|p\|_{L^\infty}\Big[1 + (1+|s|_{\alpha})^{\frac{2+\alpha}{\alpha}} \\ & + \sum_{|\beta|=1}\Big((1+\|D^{\beta}s\|_{L^\infty})^{2+\alpha} + (1+|D^{\beta}s|_{\alpha})^{\frac{2+\alpha}{1+\alpha}}\Big)\Big]\Big\}. \end{aligned}$$

The constant M depends only on the ellipticity constant c_0 (cf. (5)) and the smoothness of λ and $\partial\Omega$. Since $1 + a \le \exp(a)$ for any a, it follows that

$$\begin{aligned} \|p\|_{2,\alpha} \quad \le \quad & M\Big\{\|q\|_{0,\alpha}\exp(\|s\|_{1,\alpha}) + \|p\|_{L^\infty}\Big[1 + \exp(\frac{2+\alpha}{1+\alpha}\|s\|_{1,\alpha}) \\ & + \exp((2+\alpha)\|s\|_{1,\alpha}) + \exp(\frac{2+\alpha}{\alpha}\|s\|_{1,\alpha})\Big]\Big\}. \end{aligned}$$

Since

$$\frac{2+\alpha}{1+\alpha} < 2+\alpha < \frac{2+\alpha}{\alpha}, \quad \alpha \in (0,1)$$

it follows

$$\|p\|_{2,\alpha} \le M\left\{\|q\|_{0,\alpha}\exp(\|s\|_{1,\alpha}) + \|p\|_{L^\infty}\exp(\frac{2+\alpha}{\alpha}\|s\|_{1,\alpha})\right\}. \tag{10}$$

Here, we need to bound $\|p\|_{L^\infty}$ in terms of the data q and s. The outline for this bound is as follows. By a Sobolev embedding (cf. [1] Theorem 5.4 part II) $H^2(\Omega)$ is embedded in $C^{0,\delta}(\bar{\Omega})$ where $0 < \delta \le 2 - d/2 \le 1/2$ and $d = 2, 3$. Hence,

$$\|p\|_{L^\infty} \le \|p\|_{0,\delta} \le M\|p\|_{H^2}.$$

Here, and in the rest of the paper, M denotes a generic constant. Furthermore, $p \in H^2$ and

$$\|p\|_{H^2} \le M(\|q\|_{L^2} + \|s\|_{1,\alpha}\|p\|_{H^1}). \tag{11}$$

This bound will be derived below. Finally, it is well known, c.f. Hackbusch [14] p. 152, that

$$\|p\|_{H^1} \le M\|q\|_{L^2}.$$

The last three estimates yield

$$\|p\|_{L^\infty} \le M\|p\|_{H^2} \le M(1 + \|s\|_{1,\alpha})\|q\|_{L^2}.$$

Since Ω is bounded and q is smooth

$$\|p\|_{L^\infty} \le M\,(1 + \|s\|_{1,\alpha})\,\|q\|_{L^\infty}.$$

This estimate is used in (10) to find

$$\|p\|_{2,\alpha} \le M\|q\|_{0,\alpha}\,(1 + \|s\|_{1,\alpha})\exp(\frac{2+\alpha}{\alpha}\|s\|_{1,\alpha}).$$

Considering q as a given smooth function and α as a constant this implies the following bounds for p and $v = -\lambda(s)\nabla p$

$$\|p\|_{2,\alpha} \le M\exp(M\|s\|_{1,\alpha}), \qquad \|v\|_{1,\alpha} \le M\exp(M\|s\|_{1,\alpha}), \tag{12}$$

which will be crucial in the estimate of the saturation derived in the next section.

It remains to derive the estimate (11). A direct application of Hackbusch [14] p. 218 ff. would give a bound like

$$\|p\|_{H^2} \le C(\|q\|_{L^2} + \|p\|_{H^1}),$$

where the "constant" C depends on the coefficients of the elliptic operator

$$-\nabla\cdot(\lambda(s)\nabla p)$$

and hence on the saturation s. To avoid this implicit dependence, we rewrite the pressure equation on Poisson form. Since $p \in H^2$ it follows that

$$\Delta p = -\frac{1}{\lambda(s)}\,[q + \lambda'(s)\nabla s\cdot\nabla p] =: rhs.$$

For this type of equation it follows from standard elliptic theory (cf. [14] Theorem 9.1.16) that

$$\|p\|_{H^2} \le M\,(\|rhs\|_{L^2} + \|p\|_{H^1}) \le M\|rhs\|_{L^2}.$$

Here, the constant M depends only on the geometry of Ω. Since $\lambda(s) \ge c_0 > 0$ is smooth and $0 \le s \le 1$ the right hand side is bounded by

$$\begin{aligned}
\|rhs\|_{L^2} &\le \frac{1}{c_0}\,(\|q\|_{L^2} + \|\lambda'(s)\|_{L^\infty}\|\nabla s\cdot\nabla p\|_{L^2}) \\
&\le M\Big(\|q\|_{L^2} + \sum_{|\beta|=1}\|D^\beta s D^\beta p\|_{L^2}\Big) \\
&\le M\Big(\|q\|_{L^2} + \|s\|_{1,\alpha}\sum_{|\beta|=1}\|D^\beta p\|_{L^2}\Big).
\end{aligned}$$

¿From $(a+b)^2 \leq 2(a^2+b^2)$, $a, b \geq 0$ it follows that

$$\sum_{|\beta|=1} \|D^\beta p\|_{L^2} \leq M\|\nabla p\|_{L^2} \leq M\|p\|_{H^1},$$

thus

$$\|p\|_{H^2} \leq M\|rhs\|_{L^2} \leq M\left(\|q\|_{L^2} + \|s\|_{1,\alpha}\|p\|_{H^1}\right),$$

which is exactly the estimate (11).

5 The advection equation

In this section the saturation equation with frozen speed (8) is analyzed. It is a variable coefficient advection equation of the generic type

$$s_t + w \cdot \nabla s = h, \quad t \geq 0, x \in \Omega \tag{13}$$

with data

$$s(0,\cdot) = s^0, \quad w \cdot n = 0. \tag{14}$$

By the assumption $w \cdot n = 0$ the boundary is characteristic and we do not need to specify boundary data. Based on characteristics, we construct the classical solution and derive estimates in terms of the data s^0, w and h.

Lemma 5.1 *With data $s^0, w \in C^1(\bar\Omega)$ and $h \in C^1(\mathbb{R}^+ \times \bar\Omega)$ the Cauchy problem (13) and (14) has a unique smooth solution $s \in C^1(\mathbb{R}^+ \times \bar\Omega)$ and the following bound holds:*

$$\|s(t,\cdot)\|_1 \leq [1 + MCt\exp(MCt)]\,\|s^0\|_1 + t\|h\|_1.$$

With data $s^0, w \in C^{1,\alpha}(\bar\Omega)$ and $h \in C^{1,\alpha}(\mathbb{R}^+ \times \bar\Omega)$ it follows $s(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$ and

$$\|s(t,\cdot)\|_{1,\alpha} \leq [1 + MCt\exp(MCt)]\,\|s^0\|_{1,\alpha} + Mt\exp(MCt)\|h\|_{1,\alpha}.$$

Here, C is a bound for $\|w\|_1$ or $\|w\|_{1,\alpha}$ respectively.

Proof. The proof of this result is divided into three steps. First, the solution is constructed with C^1–data. Then, the C^1–estimate is derived. In a third step the additional regularity with $C^{1,\alpha}$–data will be proved. The uniqueness is obvious, as the equation is linear.

1. Existence: The point here is to observe that C^1–regularity of the data is enough to construct a smooth solution. The solution will be defined with the help of characteristics $c = c(t, \bar x, \bar t)$ given by

$$c_t = w(c), \quad t \in \mathbb{R}, \quad c(\bar t, \bar x, \bar t) = \bar x \in \bar\Omega.$$

As the characteristic speed is differentiable, the characteristic is unique and depends smoothly on the initial value $D_{\bar x} c(t,\cdot,\bar t) \in C(\Omega)$. Since the boundary itself is characteristic, all characteristics started in $\bar{\Omega}$ stay in $\bar{\Omega}$ and therefore they exist for all time.

We define a function $s(t, x)$ by pointwise backtracking the characteristic. For any $\bar{t} \geq 0$ and $\bar{x} \in \bar{\Omega}$ we follow the characteristic back to $t = 0$. Let $x^0 := c(0, \bar{x}, \bar{t})$ and set

$$s(\bar{t}, \bar{x}) := \int_0^{\bar{t}} h\left(t, c(t, \bar{x}, \bar{t})\right)\, dt + s^0(x^0), \quad \bar{x} \in \bar{\Omega}, \bar{t} \geq 0. \tag{15}$$

As the characteristics are unique, this is an appropriate pointwise definition. If we can show that s is differentiable, then

$$\frac{d}{dt} s(t, c(t, \bar{x}, \bar{t})) = s_t + \nabla s \cdot c_t = s_t + w \cdot \nabla s = h(t, c(t, \bar{x}, \bar{t})),$$

hence s is a solution of (13).

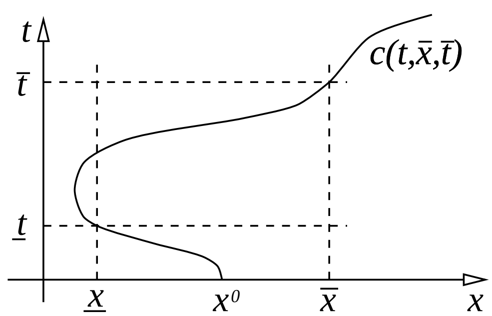
To study the differentiability of s, denote by $x(t) = c(t, \bar{x}, \bar{t})$ and $y(t) = c(t, \bar{y}, \bar{t})$ two different characteristics $\bar{x} \neq \bar{y}$. Then consider

$$s(\bar{t}, \bar{x}) - s(\bar{t}, \bar{y}) = \int_0^{\bar{t}} \int_0^1 \nabla h\left(t, \eta(t, \xi)\right)\, d\xi\, \left(x(t) - y(t)\right)\, dt + s^0(x^0) - s^0(y^0),$$

where $\eta(t, \xi) = y(t) + \xi\left(x(t) - y(t)\right)$, $\xi \in [0, 1]$. Since the characteristic depends smoothly on the initial value, we have

$$x(t) - y(t) = \int_0^1 D_x c(t, \bar{\eta}(\xi), \bar{t})\, d\xi (\bar{x} - \bar{y}), \quad t \geq 0,$$

where $\bar{\eta}(\xi) = \bar{y} + \xi\left(\bar{x} - \bar{y}\right)$. Finally, since ∇h, ∇s^0 and $D_x c$ are continuous functions and $\eta(t, \xi) \to y(t)$ as $\bar{x} \to \bar{y}$, the quotient $(s(\bar{t}, \bar{x}) - s(\bar{t}, \bar{y}))/|\bar{x} - \bar{y}|$ is continuous and the limit $\bar{x} \to \bar{y}$ exists. This shows that s is differentiable with respect to x.



Concerning the time variable we select $\bar{x} \in \bar{\Omega}$ and $0 \leq \underline{t} \leq \bar{t}$. Using the characteristic $c(t, \bar{x}, \bar{t})$ we express the time difference $s(\bar{t}, \bar{x}) - s(\underline{t}, \bar{x})$ by a space difference. We have

$$s(\bar{t}, \bar{x}) = s(\underline{t}, \underline{x}) + \int_{\underline{t}}^{\bar{t}} h\left(\tau, c(\tau, \bar{x}, \bar{t})\right)\, d\tau$$

where $\underline{x} := c(\underline{t}, \bar{x}, \bar{t})$ and thus

$$\frac{s(\bar{t},\bar{x}) - s(\underline{t},\bar{x})}{\bar{t}-\underline{t}} = \frac{s(\underline{t},\underline{x}) - s(\underline{t},\bar{x})}{\bar{t}-\underline{t}} + \frac{\displaystyle\int_{\underline{t}}^{\bar{t}} h\left(\tau, c(\tau,\bar{x},\bar{t})\right)\, d\tau}{\bar{t}-\underline{t}}.$$

By construction $c_t = w(c)$, hence

$$\lim_{\underline{t}\to\bar{t}} \frac{\bar{x}-\underline{x}}{\bar{t}-\underline{t}} = c_t(\bar{t},\bar{x},\bar{t}) = w(c(\bar{t},\bar{x},\bar{t})) = w(\bar{x}).$$

With $s(t,\cdot) \in C^1(\bar{\Omega})$, it follows

$$\lim_{\underline{t}\to\bar{t}} \frac{s(\underline{t},\underline{x}) - s(\underline{t},\bar{x})}{\bar{t}-\underline{t}} = \nabla s(\bar{t},\bar{x}) w(\bar{x}).$$

As c and h are continuous,

$$\lim_{\underline{t}\to\bar{t}} \frac{s(\bar{t},\bar{x}) - s(\underline{t},\bar{x})}{\bar{t}-\underline{t}} = \nabla s(\bar{t},\bar{x}) w(\bar{x}) + h(\bar{t},\bar{x}).$$

Therefore, s is differentiable with respect to t and $s \in C^1(\mathbb{R}^+ \times \bar{\Omega})$.

2. The estimate: ¿From the definition (15) it is obvious that

$$\|s(t,\cdot)\|_{L^\infty} \leq \|s^0\|_{L^\infty} + t\|h\|_{L^\infty}. \tag{16}$$

To estimate first order derivatives of s, we apply the gradient to equation (13). Along characteristics it follows

$$\frac{d}{dt}\nabla s + Dw^T \cdot \nabla s = \nabla h,$$

where Dw is the Jacobian of w. Hence,

$$\begin{aligned}\nabla s(t, c(t,x^0,0)) \quad &= \quad \exp\left(-\int_0^t Dw^T(c(\tau,x^0,0))\, d\tau\right) \nabla s^0(x^0) \\ &+ \quad \int_0^t \nabla h(\tau, x(\tau,x^0,0))\, d\tau,\end{aligned}$$

where c is a characteristic. Because each element of Dw is bounded by C, we have $|Dw^T| \leq MC$ and

$$\left|\exp\left(-\int_0^t Dw^T(c(\tau,x^0,0))\, d\tau\right) - I\right| \leq MCt\exp(MCt). \tag{17}$$

For $|\beta| = 1$ it follows

$$\left|D_x^\beta s(t, c(t,x^0,0))\right| \leq \|D_x^\beta s^0\|_{L^\infty} + MCt\exp(MCt) \sum_{|\beta|=1} \|D_x^\beta s^0\|_{L^\infty} + t\|D_x^\beta h\|_{L^\infty}$$

and therefore

$$\sum_{|\beta|=1} \|D_x^\beta s(t,\cdot)\|_{L^\infty} \le [1+MCt\exp(MCt)] \sum_{|\beta|=1} \|D_x^\beta s^0\|_{L^\infty} + t \sum_{|\beta|=1} \|D_x^\beta h\|_{L^\infty}.$$

In combination with (16) the bound in C^1 follows

$$\|s(t,\cdot)\|_1 \le [1+MCt\exp(MCt)]\,\|s^0\|_1 + t\|h\|_1. \tag{18}$$

Note that M depends on the dimension d.

3. Additional regularity: Next, we assume slightly more regularity for the data, namely (14), and show that the solution has the same regularity i.e. $s(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$. Consider again two different characteristics $x(t) = c(t,x^0,0) \neq c(t,y^0,0) = y(t)$, then

$$\begin{aligned}
\nabla s(t,x(t)) - \nabla s(t,y(t)) \quad = \quad & \nabla s^0(x^0) - \nabla s^0(y^0) \\
& + [\exp(A(x)) - I]\left(\nabla s^0(x^0) - \nabla s^0(y^0)\right) \\
& + [\exp(A(x)) - \exp(A(y))]\,\nabla s^0(y^0) \\
& + \int_0^t \nabla h(\tau,x(\tau)) - \nabla h(\tau,y(\tau))\,d\tau,
\end{aligned} \tag{19}$$

where $A(x) := -\int_0^t Dw^T(x(\tau))\,d\tau$. As the convex combination

$$B(\xi) = A(y) + \xi\,(A(x) - A(y))\,, \quad \xi \in [0,1]$$

is bounded

$$|B(\xi)| \le \max\left(|A(x)|, |A(y)|\right) \le t\|Dw^T\|_{L^\infty} \le MCt$$

it follows

$$|\exp\left(A(x)\right) - \exp\left(A(y)\right)| \le \exp\left(MCt\right)|A(x) - A(y)|.$$

Here, we use $w \in C^{1,\alpha}$ and find

$$|\exp\left(A(x)\right) - \exp\left(A(y)\right)| \le \exp\left(MCt\right)|Dw^T|_\alpha \int_0^t |x(\tau) - y(\tau)|^\alpha\,d\tau. \tag{20}$$

Again each element of Dw has a bounded Hölder coefficient, therefore $|Dw^T|_\alpha \le MC$. It follows from (19) using the estimates (17),(20) as well as $C^{1,\alpha}$–smoothness of s^0 and h

$$\begin{aligned}
\left|D_x^\beta s(t,x(t)) - D_x^\beta s(t,y(t))\right| \le & \left|D_x^\beta s^0\right|_\alpha |x^0 - y^0|^\alpha \\
& + MCt\exp\left(MCt\right) \sum_{|\beta|=1} \left|D_x^\beta s^0\right|_\alpha |x^0 - y^0|^\alpha \\
& + MC\exp\left(MCt\right) \sum_{|\beta|=1} \|D_x^\beta s^0\|_{L^\infty} \int_0^t |x(\tau) - y(\tau)|^\alpha\,d\tau \\
& + \sum_{|\beta|=1} \left|D_x^\beta h\right|_\alpha \int_0^t |x(\tau) - y(\tau)|^\alpha\,d\tau, \quad |\beta| = 1.
\end{aligned} \tag{21}$$

Here, we need a bound for $\gamma(\tau) = x(\tau) - y(\tau)$ in terms of $\gamma(t)$ where $0 \le \tau \le t$. By definition

$$\gamma'(t) = w(x(t)) - w(y(t)).$$

Let $\eta(t,\xi) = y(t) + \xi(x(t) - y(t))$, $\xi \in [0,1]$, then it holds

$$\gamma'(t) = w(\eta(t,1)) - w(\eta(t,0)) = \int_0^1 Dw(\eta(t,\xi))\, d\xi\ \gamma(t).$$

Now, consider the derivative

$$\frac{d}{dt}|\gamma(t)|^2 = 2 < \gamma(t), \gamma'(t) >= 2 < \gamma(t), \int_0^1 Dw(\eta(t,\xi))\, d\xi\ \gamma(t) > .$$

As Dw is bounded it follows

$$-MC|\gamma(t)|^2 \le \frac{d}{dt}|\gamma(t)|^2 \le MC|\gamma(t)|^2.$$

Using the lower bound, we find the desired estimates

$$|\gamma(\tau)| \le \exp(MC(t-\tau))|\gamma(t)|, \quad \tau \le t$$

and

$$\int_0^t |\gamma(\tau)|^\alpha\, d\tau \le t \exp(\alpha MCt)|\gamma(t)|^\alpha.$$

Inserted in (21) —treating α as a constant— we get

$$\begin{aligned} \left|D_x^\beta s(t,\cdot)\right|_\alpha \quad &\le \quad \exp(MCt)\left|D_x^\beta s^0\right|_\alpha \\ &\quad + MCt \exp(MCt) \sum_{|\beta|=1} \|D_x^\beta s^0\|_{0,\alpha} \\ &\quad + t\exp(MCt) \sum_{|\beta|=1} \left|D_x^\beta h\right|_\alpha, \quad |\beta| = 1. \end{aligned}$$

With $\exp(x) \le 1 + x\exp(x)$ it follows

$$\begin{aligned} \sum_{|\beta|=1} \left|D_x^\beta s(t,\cdot)\right|_\alpha \quad &\le \quad \sum_{|\beta|=1} \left|D_x^\beta s^0\right|_\alpha \\ &\quad + MCt \exp(MCt) \sum_{|\beta|=1} \|D_x^\beta s^0\|_{0,\alpha} \\ &\quad + Mt\exp(MCt) \sum_{|\beta|=1} \left|D_x^\beta h\right|_\alpha. \end{aligned}$$

Finally, taking into account the C^1–bound (18) results in

$$\|s(t,\cdot)\|_{1,\alpha} \le [1 + MCt\exp(MCt)]\, \|s^0\|_{1,\alpha} + Mt\exp(MCt)\|h\|_{1,\alpha}.$$

The constant M depends on the spatial dimension of the reservoir Ω and the Hölder coefficient α, but not on the data s^0, w or h. Here, the proof of Lemma 5.1 is complete. $\diamondsuit$

6 Existence of the limit

To establish the limit of the approximate solution by the Arzela–Ascoli theorem, we need to bound the family (s^n_Δ, p^n_Δ) uniformly in Δt.

Due to the time limitation (4) it follows easily that the saturation stays in the unit interval.

Lemma 6.1 *Assume the general requirements (5) hold. Then, up to time $t_\star$, the approximate saturation s_Δ defined in Section 3 remains in the unit interval*

$$s_\Delta(t,x) \in [0,1], \quad (t,x) \in [0,t_\star] \times \bar{\Omega}.$$

Proof. Initially we have

$$0 \le s^0(x) \le s^+ \ll 1.$$

The approximation is piecewise defined by (8)

$$\begin{aligned} \frac{\partial}{\partial t} s_\Delta + f'(s^n_\Delta) v^n_\Delta \cdot \nabla s_\Delta &= g^n - f(s^n_\Delta) q^n, \quad (t,x) \in (t_n, t_{n+1}) \times \Omega, \\ s_\Delta(t_n, \cdot) &= s^n_\Delta. \end{aligned}$$

By (15), the solution of this Cauchy problem is

$$s_\Delta(t,x) = \int_{t_n}^{t} h_\Delta(c(\tau,x,t))\, d\tau + s^n_\Delta(c(t_n,x,t)), \quad t_n \le t \le t_{n+1},$$

where $h_\Delta = g^n - f(s^n_\Delta) q^n$. With $s^n_\Delta \in [0,1]$, the source term is non–negative and bounded (cf. (5) *(v)*)

$$0 \le g^n - f(s^n_\Delta) q^n = (1 - f(s^n_\Delta)) g^n + f(s^n_\Delta)(g^n - q^n) \le \|g\|_{L^\infty} + \|g-q\|_{L^\infty}$$

It follows by induction

$$0 \le s_\Delta(t,x) \le s^+ + n\Delta t \left(\|g\|_{L^\infty} + \|g-q\|_{L^\infty} \right).$$

By the definition of $t_\star$ (cf. (4)) the right hand side is bounded by 1 for $n\Delta t \le t_\star$ (remember $g = q_w$ and $g - q = q_o$). ♢

A bound for first order derivatives and their Hölder coefficients is obtained by an iterative application of Lemma 5.1 to the linearized saturation equation (8). Therefore, we need to estimate the frozen speed $w_\Delta = f'(s^n_\Delta) v^n_\Delta$ and the right hand side $h_\Delta = g^n - f(s^n_\Delta) q^n$ in (8) in terms of s^n_Δ.

Lemma 6.2 *Let $s \in C^{1,\alpha}(\bar{\Omega}, [0,1])$, $f \in C^\infty([0,1], \mathbb{R})$ and $v \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R})$ be given, then*

$$\|v f(s)\|_{1,\alpha} \le M \|v\|_{1,\alpha} (1 + \|s\|_{1,\alpha} + \|s\|^2_{1,\alpha}).$$

Proof. Since s is bounded and f is smooth

$$\|v f(s)\|_{L^\infty} \le M \|v\|_{L^\infty}. \tag{22}$$

Next, consider the gradient $\nabla(vf(s)) = (\nabla v)f(s) + vf'(s)\nabla s$. Similarly it follows

$$\begin{aligned}\|\nabla(vf(s))\|_{L^\infty} &\le M(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}\|\nabla s\|_{L^\infty} \\ &\le M\|v\|_{1,\alpha}(1+\|\nabla s\|_{L^\infty}).\end{aligned}$$

Since there is a generic M on the right hand side, we have

$$\sum_{|\beta|=1} \|D^\beta(vf(s))\|_{L^\infty} \le M\|v\|_{1,\alpha}(1+\|\nabla s\|_{1,\alpha}). \tag{23}$$

Applying the product rule $|fg|_\alpha \le \|f\|_{L^\infty}|g|_\alpha + |f|_\alpha\|g\|_{L^\infty}$, where f is a scalar and g a vector to the gradient $\nabla(vf(s)) = (\nabla v)f(s) + v\nabla f(s)$, it follows that

$$|\nabla(vf(s))|_\alpha \le \|\nabla v\|_{L^\infty}|f(s)|_\alpha + M|\nabla v|_\alpha + \|v\|_{L^\infty}|\nabla f(s)|_\alpha + |v|_\alpha\|\nabla f(s)\|_{L^\infty}.$$

Observe that $|f(s)|_\alpha \le M|s|_\alpha$, $\|\nabla f(s)\|_{L^\infty} \le M\|\nabla s\|_{L^\infty}$ and

$$\begin{aligned}|\nabla f(s)|_\alpha = |f'(s)\nabla s|_\alpha &\le M|\nabla s|_\alpha + |f'(s)|_\alpha + \|\nabla s\|_{L^\infty} \\ &\le M|\nabla s|_\alpha + M|s|_\alpha + \|\nabla s\|_{L^\infty}.\end{aligned}$$

This implies

$$|\nabla(vf(s))|_\alpha \le M\|v\|_{1,\alpha}(1+\|s\|_{1,\alpha} + \|s\|^2_{1,\alpha}),$$

where we have used $|g|_\alpha \le M(\|g\|_{L^\infty} + \|\nabla g\|_{L^\infty}) \le M\|g\|_{1,\alpha}$ for any $C^{1,\alpha}$–function. Again, the same estimate holds for the sum

$$\sum_{|\beta|=1} |D^\beta(vf(s))|_\alpha \le M\|v\|_{1,\alpha}(1+\|s\|_{1,\alpha} + \|s\|^2_{1,\alpha}). \tag{24}$$

Finally, the statement in Lemma 6.2 follows by adding the inequalities (22), (23) and (24). $\diamondsuit$

Concerning the source term it holds

Lemma 6.3 *Let $s \in C^{1,\alpha}(\bar\Omega, [0,1])$, $f \in C^\infty([0,1], \mathbb{R})$ and $g, q \in C^\infty([0,t_\star] \times \bar\Omega, \mathbb{R})$ be given, then*

$$\|g - f(s)q\|_{1,\alpha} \le M(1+\|s\|_{1,\alpha} + \|s\|^2_{1,\alpha}).$$

Proof. Obviously $\|g - f(s)q\|_{1,\alpha} \le M + \|f(s)q\|_{1,\alpha}$ and it remains to bound the product $f(s)q$. As s is bounded, f is smooth and q is compactly supported $\|f(s)q\|_{L^\infty} \le M$. Furthermore,

$$\begin{aligned}\|\nabla(f(s)q)\|_{L^\infty} &\le M\left(\|\nabla f(s)\|_{L^\infty} + \|f(s)\|_{L^\infty}\right) \le M\left(1+\|\nabla s\|_{L^\infty}\right), \\ |\nabla(f(s)q)|_\alpha &\le \|\nabla f(s)\|_{L^\infty}|q|_\alpha + |\nabla f(s)|_\alpha\|q\|_{L^\infty} \\ &\quad + \|f(s)\|_{L^\infty}|\nabla q|_\alpha + |f(s)|_\alpha\|\nabla q\|_{L^\infty} \\ &\le M\left(\|\nabla s\|_{L^\infty} + |\nabla s|_\alpha + |s|_\alpha\|\nabla s\|_{L^\infty} + 1 + |s|_\alpha\right) \\ &\le M\left(1+\|s\|_{1,\alpha} + \|s\|^2_{1,\alpha}\right)\end{aligned}$$

and Lemma 6.3 is proved. $\diamondsuit$

Now, we are prepared to estimate the approximate saturation s_Δ step by step. Applying Lemma 5.1 to one time step

$$\frac{\partial}{\partial t} s_\Delta + f'(s^n_\Delta) v^n_\Delta \cdot \nabla s_\Delta = g^n - f(s^n_\Delta) q^n, \quad t_n < t < t_{n+1}$$

it follows

$$\begin{aligned}\|s_\Delta(t,\cdot)\|_{1,\alpha} \quad &\leq \quad [1 + \Delta t M C \exp(\Delta t M C)] \, \|s^n_\Delta\|_{1,\alpha} \\ &\quad\quad + \Delta t M \exp(\Delta t M C) \, \|g^n - f(s^n_\Delta) q^n\|_{1,\alpha} \,,\end{aligned}$$

where C is a bound for $\|f'(s^n_\Delta)v^n_\Delta\|_{1,\alpha}$. By Lemma 6.2 —applied componentwise to $f'(s^n_\Delta)v^n_\Delta$—

$$\|f'(s^n_\Delta)v^n_\Delta\|_{1,\alpha} \leq C \leq M \|v^n_\Delta\|_{1,\alpha} \left(1 + \|s^n_\Delta\|_{1,\alpha} + \|s^n_\Delta\|^2_{1,\alpha}\right) \,.$$

Using the bound (12) for the approximate speed v^n_Δ

$$C \leq M \exp\left(M \|s^n_\Delta\|_{1,\alpha}\right) \,.$$

Lemma 6.3 states

$$\|g^n - f(s^n_\Delta) q^n\|_{1,\alpha} \leq M \exp\left(M \|s^n_\Delta\|_{1,\alpha}\right)$$

and hence

$$\begin{aligned}\|s_\Delta(t,\cdot)\|_{1,\alpha} \quad &\leq \quad \{1 + \Delta t M \exp[M \exp(M\|s^n_\Delta\|_{1,\alpha})]\} \, \|s^n_\Delta\|_{1,\alpha} \\ &\quad\quad + \Delta t M \exp[M \exp(M\|s^n_\Delta\|_{1,\alpha})] \,, \quad t_n \leq t \leq t_{n+1} \,.\end{aligned}$$

Using $x \leq e^x$ this estimate simplifies to

$$\|s^{n+1}_\Delta\|_{1,\alpha} \leq \|s^n_\Delta\|_{1,\alpha} + \Delta t M \exp[M \exp(M\|s^n_\Delta\|_{1,\alpha})] \,.$$

Therefore $\|s^n_\Delta\|_{1,\alpha}$ is bounded by ψ^n which is defined by the one–step method

$$\psi^{n+1} = \psi^n + \Delta t M \exp[M \exp(M \psi^n)] \,, \quad \psi^0 = \|s^0\|_{1,\alpha} \,.$$

This scheme is consistent to the initial value problem

$$\psi' = M \exp[M \exp(M \psi(t))] \,, \quad \psi(0) = \|s^0\|_{1,\alpha}$$

which has a unique solution $\psi = \psi(t)$ up to some maximal $T_* > 0$. As the right hand side of this ordinary differential equation is convex and strictly increasing, the approximation obtained by the explicit Euler method stays below the exact solution

$$\|s^n_\Delta\|_{1,\alpha} \leq \psi^n \leq \psi(t_n), \quad t_n = n \cdot \Delta t \leq T_*$$

and the approximate saturation is uniformly bounded

$$\|s_\Delta(t,\cdot)\|_{1,\alpha} \leq \psi(t) \leq M, \quad t \in [0,T], \quad T := \min(t_*, T_*) \,.$$

This proves

Lemma 6.4 *Assume the general assumptions (5) hold and the approximate solution is defined by the procedure in Section 3. Then, there is a time* $T > 0$ *such that the family* $(s_\Delta, p_\Delta, v_\Delta)$ *is uniformly bounded*

$$\begin{aligned} \|s_\Delta(t,\cdot)\|_{1,\alpha} &\leq M\,, \\ \|p_\Delta(t,\cdot)\|_{2,\alpha} &\leq M\exp\left(M\|s_\Delta(t,\cdot)\|_{1,\alpha}\right)\,, \\ \|v_\Delta(t,\cdot)\|_{1,\alpha} &\leq M\exp\left(M\|s_\Delta(t,\cdot)\|_{1,\alpha}\right), \quad 0 \leq t \leq T\,. \end{aligned}$$

The generic M *does not depend on the step–size* Δt.

Based on this a-priori bounds, the convergence of a subsequence as $\Delta t \to 0$ follows.

Lemma 6.5 *Under the assumptions of Lemma 6.4 there are limits* $s(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$, $p(t,\cdot) \in C^{2,\alpha}(\bar\Omega)$ *and* $v(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$ *and a sequence* Δt_j *such that*

$$\begin{aligned} \|s_{\Delta t_j}(t,\cdot) - s(t,\cdot)\|_1 &\to 0\,, \\ \|p_{\Delta t_j}(t,\cdot) - p(t,\cdot)\|_2 &\to 0\,, \\ \|v_{\Delta t_j}(t,\cdot) - v(t,\cdot)\|_1 &\to 0 \quad \textit{as } \Delta t_j \to 0\,. \end{aligned}$$

Furthermore, in the limit the pressure equation holds

$$-\nabla\cdot(\lambda(s)\nabla p) = q \ \textit{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \ \textit{on } \partial\Omega, \quad \int_\Omega p\,dx = 0$$

and

$$\nabla\cdot v = -\nabla\cdot(\lambda(s)\nabla p) = q$$

for all $t \in [0,T]$.

A tool for the proof of Lemma 6.5 is the following version of the Arzela–Ascoli theorem (cf. [15] Appendix 4)

Theorem 6.6 (Arzela–Ascoli) *Let* $D \subset \mathbb{R}^s$ *denote a closed and bounded set and let* $u_m : D \to \mathbb{C}^n$ *be a sequence of functions such that*

i) u_m *is uniformly bounded:* $\|u_m\|_{L^\infty} \leq M$

ii) u_m *is uniformly continuous: For all* $\delta > 0$ *there exists a* $\epsilon > 0$, *independent of* m, *such that*

$$\|u_m(x) - u_m(y)\|_{L^\infty} < \delta \ \textit{if } \|x-y\| < \epsilon\,.$$

Then there is a sequence $m_j \to \infty$ *and a continuous function* $u : D \to \mathbb{C}^n$ *such that*

$$\|u_{m_j} - u\|_{L^\infty} \to 0 \ \textit{as } m_j \to \infty\,.$$

Proof of Lemma 6.5. Consider some fixed time $t \in [0,T]$. We apply Theorem 6.6 to $s_\Delta(t,\cdot)$. By Lemma 6.4 s_Δ is uniformly bounded. As first order derivatives are bounded as well, $s_\Delta(t,\cdot)$ is uniformly continuous. Therefore, there exists a sequence Δt_j and a continuous function $s(t,\cdot)$ such that

$$\|s_{\Delta t_j}(t,\cdot) - s(t,\cdot)\|_{L^\infty} \to 0 \text{ as } \Delta t_j \to 0\,.$$

Next, consider the sequence $\nabla s_{\Delta t_j}(t,\cdot)$. Again, by Lemma 6.4 $\nabla s_{\Delta t_j}(t,\cdot)$ is uniformly bounded. As $|\nabla s_{\Delta t_j}(t,\cdot)|_\alpha$ is uniformly bounded, $\nabla s_{\Delta t_j}(t,\cdot)$ is uniformly continuous. Again, by Theorem 6.6 there is a subsequence $\Delta t_k = \Delta t_{j_k}$ and a continuous function $G(t,\cdot)$ such that

$$\|\nabla s_{\Delta t_k}(t,\cdot) - G(t,\cdot)\|_{L^\infty} \to 0 \text{ as } \Delta t_k \to 0\,.$$

Moreover, $G(t,\cdot)$ is the gradient of $s(t,\cdot)$. It holds

$$s_{\Delta t_k}(t,x) - s_{\Delta t_k}(t,\bar{x}) = \int_{\bar{x}}^{x} \nabla s_{\Delta t_k}(t,\xi)\, d\xi\,.$$

Sending Δt_k to zero we find

$$s(t,x) - s(t,\bar{x}) = \int_{\bar{x}}^{x} G(t,\xi)\, d\xi\,.$$

Obviously $s(t,\cdot)$ is differentiable and $G(t,\cdot) = \nabla s(t,\cdot)$. Next, we show $s(t,\cdot) \in C^{1,\alpha}(\bar{\Omega})$. By Lemma 6.4

$$|\nabla s_{\Delta t_k}(t,x) - \nabla s_{\Delta t_k}(t,y)| \le M|x-y|^\alpha\,,$$

where M is independent of Δt_k. Because $\nabla s_{\Delta t_k}$ converges to ∇s it follows $|\nabla s(t,\cdot)|_\alpha \le M$ and hence $s(t,\cdot) \in C^{1,\alpha}(\bar{\Omega})$.

Associated with each $s_{\Delta t_k}(t,\cdot) \in C^{1,\alpha}(\bar{\Omega})$, there is a corresponding pressure $p_{\Delta t_k}(t,\cdot) \in C^{2,\alpha}(\bar{\Omega})$ defined by the Neumann problem (9). It holds

$$\|p_{\Delta t_k}(t,\cdot)\|_{2,\alpha} \le M \exp\left(M\|s_{\Delta t_k}(t,\cdot)\|_{1,\alpha}\right)\,.$$

By Lemma 6.4 the pressure is uniformly bounded

$$\|p_{\Delta t_k}(t,\cdot)\|_{2,\alpha} \le M, \quad t \in [0,T]\,.$$

Again, it follows as above the existence of a subsequence $\Delta t_\ell = \Delta t_{k_\ell}$ — the subsequence claimed in Lemma 6.5 — such that both $s_{\Delta t_\ell}$ and $p_{\Delta t_\ell}$ converge.

Obviously, the flow speed $v_{\Delta t_\ell}(t,\cdot) := -\lambda(s_{\Delta t_\ell}(t,\cdot))\nabla p_{\Delta t_\ell}(t,\cdot)$ associated with the saturation and the pressure converges in C^1 to $v(t,\cdot) := -\lambda(s(t,\cdot))\nabla p(t,\cdot)$. It remains to show that the pressure equation holds in the limit. By construction we have

$$-\nabla\cdot(\lambda(s_{\Delta t_\ell})\nabla p_{\Delta t_\ell}) = q \text{ in } \Omega, \quad \frac{\partial p_{\Delta t_\ell}}{\partial n} = 0 \text{ on } \bar{\Omega} \quad \text{and} \quad \int_\Omega p_{\Delta t_\ell}\, dx = 0\,. \tag{25}$$

As λ is smooth $\nabla\cdot(\lambda(s_{\Delta t_\ell})\nabla p_{\Delta t_\ell})$ converges pointwise to $\nabla\cdot(\lambda(s)\nabla p)$ as $\Delta t_\ell \to 0$. Therefore (25) holds in the limit. Finally, by definition $\nabla \cdot v(t,\cdot) = q$ and Lemma 6.5 is proved. $\diamondsuit$

In order to verify the saturation equation, we have to show that s is differentiable with respect to time. Observe that this is not true for the approximation s_Δ, as the frozen speed $f'(s^n_\Delta)v^n_\Delta$ has a jump at discrete time levels $t_n = n\Delta t$. It is the purpose of the next section to verify that this jump is of order Δt and hence the limit v is continuous and s is differentiable in time.

7 The saturation equation

The approximate saturation is piecewise defined by

$$\frac{\partial}{\partial t} s_\Delta + f'(s^n_\Delta)v^n_\Delta \cdot \nabla s_\Delta = g^n - f(s^n_\Delta)q^n, \quad (t,x) \in (t_n, t_{n+1}) \times \Omega \,.$$

At t_n the time derivative $\frac{\partial}{\partial t} s_\Delta$ is discontinuous, since the frozen speed $w_\Delta = f'(s_\Delta)v_\Delta$ is discontinuous. We will show that s_Δ is Lipschitz and the gradient ∇s_Δ is α–Hölder continuous in time.

Lemma 7.1 *Assume (5) holds. Then the approximate saturation s_Δ has the following regularity: For any $t \in [0,T]$, $s_\Delta(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$. Furthermore*

i) $\|s_\Delta(t,\cdot) - s_\Delta(\tau,\cdot)\|_{L^\infty} \le M|t-\tau|$,

ii) $\|\nabla s_\Delta(t,\cdot) - \nabla s_\Delta(\tau,\cdot)\|_{L^\infty} \le M|t-\tau|^\alpha, \quad t,\tau \in [0,T]$.

The constant M does not depend on Δt.

We shall first apply this result before we prove it. An immediate consequence from the uniform estimates and Lemma 6.5 is

Corollary 7.2 *The regularity of s_Δ carries over to the limit, i.e. $s(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$ and*

i) $\|s(t,\cdot) - s(\tau,\cdot)\|_{L^\infty} \le M|t-\tau|$,

ii) $\|\nabla s(t,\cdot) - \nabla s(\tau,\cdot)\|_{L^\infty} \le M|t-\tau|^\alpha$.

Note that this is a preliminary result which will be improved below. It implies that the speed v will be continuous in time. Then the saturation s is differentiable in space and time.

Proof of Corollary 7.2. We know already that $s(t,\cdot) \in C^{1,\alpha}(\bar\Omega)$. Furthermore, by Lemma 6.5 there is a sequence $s_{\Delta t_j}$ converging to s. By Lemma 7.1 *i)* it holds

$$\|s_{\Delta t_j}(t,\cdot) - s_{\Delta t_j}(\tau,\cdot)\|_{L^\infty} \le M|t-\tau| \,.$$

As M is independent of Δt, the limit s is Lipschitz in time. In the same way Hölder continuity of the gradient follows from Lemma 7.1 *ii)*. This proves the Corollary. $\diamondsuit$

As indicated above, time–Lipschitz continuity of s gives additional regularity for the pressure gradient and the flow speed. Consider two time levels $t,\tau \in [0,T]$ and denote by $S = s(t,\cdot)$, $P = p(t,\cdot)$, $V = v(t,\cdot)$ and $\overline{S} = s(\tau,\cdot)$, $\overline{P} = p(\tau,\cdot)$, $\overline{V} = v(\tau,\cdot)$ the saturation, pressure and speed respectively. By definition it holds

$$V - \overline{V} = -\lambda(S)(\nabla(P - \overline{P})) - (\lambda(S) - \lambda(\overline{S}))\nabla\overline{P}\,. \tag{26}$$

Obviously time–continuity of v depends on the continuity of the pressure gradient. By Lemma 6.5 the pressure equation holds for all $t \in [0,T]$, hence

$$-\nabla\cdot(\lambda(S)\nabla P) = q = -\nabla\cdot(\lambda(\overline{S})\nabla\overline{P})$$

and

$$-\nabla\cdot(\lambda(S)\nabla(P - \overline{P})) = \nabla\cdot\left[(\lambda(S) - \lambda(\overline{S}))\nabla\overline{P}\right]\,.$$

To estimate the pressure difference, we multiply by $P - \overline{P}$, integrate in Ω and apply the Gauss formula. Due to the Neumann boundary condition $\partial p/\partial n = 0$ it follows

$$\int_\Omega \lambda(S)\nabla(P - \overline{P})\cdot\nabla(P - \overline{P})\,d\Omega = \int_\Omega (\lambda(S) - \lambda(\overline{S}))\nabla\overline{P}\cdot\nabla(P - \overline{P})\,d\Omega.$$

Because of the ellipticity assumption $\lambda \geq c_0 > 0$, we find

$$c_0\|\nabla(P - \overline{P})\|_{L^2} \leq M\|S - \overline{S}\|_{L^2}.$$

By (26) and Corollary 7.2 *i)* it follows

$$\|v(t,\cdot) - v(\tau,\cdot)\|_{L^2} \leq M\|s(t,\cdot) - s(\tau,\cdot)\|_{L^2} \leq M|t - \tau|.$$

Recall that by Lemma 6.4 $\|v_\Delta(t,\cdot)\|_{1,\alpha}$ is uniformly bounded. By Lemma 6.5 this bound carries over to the limit $\|v(t,\cdot)\|_1 \leq C$ uniformly in $0 < t < T$. Therefore, a technical argument ([19], Lemma 11.1) implies

Lemma 7.3 *The velocity $v(t,\cdot)$ is Hölder continuous in time*

$$\|v(t,\cdot) - v(\tau,\cdot)\|_{L^\infty} \leq M|t - \tau|^{2/(2+d)}, \quad t,\tau \in [0,T]\,.$$

Now, we are in the position to conclude that s is differentiable and the saturation equation holds. The approximate saturation is defined by the initial value problem

$$\begin{aligned} \frac{\partial}{\partial t}s_\Delta + w_\Delta\cdot\nabla s_\Delta &= h_\Delta \quad \text{in } (t_n,t_{n+1})\times\Omega\,, \\ s_\Delta(t_n,\cdot) &= s_\Delta^n\,. \end{aligned}$$

Piecewise integration yields

$$s_\Delta(t,\cdot) - s_\Delta(\tau,\cdot) = \int_\tau^t h_\Delta(\xi,\cdot) - w_\Delta(\xi,\cdot)\cdot\nabla s_\Delta(\xi,\cdot)\,d\xi\,.$$

Consider the convergent subsequence of Lemma 6.5. As $s_{\Delta t_j}$ and $v_{\Delta t_j}$ are uniformly bounded and converge pointwise, we may compute the limit $\Delta t_j \to 0$ under the integral sign and find

$$s(t,\cdot) - s(\tau,\cdot) = \int_\tau^t h(\xi,\cdot) - w(\xi,\cdot)\cdot\nabla s(\xi,\cdot)\,d\xi\,, \tag{27}$$

where $h = g - f(s)q$ and $w = f'(s)v$. By Corollary 7.2 s and hence h are Lipschitz in time. Furthermore, the gradient of s is α–Hölder continuous. By Lemma 7.3 v and hence w are continuous in time. Therefore, the complete expression under the integral in (27) is continuous in time. It follows that s is differentiable,

$$s_t + f'(s)v\cdot\nabla s = g - f(s)q\,,$$

and s_t is continuous[1] in time. Since by Lemma 6.5 $q = \nabla\cdot v$, the saturation equation holds in conservative form

$$s_t + \nabla\cdot[f(s)v] = g \quad \text{in } [0,T]\times\Omega\,.$$

Except of Lemma 7.1, the existence part of Theorem 2.1 is proved.

Proof of Lemma 7.1. Smoothness in space i.e. $s_\Delta(t,\cdot)\in C^{1,\alpha}(\bar\Omega)$ has been discussed already, see Lemma 6.4. The point here is to focus on the time variable. Similar as in the proof of Lemma 5.1 we investigate the smoothness in time by transforming time differences into space differences using characteristics. By definition it holds

$$\frac{\partial}{\partial t}s_\Delta + w_\Delta\cdot\nabla s_\Delta = h_\Delta \text{ in } (t_n,t_{n+1})\times\Omega\,,\quad s_\Delta(t_n,\cdot) = s_\Delta^n\,,$$

where $w_\Delta(t,x) = f'(s_\Delta^n(x))v_\Delta^n(x)$ and $h_\Delta(t,x) = g^n(x) - f(s_\Delta^n(x))q^n(x)$ in $[t_n,t_{n+1})\times\Omega$. Note that characteristics $c = c(t,\bar x,\bar t)$ are uniquely defined by

$$c_t = w_\Delta(t,c),\quad t\in[0,T],\quad c(\bar t,\bar x,\bar t) = \bar x\in\bar\Omega\,.$$

Along characteristics it holds

$$\frac{d}{dt}s_\Delta(t,c(t,\bar x,\bar t)) = h_\Delta(t,c(t,\bar x,\bar t)),\quad t_n < t < t_{n+1}$$

and

$$s_\Delta(\bar t,\bar x) = s_\Delta(\underline{t},\underline{x}) + \int_{\underline{t}}^{\bar t} h_\Delta(\tau,c(\tau,\bar x,\bar t))\,d\tau,\quad 0\le\underline{t}\le\bar t\le T$$

where $\underline{x} := c(\underline{t},\bar x,\bar t)$. Now, the time difference is expressed by a space difference and the evolution along the characteristic

$$s_\Delta(\bar t,\bar x) - s_\Delta(\underline{t},\bar x) = s_\Delta(\underline{t},\underline{x}) - s_\Delta(\underline{t},\bar x) + \int_{\underline{t}}^{\bar t} h_\Delta(\tau,c(\tau,\bar x,\bar t))\,d\tau\,.$$

[1] In fact s_t is Hölder continuous in time with exponent $\min(\alpha, 2/(2+d))$.

As s_Δ is uniformly bounded, h_Δ is bounded as well and the integral is of order $\bar{t} - \underline{t}$

$$\left| \int_{\underline{t}}^{\bar{t}} h_\Delta(\tau, c(\tau, \bar{x}, \bar{t}))\, d\tau \right| \leq M(\bar{t} - \underline{t}).$$

Furthermore, w_Δ is bounded and $|\bar{x} - \underline{x}| = \mathcal{O}(\bar{t} - \underline{t})$, therefore

$$\frac{|s_\Delta(\bar{t}, \bar{x}) - s_\Delta(\underline{t}, \bar{x})|}{\bar{t} - \underline{t}} \leq M \frac{|s_\Delta(\underline{t}, \underline{x}) - s_\Delta(\underline{t}, \bar{x})|}{|\bar{x} - \underline{x}|} + M \leq M \left(1 + \|\nabla s_\Delta(\underline{t}, \cdot)\|_{L^\infty}\right) .$$

Obviously, $s_\Delta(\cdot, x)$ is Lipschitz in time.

Concerning the gradient we have

$$\frac{\partial}{\partial t} \nabla s_\Delta + \begin{pmatrix} w_\Delta \cdot (\nabla s_\Delta)_x \\ w_\Delta \cdot (\nabla s_\Delta)_y \end{pmatrix} + \begin{pmatrix} (w_\Delta)_x \cdot \nabla s_\Delta \\ (w_\Delta)_y \cdot \nabla s_\Delta \end{pmatrix} = \nabla h_\Delta, \quad t_n < t < t_{n+1} .$$

Along characteristics it follows

$$\frac{d}{dt} \nabla s_\Delta = \nabla h_\Delta - J_\Delta^T \cdot \nabla s_\Delta, \quad t_n < t < t_{n+1} ,$$

where J_Δ is the Jacobian $D_x w_\Delta$. This ordinary differential equation can easily be solved

$$\begin{aligned} \nabla s_\Delta(t, c(t, \bar{x}, \bar{t})) &= \exp\left(- \int_{t_n}^{t} J_\Delta^T(\tau, c(\tau, \bar{x}, \bar{t}))\, d\tau \right) \nabla s_\Delta(t_n, c(t_n)) \\ &\quad + \int_{t_n}^{t} \nabla h_\Delta(\tau, c(\tau, \bar{x}, \bar{t}))\, d\tau, \quad t_n \leq t \leq t_{n+1} . \end{aligned}$$

Because spatial derivatives of w_Δ and h_Δ are uniformly bounded

$$\begin{aligned} \nabla s_\Delta(\bar{t}, \bar{x}) &= \exp\left(- \int_{\underline{t}}^{\bar{t}} J_\Delta^T(\tau, c(\tau, \bar{x}, \bar{t}))\, d\tau \right) \nabla s_\Delta(\underline{t}, \underline{x}) \\ &\quad + \int_{\underline{t}}^{\bar{t}} \nabla h_\Delta(\tau, c(\tau, \bar{x}, \bar{t}))\, d\tau + \mathcal{O}(\bar{t} - \underline{t}) , \end{aligned}$$

where the characteristic connects $\underline{x} := c(\underline{t}, \bar{x}, \bar{t})$ and $\bar{x} = c(\bar{t}, \bar{x}, \bar{t})$. It follows

$$|\nabla s_\Delta(\bar{t}, \bar{x}) - \nabla s_\Delta(\underline{t}, \underline{x})| = \mathcal{O}(\bar{t} - \underline{t}),$$

$$|\nabla s_\Delta(\bar{t}, \bar{x}) - \nabla s_\Delta(\underline{t}, \bar{x})| = |\nabla s_\Delta(\underline{t}, \underline{x}) - \nabla s_\Delta(\underline{t}, \bar{x})| + \mathcal{O}(\bar{t} - \underline{t})$$

and

$$\frac{|\nabla s_\Delta(\bar{t}, \bar{x}) - \nabla s_\Delta(\underline{t}, \bar{x})|}{|\bar{t} - \underline{t}|^\alpha} \leq M \left(|\nabla s_\Delta(\underline{t}, \cdot)|_\alpha + |\bar{t} - \underline{t}|^{1-\alpha} \right) .$$

As $|\nabla s_\Delta(\underline{t}, \cdot)|_\alpha$ is bounded, $\nabla s(\cdot, x)$ is α–Hölder continuous.
This concludes the proof of Lemma 7.1. $\diamondsuit$

8 Uniqueness and stability

It remains to show that for a pair of initial data s^0 and S^0 satisfying (5), the associated solutions of the system

$$\begin{aligned} s_t + \nabla\cdot[f(s)v] &= g\,, \\ -\nabla\cdot[\lambda(s)\nabla p] &= q\,, \\ v = -\lambda(s)\nabla p, &\quad (t,x)\in[0,T]\times\Omega \end{aligned}$$

subject to the boundary conditions (3), are stable in the sense of (6). Obviously, uniqueness is a consequence of these estimates.

In the following we derive an energy estimate for the saturation. For both s and S the saturation equation (in non–conservative form) holds, therefore

$$(s-S)_t + w\cdot\nabla(s-S) + (w-W)\cdot\nabla S = h-H,$$

where $w = f'(s)v$, $W = f'(S)V$, $h = g - f(s)q$ and $H = g - f(S)q$. Now consider

$$\frac{1}{2}\frac{d}{dt}\|(s-S)(t,\cdot)\|^2_{L^2} = \int_\Omega (s-S)(s-S)_t\,d\Omega = (I)+(II)+(III),$$

with

$$\begin{aligned} (I) &:= -\int_\Omega (s-S)w\cdot\nabla(s-S)\,d\Omega, \\ (II) &:= -\int_\Omega (s-S)(w-W)\cdot\nabla S\,d\Omega, \\ (III) &:= \int_\Omega (s-S)(h-H)\,d\Omega. \end{aligned}$$

The goal is to bound each term by $M\|(s-S)(t,\cdot)\|^2_{L^2}$.

Concerning (I), we rewrite

$$w\cdot\nabla(s-S) = \nabla\cdot(w(s-S)) - \nabla\cdot w(s-S)$$

and find

$$(I) = -\int_\Omega (s-S)\nabla\cdot(w(s-S))\,d\Omega + \int_\Omega (s-S)\nabla\cdot w(s-S)\,d\Omega.$$

By the Gauss formula and the boundary condition $w\cdot n = 0$ it follows

$$\int_\Omega (s-S)\nabla\cdot(w(s-S))\,d\Omega = -\int_\Omega \nabla(s-S)\cdot w(s-S)\,d\Omega = (I)$$

and

$$|(I)| = \frac{1}{2}\left|\int_\Omega (s-S)^2\nabla\cdot w\,d\Omega\right| \le M\|(s-S)(t,\cdot)\|^2_{L^2}. \tag{28}$$

This constant M depends on $\|\nabla \cdot w\|_{L^\infty}$ and hence on $\|\nabla s\|_{L^\infty}$, $\|\nabla p\|_{L^\infty}$ and $\|\Delta p\|_{L^\infty}$.

Concerning the second term (II), consider

$$w - W = f'(s)(v - V) + (f'(s) - f'(S))V.$$

Here, we need to bound $\|(v-V)(t,\cdot)\|_{L^2}$ in terms of $\|(s-S)(t,\cdot)\|_{L^2}$. ¿From the pressure equation it follows

$$\nabla \cdot (\lambda(s)\nabla(p - P)) = \nabla \cdot ((\lambda(s) - \lambda(S))\nabla P).$$

We multiply by $p - P$, integrate in Ω and apply the Gauss formula. Again, the boundary condition for the pressure gives

$$\int_\Omega \lambda(s)|\nabla(p-P)|^2\, d\Omega = \int_\Omega (\lambda(s) - \lambda(S))\nabla P \cdot \nabla(p-P)\, d\Omega.$$

The lower bound for λ (cf. (5) *(ii)*) implies

$$c_0\|\nabla(p-P)(t,\cdot)\|_{L^2} \le M\|(s-S)(t,\cdot)\|_{L^2}. \tag{29}$$

Because of $v - V = -\lambda(s)\nabla(p-P) - (\lambda(s) - \lambda(S))\nabla P$ it follows

$$\|(v-V)(t,\cdot)\|_{L^2} \le M\|(s-S)(t,\cdot)\|_{L^2} \tag{30}$$

and

$$\|(w-W)(t,\cdot)\|_{L^2} \le M\|(s-S)(t,\cdot)\|_{L^2}.$$

Now it is obvious that

$$|(II)| \le M\|(s-S)(t,\cdot)\|_{L^2}^2, \tag{31}$$

where M depends on $\|\nabla P\|_{L^\infty}$.

¿From $h - H = q(f(S) - f(s))$ it follows that

$$|(III)| \le M\|(s-S)(t,\cdot)\|_{L^2}^2. \tag{32}$$

Finally, composing (28), (31) and (32) we find the energy estimate

$$\frac{d}{dt}\|(s-S)(t,\cdot)\|_{L^2}^2 \le M\|(s-S)(t,\cdot)\|_{L^2}^2, \quad t \in [0,T],$$

where $[0,T]$ is the time interval by Lemma 6.4. Gronwall's Lemma implies the stability estimate for the saturation

$$\|(s-S)(t,\cdot)\|_{L^2} \le e^{Mt}\|(s^0 - S^0)\|_{L^2}. \tag{33}$$

Together with (30) the corresponding estimate for the flow speed follows

$$\|(v-V)(t,\cdot)\|_{L^2} \le Me^{Mt}\|(s^0 - S^0)\|_{L^2}.$$

By (29) and (33) the gradient of the pressure difference is bounded in terms of the data

$$\|\nabla(p-P)(t,\cdot)\|_{L^2} \leq M e^{Mt} \|(s^0 - S^0)\|_{L^2}.$$

Using Poincaré's inequality (cf. [7] Sect. 7.2 Proposition 2) this bound turns into the stability estimate for the pressure

$$\|(p-P)(t,\cdot)\|_{L^2} \leq M e^{Mt} \|(s^0 - S^0)\|_{L^2}$$

and Theorem 2.1 is proved.

9 Convergence of the approximate solution

Having proved the uniqueness of smooth solutions of the hyperbolic–elliptic system (2), (3) it follows by a standard argument, that not only the subsequence obtained by Arzela–Ascoli (see Lemma 6.5), but also the entire family of approximate solutions (s_Δ, p_Δ) converges to this unique limit. The argument is stated next.

Lemma 9.1 *Let b_j denote a sequence in a normed space such that*

i) each subsequence b_{j_k} has a convergent subsequence.

ii) All convergent subsequences of b_j converge to the same limit.

Then $b_j \to b$ as $j \to \infty$.

Proof. If b_j does not converge to b, then there is an $\epsilon > 0$ and a sequence $j_k \to \infty$, such that $\|b_{j_k} - b\| \geq \epsilon$ for all j_k. However, there is a convergent subsequence of b_{j_k} with limit b, a contradiction. $\diamondsuit$

Let us apply Lemma 9.1 to prove Theorem 3.1.

Proof of Theorem 3.1. Consider the approximate solutions (s_Δ, p_Δ) defined in Section 3. The complete family (s_Δ, p_Δ) and hence every subsequence $(s_{\Delta t_j}, p_{\Delta t_j})$ is uniformly bounded by Lemma 6.4. By Lemma 6.5 every subsequence has a convergent subsequence $(s_{\Delta t_{j_k}}, p_{\Delta t_{j_k}})$ and in the limit, the pressure equation holds. Furthermore, by Section 7, the limit of $s_{\Delta t_{j_k}}$ satisfies the saturation equation. As by Theorem 2.1 the solution of the coupled system (2), (3) is unique, Lemma 9.1 applies and Theorem 3.1 is proved. $\diamondsuit$

10 Summary

The present paper presents an existence, uniqueness and stability result for a hyperbolic–elliptic model of two–phase reservoir flow. Furthermore, a widely used operator splitting approach is shown to converge to this solution. The proof of the existence result Theorem 2.1 can be summarized as follows.

The iterative procedure in Section 3 defines a family of approximate solutions (s_Δ, p_Δ). By Section 4 the pressure and the flow speed $v_\Delta = -\lambda(s_\Delta)\nabla p_\Delta$ are well–defined

$$p_\Delta(t_n, \cdot) \in C^{2,\alpha}(\bar\Omega), \qquad v_\Delta(t_n, \cdot) \in C^{1,\alpha}(\bar\Omega)$$

and bounded in terms of the source $\|q\|_{0,\alpha}$ and the saturation $\|s_\Delta(t_n, \cdot)\|_{1,\alpha}$. Lemma 5.1 states that the approximate saturation

$$s_\Delta(t, \cdot) \in C^{1,\alpha}(\bar\Omega), \quad t_n \le t \le t_{n+1}$$

is piecewise well–defined. In combination with Lemmas 6.2 and 6.3 $\|s_\Delta(t, \cdot)\|_{1,\alpha}$ is bounded in terms of the data $\|s_\Delta(t_n, \cdot)\|_{1,\alpha}$ and $\|v_\Delta(t_n, \cdot)\|_{1,\alpha}$. An iterative application of these bounds leads to the step–size independent a–priori bounds stated in Lemma 6.4. By the Arzela–Ascoli theorem the existence of smooth limits $s(t, \cdot) \in C^{1,\alpha}(\bar\Omega)$, $p(t, \cdot) \in C^{2,\alpha}(\bar\Omega)$ and $v(t, \cdot) \in C^{1,\alpha}(\bar\Omega)$ follows. Furthermore, the pressure equation holds in the limit cf. Lemma 6.5.

As the approximate saturation is not differentiable in time, it is not obvious that the limit is differentiable. Lemma 7.1 states that the approximate saturation $s_\Delta(t, \cdot)$ is Lipschitz continuous in time. This result carries over to the limit s, see Corollary 7.2, but still is not sufficient. To verify the differentiability of s, one has to investigate the smoothness of v and hence ∇p with respect to time. ¿From the pressure equation it follows $\|\nabla p(t, \cdot) - \nabla p(\tau, \cdot)\|_{L^2} = \mathcal{O}(|t-\tau|)$. Therefore, the speed $v = \lambda(s)\nabla p$ is continuous in time (Lemma 7.3). As the approximate saturation is piecewise differentiable and continuous, the integral equation holds

$$s_\Delta(t, \cdot) - s_\Delta(\tau, \cdot) = \int_\tau^t h_\Delta(\xi, \cdot) - w_\Delta(\xi, \cdot) \cdot \nabla s_\Delta(\xi, \cdot)\, d\xi\,.$$

Passing to the limit the saturation equation holds in a weak sense. Since the expression under the integral sign

$$h - w \cdot \nabla s = g - f(s)q - f'(s)v \cdot \nabla s$$

is time continuous, it follows that s is differentiable with respect to time and the saturation equation holds pointwise. Uniqueness follows from the stability estimates (6) which are derived from an energy estimate for the saturation.

Finally, this uniqueness result implies the convergence of the approximate solution (s_Δ, p_Δ) towards the unique limit (s, p) as $\Delta t \to 0$.

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