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Asymptotic behavior of solutions of a partial functional differential equation *

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Abstract

The asymptotic behavior of solutions of an asymptotically autonomous partial functional differential equation is investigated. The aim of the present paper is to extend our earlier result for ordinary functional differential equations and difference equations to partial functional differential equations.

1 Introduction and preliminaries

Let X be a Banach space with norm $\|\cdot\|_X$. For a fixed $r>0$ define the space

$$C := C([-r,0],X) := \{u : [-r,0] \to X \ : \ u \text{ is continuous } \}.$$

Equipped with norm $\|u\| := \sup\{\|u(\theta)\|_X \ : \ \theta \in [-r,0]\}$, C is a Banach space. Consider also $L(C,X)$ the space of continuous linear mappings of C into X. For the sake of simplicity the induced operator norm on $L(C,X)$ will also be denoted by $\|\cdot\|$. Let $A_T : \mathrm{Dom}(A_T) \subset X \to X$ be a linear operator which generates a compact semigroup $T(t)$ on X. Let $F \in L(C,X)$ be given by

$$F(\phi) = \int_{-r}^{0} d\eta(\theta)\phi(\theta), \quad \phi \in C,$$

where $\eta : [-r,0] \to L(X,X)$ is of bounded variation. We consider the abstract linear autonomous functional differential equation

$$\dot{u}(t) = A_T u(t) + F(u_t) \tag{1}$$

where $u_t \in C$ is defined as $u_t(\theta) := u(t+\theta)$, $\theta \in [-r,0]$. Denote the solution operator of (1) by $U : \mathbb{R}_+ \times C \to C$. Consider also a non-autonomous perturbation of (1):

$$\dot{u}(t) = A_T u(t) + F(u_t) + G(t,u_t) \tag{2}$$

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where $G : \mathbb{R}_+ \times C \to X$ is continuous and linear for each fixed $t \in \mathbb{R}_+$, i.e. $G(t, \cdot) \in L(C, X)$.

It is natural to ask whether there is any "qualitative similarities" between (1) and (2) if the non-autonomous perturbation becomes small at $t = \infty$ in some sense.

Some results related to this question for ordinary functional differential equations were obtained in [1]. The discrete counterpart of ordinary functional differential equations, i.e. difference equations, was treated in [2]. The aim of the present work is to extend the results in [1] to partial functional differential equations. For each complex number λ we define the X-valued operator $\Delta(\lambda)$ by

$$\Delta(\lambda) = A_T x - \lambda x + F(e^{\lambda \cdot} x),\ x \in \mathrm{Dom}(A_T),$$

where $e^{\lambda \cdot} x \in C$ is defined by $(e^{\lambda \cdot} x)(\theta) = e^{\lambda\theta} x$, $\theta \in [-r, 0]$ (note that we use C to denote its complexification). A complex number λ is said to be a characteristic value of (1) if there exists $x \in \mathrm{Dom}(A_T)\backslash\{0\}$ solving the characteristic equation $\Delta(\lambda)x = 0$. The multiplicity of a characteristic value λ is defined as $\dim\ker\Delta(\lambda)$. Denote the set of characteristic values of (1) by Λ and set $\Lambda_\gamma := \{\lambda \in \Lambda : \mathrm{Re}\lambda \geq \mathrm{Re}\gamma\}$. It is known [5] that for all $\gamma \in \mathbf{C}$, Λ_γ is a finite set.

Pick a characteristic value $\lambda_r \in \Lambda$. For the rest of this article assume that λ_r is simple (has multiplicity 1) and all other characteristic value with real part equal to $\mathrm{Re}\lambda_r$ are simple. Define $u_r \in C$ by $u_r := e^{\lambda_r \cdot} x_r$, where $x_r \in \ker\Delta(\lambda_r)$. Let $\kappa := \max\{\mathrm{Re}\lambda : \lambda \in \Lambda\backslash\Lambda_{\lambda_r}\}$ and note that $\kappa < \mathrm{Re}\lambda_r$. We use the symbols "$o$" and "$O$" to indicate asymptotic behavior in the usual way.

2 Main result

Theorem 1 *Assume that for all t large enough the following inequalities are satisfied*

$$\int_t^\infty \|G(\tau, u_r)\|_X \, d\tau = O(\alpha(t)),$$

$$\|G(t, u_r)\|_X = O(\alpha(t)),$$

$$\int_t^\infty \|G(\tau, \cdot)\|\alpha(\tau)\, d\tau = O(\beta(t)),$$

$$\|G(t, \cdot)\|\alpha(t) = O(\beta(t)),$$

where α and β are non-increasing functions with zero limit at infinity, $\beta(t) = o(\alpha(t))$ and there is a ρ, $0 < \rho < Re\lambda_r - \kappa$ such that $e^{\rho t}\alpha(t)$ and $e^{\rho t}\beta(t)$ are non-decreasing functions. Then there is a σ and a solution $u(t)$ of (2) of the form

$$u(t) = e^{\lambda_r t}(x_r + u^*(t)),\ t \geq \sigma,$$

where $\|u^*(t)\|_X = O(\alpha(t))$.

Proof. The idea of the proof is to build a fixed-point setting in a certain Banach space whose fixed point is a solution of (2) and satisfies the desired asymptotic behavior. We construct such a fixed-point setting with the help of a decomposed form of a variation-of-constants formula.

Define the space

$$\tilde{C} := \{u : [-r,0] \to X \ : \ u|_{[-r,0)} \text{ is continuous and } \lim_{\theta\to 0-} u(\theta) \in X \text{ exists } \}.$$

In this space we use the supremum norm. Extend the domain of $U(t)$ to $\tilde{C}$. Let $X_0 : [-r,0] \to L(X,X)$ $X_0(\theta) = 0$ if $-r \le \theta < 0$ and $X_0(0) = Id$. Denote the generalized eigenspaces of $U(t)$ corresponding to Λ_{λ_r} and $\Lambda\backslash\Lambda_{\lambda_r}$ by PC and QC, respectively. Denote the projections onto these subspaces by P and Q, respectively. Projections P and Q can also be applied to $u \in \tilde{C}$. Define $X_0^P := PX_0$ and $X_0^Q := QX_0$.

Consider the equation

$$u_t = e^{\lambda_r t}u_r - \int_t^\infty U(t-\tau)X_0^P G(\tau,u_\tau)\,d\tau + \int_\sigma^t U(t-\tau)X_0^Q G(\tau,u_\tau)\,d\tau\,. \quad (3)$$

It is easy to see that a solution of equation (3) also solves equation (2). Introduce a new variable v_t as

$$v_t := e^{-\lambda_r t}u_t - u_r.$$

Note that the above transformation is meaningless in equation (2). It is easy to see that our integral equation has the form

$$v_t = F(t) + \mathcal{F}v_t,$$

where

$$\begin{aligned} F(t) &= -\int_t^\infty e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^P G(\tau,u_r)\,d\tau \\ &\quad + \int_\sigma^t e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^Q G(\tau,u_r)\,d\tau \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}v_t &= -\int_t^\infty e^{\lambda_r(t-\tau)}U(t-\tau)X_0^P G(\tau,v_\tau)\,d\tau \\ &\quad + \int_\sigma^t e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^Q G(\tau,v_\tau)\,d\tau. \end{aligned}$$

Introduce the Banach space

$$Y := \{y : [\sigma,\infty) \to C([-r,0],X) \ : \ y \text{ is continuous and } \|y(t)\|_X = O(\alpha(t))\}$$

with norm $|y|_Y = \sup_{t\ge\sigma}\{\|y(t)\|_X/\alpha(t)\}$. We will show that equation $y = F + \mathcal{F}y$ has a (unique) solution y^* on Y if σ is sufficiently large. With this solution in hand define $u_t := e^{\lambda_r t}(u_r + y^*(t))$. Then $u(t) = u_t(0)$ is a solution of (2) with the desired asymptotic behavior.

Lemma 1 $\|U(t)X_0^P\| \le K_1 e^{\mathrm{Re}\lambda_r t}$ *for* $t \le 0$.

Proof. Let P_0C be the generalized eigenspace of $U(t)$ corresponding to characteristic values with real part $\mathrm{Re}\lambda_r$. Then PC decomposes further as $PC = P_0C \oplus P_1C$. Denote the corresponding projections by P_0 and P_1, respectively. The domain of these projections extend to $\tilde{C}$ as well. Define $X_0^{P_0} := P_0X_0$ and $X_0^{P_1} := P_1X_0$. Since P_1C is the generalized eigenspace of $U(t)$ corresponding to characteristic values with real part strictly greater than $\mathrm{Re}\lambda_r$,

$$\|U(t)X_0^{P_1}\| \le Ke^{\mathrm{Re}\lambda_r t} \text{ for } t \le 0\,.$$

On the other hand if Φ_0 is a basis of P_0C then there is a constant matrix B_0 such that

$$U(t)\Phi_0 = \Phi_0 e^{B_0 t}$$

and the eigenvalues of B_0 are the characteristic values with real part $\mathrm{Re}\lambda_r$, see [5, Theorem 2.3,p. 77.]. Since these characteristic values are simple, from the Jordan form of B_0 one sees that there is a constant $\tilde{K}$ such that

$$\|U(t)X_0^{P_0}\| \le \tilde{K}e^{\mathrm{Re}\lambda_r t}.$$

♢

It is known that there are constants $K_2 \ge 1$ and $\rho_1 > 0$ such that

$$\|U(t)X_0^Q\| \le K_2 e^{(\mathrm{Re}\lambda_r - \rho_1)t} \text{ for } t \ge 0\,,$$

furthermore, we can assume that $\rho_1 > \rho$.

Lemma 2 $F \in Y$.

Proof. On the one hand

$$\begin{aligned}
&\|\int_\sigma^t e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^Q G(\tau,u_r)\,d\tau\|_X \\
&\le \int_\sigma^t e^{-\mathrm{Re}\lambda_r(t-\tau)}K_2 e^{(\mathrm{Re}\lambda_r-\rho_1)(t-\tau)}\|G(\tau,u_r)\|_X\,d\tau \\
&= \int_\sigma^t K_2 e^{-\rho_1(t-\tau)}e^{-\rho\tau}e^{\rho\tau}\|G(\tau,u_r)\|_X\,d\tau \\
&\le \sup_{\sigma\le\tau\le t}\{e^{\rho\tau}\|G(\tau,u_r)\|_X\}K_2 e^{-\rho_1 t}\int_\sigma^t e^{(\rho_1-\rho)\tau}\,d\tau \\
&= O(\alpha(t))\,.
\end{aligned}$$

On the other hand (using Lemma 1)

$$\begin{aligned}
&-\int_t^\infty e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^P G(\tau,u_r)\,d\tau\|_X \\
&\le \int_t^\infty e^{-\mathrm{Re}\lambda_r(t-\tau)}K_1 e^{\mathrm{Re}\lambda_r(t-\tau)}\|G(\tau,u_r)\|_X\,d\tau \\
&= O(\alpha(t))\,.
\end{aligned}$$

$\diamondsuit$

Let $\delta(\sigma) := \sup_{t\geq\sigma}\{\beta(t)/\alpha(t)\}$. Since $\beta(t) = o(\alpha(t))$, δ is well defined and tends to zero as σ tends to infinity.

Lemma 3 *If $y \in Y$ then $\mathcal{F}y \in Y$ and $|\mathcal{F}y|_Y \leq N\delta(\sigma)|y|_Y$, where N is independent of y and σ.*

Proof. On the one hand

$$\begin{aligned}
&\| \int_\sigma^t e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^Q G(\tau, y(\tau))\,d\tau\|_X \\
&\leq \sup_{\sigma\leq\tau\leq t}\{\|y(\tau)\|_X/\alpha(\tau)\}\int_\sigma^t e^{-\mathrm{Re}\lambda_r(t-\tau)}K_2 e^{(\mathrm{Re}\lambda_r-\rho_1)(t-\tau)}\|G(\tau,\cdot)\|\alpha(\tau)\,d\tau \\
&= \sup_{\sigma\leq\tau\leq t}\{\|y(\tau)\|_X/\alpha(\tau)\}\int_\sigma^t K_2 e^{-\rho_1(t-\tau)}e^{-\rho\tau}e^{\rho\tau}\|G(\tau,\cdot)\|\alpha(\tau)\,d\tau \\
&\leq \sup_{\sigma\leq\tau\leq t}\{\|y(\tau)\|_X/\alpha(\tau)\}K_2 \sup_{\sigma\leq\tau\leq t}\{e^{\rho\tau}\|G(\tau,\cdot)\|\alpha(\tau)\}e^{-\rho_1 t}\int_\sigma^t e^{(\rho_1-\rho)\tau}\,d\tau \\
&\leq K_3|y|_Y\beta(t)
\end{aligned}$$

where constant K_3 is independent of both y and σ. On the other hand (using Lemma 1 again)

$$\begin{aligned}
&\| -\int_t^\infty e^{-\lambda_r(t-\tau)}U(t-\tau)X_0^P G(\tau,y(\tau))\,d\tau\|_X \\
&\leq \sup_{\tau\geq t}\{\|y(\tau)\|_X/\alpha(\tau)\}K_4\int_t^\infty \|G(\tau,\cdot)\|\alpha(\tau)\,d\tau \\
&\leq K_5|y|_Y\beta(t)\,,
\end{aligned}$$

where the constant K_5 is independent of σ and y. These completes the present proof. $\diamondsuit$

Now choose a σ for which $N\delta(\sigma) < 1$. From Lemmas 2 and 3 it follows that operator $F + \mathcal{F}(\cdot)$ maps Y into itself and is a contraction on it. Applying the Contraction Mapping Principle the desired result follows.

Remarks

First observe that if $\|G(t,u_r)\|_X$ and $\|G(t,\cdot)\|\alpha(t)$ are non-increasing functions then conditions

$$\|G(t,u_r)\|_X = O(\alpha(t))$$

and

$$\|G(t,\cdot)\|\alpha(t) = O(\beta(t))$$

can be omitted.

Similar results for ordinary functional differential equations can be obtained under the condition $\|G(t,\cdot)\| \in L_p$ with $1 \le p < \infty$. The case case $p = 1$ can be found in [3, Theorem 5.2 p218.]; this result was recently extended to case $1 \le p \le 2$, see [4]. Since our conditions require the smallness of $G(t,\cdot)$ only on u_r it is reasonable to expect that the conditions of Theorem 1 can be satisfied even if $\|G(t,\cdot)\|$ is not in L_p. In fact this is the case in the following example. Consider a partial functional differential equation (2) such that λ_r is a simple characteristic value and assume that all other characteristic values with real part equal to $\mathrm{Re}\lambda_r$ are simple. Choose a positive constant δ with $0 < \delta < \mathrm{Re}\lambda_r - \kappa$ and let $1 \le p < \infty$. Fix $x \in X$ with $\|x\|_X = 1$, define

$$G(t, u_r) = \frac{1}{e^{\delta t}} x \,,$$

and extend $G(t,\cdot)$ by using the Hahn-Banach Theorem in such a way that

$$\|G(t,\cdot)\| = \frac{1}{t^{1/p}}$$

holds. Let

$$\alpha(t) = \frac{1}{e^{\delta t}}$$

and

$$\beta(t) = \frac{1}{t^{1/p} e^{\delta t}} \,.$$

Then α and β are non-increasing functions with zero limit at infinity and $\beta(t) = o(\alpha(t))$. Furthermore, $\int_t^\infty \|G(\tau, u_r)\|_X \, d\tau = O(\alpha(t))$ and

$$\begin{aligned} \int_t^\infty \|G(\tau,\cdot)\| \alpha(\tau) \, d\tau &= \int_t^\infty \frac{1}{\tau^{1/p} e^{\delta\tau}} \, d\tau \\ &\le \frac{1}{t^{1/p}} \int_t^\infty e^{-\delta\tau} \, d\tau \\ &= O(\beta(t)) \,. \end{aligned}$$

Choose a constant ρ with $\delta < \rho < \mathrm{Re}\lambda_r - \kappa$. Then $e^{\rho t}\alpha(t)$ and $e^{\rho t}\beta(t)$ are non-decreasing functions (for t large enough). Thus the conditions of Theorem 1 are satisfied for λ_r but $\|G(t,\cdot)\|$ does not belong to L_p.

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