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Existence results for neutral functional differential and integrodifferential inclusions in Banach spaces *

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Abstract

In this paper we investigate the existence of solutions on a compact interval for the first and second order initial-value problems for neutral functional differential and integrodifferential inclusions in Banach spaces. We shall use of a fixed point theorem for condensing maps introduced by Martelli.

1 Introduction

Existence of solutions on compact intervals for neutral functional differential equations has received much attention in recent years. We refer for instance to the books of Erbe, Qingai and Zhang [4], Hale [5] and Henderson [6], the paper of Ntouyas [13] and Ntouyas, Sficas and Tsamatos [17]. For other results on functional differential and integrodifferential equations, we mention for instance the paper of Hristova and Bainov [7], Nieto, Jiang and Jurang [12], Ntouyas [14], [15] and Ntouyas and Tsamatos [16].

In the above mentioned papers the main tools used for the existence of solutions are the monotone iterative method combined with upper and lower solutions or the topological transversality theory of Granas. For more details on these theories we refer the interesting reader to the book of Ladde, Lakshmikantham and Vatsala [9] and the monograph of Dugundji and Granas [3].

This paper is organized as follows. In section 2, we introduce some definitions and preliminary facts from multi-valued analysis which are used later. In section 3, we give an existence result of solutions on compact intervals to the initial value problem (IVP for short) of the first order neutral functional differential inclusion

$$\frac{d}{dt}[y(t) - f(t, y_t)] \in F(t, y_t), \quad \text{a.e.} \quad t \in J = [0, T], \tag{1.1}$$

$$y_0 = \phi, \tag{1.2}$$

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where $F : J \times C(J_0, E) \to 2^E$ ($J_0 = [-r, 0]$) is a bounded, closed, convex multi-valued map, $f : J \times C(J_0, E) \to E$, $\phi \in C(J_0, E)$, and E a real Banach space with the norm $|\cdot|$.

For any continuous function y defined on the interval $J_1 = [-r, T]$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

Section 4 is devoted to the study of the existence of solutions to the first order IVP for neutral functional integrodifferential inclusion of the form

$$\frac{d}{dt}[y(t) - f(t, y_t)] \in \int_0^t K(t, s)F(s, y_s)ds, \quad t \in J = [0, T], \tag{1.3}$$

$$y_0 = \phi, \tag{1.4}$$

where F, f, ϕ are as in the problem (1.1)-(1.2) and $K : D \to \mathbb{R}$, $D = \{(t, s) \in J \times J : t \geq s\}$.

In Section 5, we give an existence theorem for solutions to the second order IVP for neutral functional differential inclusions of the form

$$\frac{d}{dt}[y'(t) - f(t, y_t)] \in F(t, y_t), \quad t \in J = [0, T], \tag{1.5}$$

$$y_0 = \phi, \ y'(0) = \eta. \tag{1.6}$$

where F, f, ϕ are as in the problem (1.1)-(1.2) and $\eta \in E$.

The strategy is to reduce the existence of solutions to problems (1.1)-(1.2), (1.3)-(1.4) and (1.5)-(1.6) to the search for fixed points of a suitable multi-valued map on the Banach space $C(J_1, E)$. To prove the existence of fixed points, we shall rely on a fixed point theorem for condensing maps introduced by Martelli [11].

2 Preliminaries

This section presents notation, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper.

Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [19]).

Let $L^1(J,E)$ denote the Banach space of continuous functions $y : J \to E$ which are Bochner integrable and have norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt \quad \text{for all} \quad y \in L^1(J,E).$$

Let $(X, \|\cdot\|)$ be a Banach space. Then a multi-valued map $G : X \to 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*,\ y_n \to y_*,\ y_n \in Gx_n$ imply $y_* \in Gx_*$).

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multi-valued map $G : J \to BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

belongs to $L^1(J,\mathbb{R})$. For more details on multi-valued maps see the books of Deimling [2] and Hu and Papageorgiou [8].

An upper semi-continuous map $G : X \to X$ is said to be condensing [1] if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [1].

We remark that a completely continuous multi-valued map is the easiest example of a condensing map.

Our existence results will be proved using the following fixed point result.

Lemma 2.1 *[11]. Let X be a Banach space and $N : X \to BCC(X)$ a condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in Ny \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3 Existence results for differential inclusions

In this section we give an existence result for the problem (1.1)-(1.2). For the study of this problem we first list the following hypotheses:

(H1) There exists constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that

$$|f(t,u)| \leq c_1\|u\| + c_2, \quad t \in J,\ u \in C(J_0,E);$$

(H2) $F : J \times C(J_0,E) \to BCC(E); (t,u) \longmapsto F(t,u)$ is measurable with respect to t for each $u \in C(J_0,E)$, u.s.c. with respect to u for each $t \in J$ and for each fixed $u \in C(J_0,E)$ the set

$$S_{F,u} = \Big\{ g \in L^1(J,E) :\ g(t) \in F(t,u) \text{ for a.e. } \ t \in J \Big\}$$

is nonempty;

(H3) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0,E)$, where $p \in L^1(J,\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^T p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)};$$

where $c = \frac{1}{1-c_1}[(1+c_1)\|\phi\| + 2c_2]$;

(H4) The function f is completely continuous and for any bounded set $A \subseteq C(J_1,E)$ the set $\{t \to f(t,y_t) : y \in A\}$ is equicontinuous in $C(J,E)$;

(H5) For each bounded $B \subset C(J_1,E)$, $u \in B$ and $t \in J$ the set

$$\Big\{ \int_0^t g(s)ds : g \in S_{F,u} \Big\}$$

is relatively compact.

Remark 3.1 *(i) If $dimE < \infty$, then for each $u \in C(J_0,E)$, $S_{F,u} \neq \emptyset$ (see Lasota and Opial [10]).*
(ii) $S_{F,u}$ is nonempty if and only if the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t,u)\}$$

belongs to $L^1(J,\mathbb{R})$ (see Papageorgiou [18]).

Definition 3.2 *By a solution to the IVP (1.1)-(1.2) it mean a function $y : J_1 \to E$ such that $y_0 = \phi$, $y_t \in C(J_0,E)$, the function $y(t) - f(t,y_t)$ is absolutely continuous and the inclusion (1.1) hold a.e. on J.*

The following Lemma is crucial in the proof of our existence results.

Lemma 3.3 *[10]. Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC(C(I,X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Now, we are in a position to state and prove our main theorem for this section

Theorem 3.4 *Assume that hypotheses (H1)-(H5) hold. Then the IVP (1.1)-(1.2) has at least one solution on* J_1.

Proof. Let $C(J_1, E)$ be the Banach space of continuous functions from J_1 into E endowed with the sup-norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r, T]\}, \text{ for } y \in C(J_1, E).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \longrightarrow 2^{C(J_1,E)}$ defined by:

$$Ny := \left\{ h \in C(J_1, E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ \phi(0) - f(0, \phi) + f(t, y_t) + \int_0^t g(s)ds, & \text{if } t \in J \end{cases} \right\}$$

where

$$g \in S_{F,y} = \left\{ g \in L^1(J, E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J \right\}.$$

Remark 3.5 *It is clear that the fixed points of* N *are solutions to (1.1)-(1.2).*

We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: *Ny is convex for each $y \in C(J_1, E)$.* Indeed, if h_1, h_2 belong to Ny, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_1(t) = \phi(0) - f(0, \phi) + f(t, y_t) + \int_0^t g_1(s)ds$$

and

$$h_2(t) = \phi(0) - f(0, \phi) + f(t, y_t) + \int_0^t g_2(s)ds.$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have

$$(kh_1 + (1-k)h_2)(t) = \phi(0) - f(0, \phi) + f(t, y_t) + \int_0^t [kg_1(s) + (1-k)g_2(s)]ds.$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$kh_1 + (1-k)h_2 \in Ny$$

which finish the proof of Step 1.

We next shall prove that N is a completely continuous operator. Using (H4) it suffices to show that the operator $N_1 : C(J_1, E) \longrightarrow 2^{C(J_1,E)}$ defined by:

$$N_1 y := \left\{ h \in C(J_1, E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ \int_0^t g(s)ds, & \text{if } t \in J \end{cases} \right\}$$

is completely continuous.

Step 2: *N_1 maps bounded sets into bounded sets in $C(J_1,E)$.* Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h\in N_1y, y\in B_q=\{y\in C(J_1,E):\|y\|_\infty\le q\}$ one has $\|h\|_\infty\le\ell$.

If $h\in N_1y$, then there exists $g\in S_{F,y}$ such that for each $t\in J$ we have

$$h(t)=\int_0^t g(s)ds.$$

By (H3) we have for each $t\in J$

$$\begin{aligned}\|h(t)\| &\le \int_0^t \|g(s)\|ds\\ &\le \sup_{y\in[0,q]}\psi(y)\Big(\int_0^t p(s)ds\Big)\\ &\le \sup_{y\in[0,q]}\psi(y)\Big(\int_0^t p(s)ds\Big).\end{aligned}$$

Then for each $h\in N(B_q)$ we have

$$\|h\|_\infty\le \sup_{y\in[0,q]}\psi(y)\Big(\int_0^T p(s)ds\Big):=\ell.$$

Step 3: *N_1 maps bounded sets into equicontinuous sets of $C(J_1,E)$.* Let $t_1,t_2\in J, t_1<t_2$ and $B_q=\{y\in C(J_1,E):\|y\|_\infty\le q\}$ be a bounded set of $C(J_1,E)$.

For each $y\in B_q$ and $h\in N_1y$, there exists $g\in S_{F,y}$ such that

$$h(t)=\int_0^t g(s)ds.$$

Thus

$$\begin{aligned}\|h(t_2)-h(t_1)\| &\le \Big\|\int_{t_1}^{t_2} g(s)ds\Big\|\\ &\le \sup_{y\in[0,q]}\psi(y)\Big(\int_{t_1}^{t_2} p(s)ds\Big).\end{aligned}$$

As $t_2\to t_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $t_1<t_2\le 0$ and $t_1\le 0\le t_2$ are obvious.

As a consequence of Step 2, Step 3, (H4) and (H5) together with the Ascoli-Arzela theorem we can conclude that $N:C(J_1,E)\to 2^{C(J_1,E)}$ is a compact multivalued map, and therefore, a condensing map.

Step 4: *N has a closed graph.* Let $y_n \to y_*$, $h_n \in Ny_n$, and $h_n \to h_*$. We shall prove that $h_* \in Ny_*$. $h_n \in Ny_n$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = \phi(0) - f(0,\phi) + f(t, y_{nt}) + \int_0^t g_n(s)ds, \quad t \in J.$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = \phi(0) - f(0,\phi) + f(t, y_{*t}) + \int_0^t g_*(s)ds, \quad t \in J.$$

Since f is continuous we have that

$$\|(h_n - \phi(0) + f(0,\phi) - f(t, y_{nt})) - (h_* - \phi(0) + f(0,\phi) - f(t, y_{*t}))\|_\infty \to 0,$$

as $n \to \infty$. Consider the linear continuous operator

$$\begin{aligned} \Gamma : L^1(J,E) &\to C(J,E) \\ g \mapsto \Gamma(g)(t) &= \int_0^t g(s)ds. \end{aligned}$$

From Lemma 3.3, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$(h_n(t) - \phi(0) + f(0,\phi) - f(t, y_{nt})) \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y^*$, it follows from Lemma 3.3 that

$$(h_*(t) - \phi(0) + f(0,\phi) - f(t, y_t)) = \int_0^t g_*(s)ds$$

for some $g_* \in S_{F,y^*}$.

Step 5: *The set* $\Omega := \{y \in C(J_1, E) : \lambda y \in Ny$ *for some* $\lambda > 1\}$ *is bounded.* Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that for $t \in J$,

$$y(t) = \lambda^{-1}\phi(0) - \lambda^{-1}f(0,\phi) + \lambda^{-1}f(t, y_t) + \lambda^{-1}\int_0^t g(s)ds\,.$$

This implies by (H1), (H3) that for each $t \in J$ we have

$$\|y(t)\| \leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|y_t\| + \int_0^t p(s)\psi(\|y_s\|)ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(t)| : -r \leq s \leq t\}, \quad t \in J.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned}\mu(t) &\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|y_t\| + \int_0^t p(s)\psi(\|y_s\|)ds \\ &\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\mu(t) + \int_0^t p(s)\psi(\mu(s))ds.\end{aligned}$$

Thus

$$\mu(t) \leq \frac{1}{1-c_1}\Big\{(1+c_1)\|\phi\| + 2c_2 + \int_0^t p(s)\psi(\mu(s))ds.\Big\}$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.
Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \frac{1}{1-c_1}\Big\{(1+c_1)\|\phi\| + 2c_2\Big\} \text{ and } \mu(t) \leq v(t), \quad t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq p(t)\psi(v(t)), \ t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant b such that $v(t) \leq b, \ t \in J$, and hence $\mu(t) \leq b, \ t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \leq t \leq T\} \leq b,$$

where b depends only on T and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1.1)-(1.2). $\diamondsuit$

4 Existence results for integrodifferential inclusions

In this section we consider the solvability of IVP (1.3)-(1.4). Let us state the following hypotheses:

(H6) For each $t \in J$, $K(t, s)$ is measurable on J and

$$K(t) = \text{ess sup}\{|K(t, s)|, \quad 0 \leq s \leq t\},$$

is bounded on J;

(H7) The map $t \longmapsto K_t$ is continuous from J to $L^\infty(J, \mathbb{R})$; here $K_t(s) = K(t,s)$;

(H8) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$T \sup_{t\in J} K(t) \int_0^T p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)};$$

where $c = \frac{1}{1-c_1}\Big\{(1+c_1)\|\phi\| + 2c_2\Big\}$;

(H9) For each bounded $B \subset C(J_1, E)$, $u \in B$ and $t \in J$ the set

$$\Big\{\int_0^t \int_0^s K(s,\sigma)g(\sigma)d\sigma ds : g \in S_{F,u}\Big\}$$

is relatively compact.

Definition 4.1 *By a solution to the IVP (1.3)-(1.4) it mean a function* $y : J_1 \to E$ *such that* $y_0 = \phi$, $y_t \in C(J_0, E)$, *the function* $y(t) - f(t, y_t)$ *is absolutely continuous and the inclusion (1.3) hold a.e. on* J.

Now, we are able to state and prove our main theorem.

Theorem 4.2 *Assume that hypotheses (H1), (H2), (H4), (H6)-(H9) are satisfied. Then the IVP (1.3)-(1.4) has at least one solution on* J_1.

Proof. Let $C(J_1, E)$ be the Banach space of continuous functions from J_1 into E endowed with the sup-norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r, T]\}, \text{ for } y \in C(J_1, E).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \longrightarrow 2^{C(J_1,E)}$ defined by:

$$Ny := \left\{ h \in C(J_1, E) : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in J_0 \\ \phi(0) - f(0,\phi) + f(t, y_t) & \\ \quad + \int_0^t \int_0^s K(s,u)g(u)duds, & \text{if } t \in J \end{array} \right. \right\}$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J, E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J\Big\}.$$

Remark 4.3 *It is clear that the fixed points of* N *are solutions to (1.3)-(1.4).*

As in Theorem 3.4 we can show that N is a completely continuous multi-valued map, u.s.c. with convex closed values, and therefore a condensing map.

Here we repeat the proof that the set

$$\Omega := \{y \in C(J_1, E) : \lambda y \in Ny, \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that for $t \in J$,

$$y(t) = \lambda^{-1}\phi(0) - \lambda^{-1}f(0,\phi) + \lambda^{-1}f(t,y_t) + \lambda^{-1}\int_0^t \int_0^s K(s,u)g(u)\,du\,ds.$$

This implies by (H1), (H6)-(H8) that for each $t \in J$ we have

$$\begin{aligned}
\|y(t)\| &\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|y_t\| + \Big\|\int_0^t \int_0^s K(s,u)g(u)\,du\,ds\Big\| \\
&\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|y_t\| + \int_0^t \int_0^s |K(s,u)|p(u)\psi(\|y_u\|)\,du\,ds \\
&\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|y_t\| + T\sup_{t\in J} K(t)\int_0^t p(s)\psi(\|y_s\|)ds.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(t)| : -r \leq s \leq t\}, \quad t \in J.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned}
\mu(t) &\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\|y_t\| + T\sup_{t\in J} K(t)\int_0^t p(s)\psi(\|y_s\|)ds \\
&\leq \|\phi\| + c_1\|\phi\| + 2c_2 + c_1\mu(t) + T\sup_{t\in J} K(t)\int_0^t p(s)\psi(\mu(s))ds.
\end{aligned}$$

Thus

$$\mu(t) \leq \frac{1}{1-c_1}\Big\{(1+c_1)\|\phi\| + 2c_2 + T\sup_{t\in J} K(t)\int_0^t p(s)\psi(\mu(s))ds\Big\}.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \frac{1}{1-c_1}\Big\{(1+c_1)\|\phi\| + 2c_2\Big\} \text{ and } \mu(t) \leq v(t), \ t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq T\sup_{t\in J} K(t)p(t)\psi(v(t)), \ t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq T \sup_{t\in J} K(t) \int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant b such that $v(t) \leq b,\ t \in J$, and hence $\mu(t) \leq b,\ t \in J$. Since for every $t \in J, \|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \leq t \leq T\} \leq b,$$

where b depends only on T and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1.3)-(1.4). ♢

5 Second order differential inclusions

In this section we consider the solvability of IVP (1.5)-(1.6). For the study of this problem we first list the following hypotheses:

(H10) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^T M(s)ds < \int_c^\infty \frac{d\tau}{u + \psi(\tau)};$$

where $c = \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T$ and $M(t) = max\{1, c_1, p(t)\}$;

(H11) for each bounded $B \subset C(J_1, E)$, $y \in B$ and $t \in J$ the set

$$\Big\{\int_0^t \int_0^u g(u)\, du\, ds : g \in S_{F,y}\Big\}$$

is relatively compact.

Definition 5.1 *By a solution to the IVP (1.5)-(1.6) we shall mean a differentiable function $y : J_1 \to E$ such that $y_0 = \phi, y'(0) = \eta, y_t \in C(J_0, E)$, the function $y'(t) - f(t, y_t)$ is absolutely continuous and the inclusion (1.5) hold a.e. on J.*

Now, we are in a position to state and prove our main theorem in this section.

Theorem 5.2 *Assume that hypotheses (H1), (H2), (H4), (H10), (H11) hold. Then the IVP (1.5)-(1.6) has at least one solution on J_1.*

Proof. Let $C(J_1, E)$ be the Banach space of continuous functions from J_1 into E endowed with the sup norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r, T]\}, \text{ for } y \in C(J_1, E).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \longrightarrow 2^{C(J_1,E)}$ defined by:

$$Ny := \left\{ h \in C(J_1, E) : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in J_0 \\ \phi(0) + [\eta - f(0, \phi)]t & \\ + \int_0^t f(s, y_s)ds + \int_0^t \int_0^s g(u)duds, & \text{if } t \in J \end{array} \right. \right\}$$

Remark 5.3 *It is clear that the fixed points of N are solutions to (1.5)-(1.6).*

As in Theorem 3.4 we can show that N is a completely continuous multivalued map, u.s.c. with convex closed values.

Now we prove only that the set

$$\Omega := \{y \in C(J_1, E) : \lambda y \in Ny, \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that for $t \in J$,

$$y(t) = \lambda^{-1}\phi(0) + \lambda^{-1}[\eta - f(0, \phi)]t + \lambda^{-1} \int_0^t f(s, y_s)ds + \lambda^{-1} \int_0^t \int_0^u g(u)\, du\, ds\,.$$

This implies by (H1), (H3) that for each $t \in J$ we have

$$\begin{aligned} \|y(t)\| &\le \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T \\ &\quad + c_1 \int_0^t \|y_s\|ds + \int_0^t \int_0^s p(u)\psi(\|y_u\|)\, du\, ds \\ &\le \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T \\ &\quad + \int_0^t M(s)\|y_s\|ds + \int_0^t M(s) \int_0^s \psi(\|y_u\|)\, du\, ds\,, \end{aligned}$$

where $M(t) = max\{1, c_1, p(t)\}$. We consider the function μ defined by

$$\mu(t) = \sup\{|y(t)| : -r \le s \le t\}, \quad t \in J.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\mu(t) \le \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T + \int_0^t M(s)\mu(s)ds + \int_0^t M(s) \int_0^s \psi(\mu(u))\, du\, ds.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds. Denoting by $u(t)$ the right hand side of the above inequality we have

$$u(0) = \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T = c, \quad \mu(t) \le u(t), \quad t \in J$$

and

$$\begin{aligned} u'(t) &\le M(t)\mu(t) + M(t)\int_0^t \psi(\mu(s))ds \\ &\le M(t)\left[u(t) + \int_0^t \psi(u(s))ds\right], \quad t \in J. \end{aligned}$$

Put

$$v(t) = u(t) + \int_0^t \psi(u(s))ds, \quad t \in J.$$

$$u(0) = \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T = c, \quad \mu(t) \le u(t), \quad t \in J$$

and

$$\begin{aligned} v'(t) &= u'(t) + \psi(u(t)) \\ &\le M(t)[v(t) + \psi(v(t))], \quad t \in J. \end{aligned}$$

This implies that for each $t \in J$,

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \le \int_0^T M(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}.$$

This inequality implies that there exists a constant b such that $v(t) \le b, \ t \in J$, and hence $\mu(t) \le b, \ t \in J$. Since for every $t \in J, \|y_t\| \le \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \le t \le T\} \le b,$$

where b depends only on T and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1.5)-(1.6). $\diamondsuit$

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