# A note on the saddle conic of quadratic planar differential systems [*]

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## Abstract

We give some properties of the saddle conic of quadratic differential systems. We also deduce semi-algebraic conditions for the existence of one, two or three saddle points (in terms of affine invariants).

## 1 Motivations and introduction

As shown in the report by Reyn [12], many publications are devoted to the qualitative analysis of the planar quadratic differential system

$$\frac{dx}{dt} = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 \tag{1}$$
$$\frac{dy}{dt} = b_{0,0} + b_{1,0}x + b_{0,1}y + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2,$$

where $a_{i,j}$ and $b_{i,j}$ are real-valued coefficients. Note that these systems form a 12-dimensional linear space, denoted by $\mathcal{A}$, and that a complete qualitative study requires determining the partition of the phase plane into trajectories. Following Leontovitch & Maier [1], this partition is completely defined by the number and the nature of critical points, the separatrix structure, and the location of closed trajectories (the famous 16th Hilbert problem). This work deals with the first of these three questions.

In [14, 15], Baltag and Vulpe established a complete tableau of the number and multiplicity of the critical points in the plane, including those at infinity. Their conditions are algebraic and semi-algebraic (equalities and inequalities) given in terms of center-affine invariants and covariants. From the symbolic computation point of view, this approach is very interesting since it provides a simple algorithm giving the number and the multiplicity of critical points without solving (with radicals) algebraic equations.

The research on the nature of critical points falls into two different categories. The first direction concerns the famous center-focus problem. Based on Dulac's early work (1908) and on Kapteyn's work (1912), a series of contributions has been presented (see [4] for the evolution of this question). Finally,

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explicit conditions (in terms of invariants and covariants) for finding systems (1) with one or two centers were obtained in [6]. The second direction is centered around the question of the coexistence of critical points of different types. It was initiated by Berlinskii [2] who established, among others, Theorem 1 below. However, he did not characterize the possible situations by algebraic or semi-algebraic conditions on the coefficients of (1). P. Curtz gave the first set of sufficient conditions expressed in terms of coefficients of systems (1) for the existence of saddle points.

The study of the second direction problem leads us to consider the determinant of the Jacobian of the vector field associated to the system (1),

$$q(x,y) = \left| \begin{array}{cc} a_{1,0} + 2a_{2,0}x + a_{1,1}y & a_{0,1} + a_{1,1}x + 2a_{0,2}y \\ b_{1,0} + 2b_{2,0}x + b_{1,1}y & a_{0,1} + b_{1,1}x + 2b_{0,2}y \end{array} \right|.$$

It is clear that a critical point $(x_0, y_0)$ is a saddle point if and only if $q(x_0, y_0) < 0$. We call the algebraic curve $q(x,y) = 0$ the saddle conic because it induces a partition of the phase plane into three regions characterized by the relations $q(x_0, y_0) < 0$, $q(x_0, y_0) > 0$ and $q(x_0, y_0) = 0$ and the first one contains the saddle points and the second one, the anti-saddle points.

In this note, we establish some algebraic and geometric properties of the polynomial $q(x,y)$ and give affine conditions for the existence of one, two, or three saddle points for system (1). All computations are made with Maple and the package **SIB** [7] which contains minimal systems of generators of center-affine and affine covariants of systems (1).

## 2   Review of invariants and covariants of differential systems

Planar quadratic differential systems with real coefficients form a $\mathbb{R}$-vector-space of dimension 12 (precisely, isomorphic to $\mathbb{R}^2 \oplus \mathbb{R}^2 \bigotimes (\mathbb{R}^2)^\star \oplus S_2 \bigotimes (\mathbb{R}^2)^\star$ where $(\mathbb{R}^2)^\star$ is the dual of $\mathbb{R}^2$ and $S_2$ the space of algebraic quadratic forms). Using Einstein notation, they can be written in the condensed form

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = 1, 2). \tag{2}$$

where $x = (x^1, x^2)^T \in \mathbb{R}^2$ (the letter $T$ means transposed) and $a^j_\alpha x^\alpha = a^j_1 x^1 + a^j_2 x^2$, $a^j_{\alpha\beta} x^\alpha x^\beta = a^j_{11}(x^1)^2 + 2a^j_{12}x^1 x^2 + a^j_{22}(x^2)^2$. The Einstein notation will be adopted in the whole paper: We suppress the symbol $\sum$ (sum) in all contractions.

In addition, let $Aff(2, \mathbb{R})$ be the group of affine transformations

$$x \mapsto y = P^{-1}(x - p) \tag{3}$$

with

$$P = \left( \begin{array}{cc} p^1_1 & p^1_2 \\ p^2_1 & p^2_2 \end{array} \right), \quad \det(P) \neq 0 \quad \text{and} \quad p = (p^1, p^2)^T.$$

It acts rationally over $\mathcal{A}$ following the rational representation

$$\rho : G \mapsto GL(\mathcal{A})$$

where $GL(\mathcal{A})$ is the group of automorphisms of $\mathcal{A}$. Putting $\rho(P,p)(a) = b$, this representation is defined by the formulae:

$$b^j = q_i^j (a^i + a_\alpha^i p^\alpha + a_{\alpha\beta}^i p^\alpha p^\beta),$$
$$b_\alpha^j = q_i^j p_\alpha^\beta (a_\beta^i + 2a_{\beta\gamma}^i p^\gamma),$$
$$b_{\alpha\beta}^j = q_i^j p_\alpha^\gamma p_\beta^\delta a_{\gamma\delta}^i,$$

where $Q = (q_i^j)$ is the inverse matrix of $P$.

Let $\mathbb{R}[a,x]$ denote the algebra of polynomials whose indeterminates are components of a generic vector $a$ of $\mathcal{A} \times \mathbb{R}^2$: $a^1, a^2, a_1^1, a_2^1, \ldots, a_{22}^1, a_{22}^2, x^1, x^2$. The representation of the group $Aff(2,\mathbb{R})$ on $GL(\mathcal{A} \times \mathbb{R}^2)$ is the direct sum of $\rho$ and $Aff(2,\mathbb{R})$. It is denoted r.

**Definition.** A polynomial function $K \in \mathbb{R}[a,x]$ is said to be a $Aff(2,\mathbb{R})$-covariant of $\mathcal{A}$ if there exists a function $\lambda : Aff(2,\mathbb{R}) \to \mathbb{R}$ such that

$$\forall g \in G, \quad (K \circ \mathrm{r})(g) = \lambda(g).K.$$

If $\lambda(g) \equiv 1$, then the invariant is said absolute. Otherwise, it is said relative. An $Aff(2,\mathbb{R})$-invariant is an $Aff(2,\mathbb{R})$-covariant which does not depend on $x$.

It can be proved [3] that the function $\lambda$ is a character group of $Aff(2,\mathbb{R})$ and equal to $\det(Q)^{-\kappa}$, where the integer $\kappa$ is called the weight of the covariant (or invariant).

The above definitions hold for any subgroup of $Aff(2,\mathbb{R})$, in particular for the center-affine group denoted $Gl(2,\mathbb{R})$ (put in the affine group $p \equiv 0$) or the special group denoted $Sl(2,\mathbb{R})$ ($\det(P) = 1$ and $p \equiv 0$).

The sets of $Sl(2,\mathbb{R})$-covariants or invariants and homogeneous $Gl(2,\mathbb{R})$-covariants (called also center-affine covariants) or $Gl(2,\mathbb{R})$-invariants (center-affine invariants) are the same. The algebras of $Sl(2,\mathbb{R})$-invariants and $Sl(2,\mathbb{R})$-covariants are finitely generated.

In [7] a package denoted **SIB** is elaborated with Maple. It contains minimal systems of generators of the algebras of center-affine (denoted $J_1, \ldots, J_{36}$, $K_1, \ldots, K_{33}$) and affine covariants (denoted by $Q_1, \ldots, Q_{36}$).

## 3  Algebraic Properties of the Saddle Conic

Let us introduce, for the systems (2), the following quantities:

$$A_{00} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix}, \; A_{i0} = \begin{vmatrix} a_{1i}^1 & a_2^1 \\ a_{1i}^2 & a_2^2 \end{vmatrix}, \; A_{0i} = \begin{vmatrix} a_1^1 & a_{2i}^1 \\ a_1^2 & a_{2i}^2 \end{vmatrix}, \; A_{ij} = \begin{vmatrix} a_{1i}^1 & a_{2j}^1 \\ a_{1i}^2 & a_{2j}^2 \end{vmatrix}.$$

The saddle conic of (2) has the expression (we represent here by $x$ the previous vector $(x, y)$):

$$\begin{aligned} q(x) &= A_{00} + 2(A_{10} + A_{01})x^1 + 2(A_{20} + A_{02})x^2 \\ &\quad + 4[A_{11}(x^1)^2 + (A_{12} + A_{21})x^1x^2 + A_{22}(x^2)^2]. \end{aligned}$$

It contains all the information about the distribution of saddles and antisaddles in the phase plane. Following [7], the polynomial $q(x, y)$ is an affine *absolute covariant*,

$$q(x, y) = \frac{1}{2}(Q_1^2 - Q_2) = \frac{1}{2}[(J_1^2 - J_2) + 4(J_1 K_1 - K_3) + 4(K_1^2 - K_7)],$$

where $Q_1$ and $Q_2$ are affine covariants. Let us consider its two discriminants,

$$4\delta_1 = \begin{vmatrix} 4A_{11} & 2(A_{12} + A_{21}) \\ 2(A_{12} + A_{21}) & 4A_{22} \end{vmatrix}, \tag{4}$$

$$2\delta_2 = \begin{vmatrix} 4A_{11} & 2(A_{12} + A_{21}) & A_{01} + A_{10} \\ 2(A_{12} + A_{21}) & 4A_{22} & A_{02} + A_{20} \\ A_{01} + A_{10} & A_{02} + A_{20} & A_{00} \end{vmatrix}. \tag{5}$$

With the help of package **SIB**, we obtain the affine invariants

$$2\delta_1 = 2J_7 - J_8 - J_9,$$

$$\delta_2 = 4J_1(J_{12} - J_{11}) - J_1^2(J_8 + J_9 - 2J_7) + 2J_2(J_9 - J_7) + 2(J_4 - J_5)(2J_3 - J_4 - J_5).$$

The total degree of $\delta_1$ is 4 and that of $\delta_2$ is 6.

**Lemma 1 ([5], p. 56)** *The quadratic homogeneous parts of the equations* (2) *have a common factor if, and only if, $\delta_1 = 0$.*

**Proof.**   The resultant of the polynomials $a_{11}^1(x^1)^2 + 2a_{12}^1 x^1 x^2 + a_{22}^1(x^2)^2$ and $a_{11}^2(x^1)^2 + 2a_{12}^2 x^1 x^2 + a_{22}^2(x^2)^2$ is equal to $\delta_1$.

**Lemma 2** *The differential system (2) can be reduced by a rotation into the form*

$$\begin{aligned} \frac{dx^1}{dt} &= a^1 + a_\alpha^1 x^\alpha, \\ \frac{dx^2}{dt} &= a^2 + a_\alpha^2 x^\alpha + a_{\alpha\beta}^2 x^\alpha x^\beta \end{aligned}$$

*if, and only if, $A_{11}(x^1)^2 + (A_{12} + A_{21})x^1x^2 + A_{22}(x^2)^2 = K_1^2 - K_7 = 0$.*

**Proof.**   The necessary condition is trivial. Suppose that $A_{11}(x^1)^2 + (A_{12} + A_{21})x^1x^2 + A_{22}(x^2)^2 = 0$. That means that

$$\begin{vmatrix} a_{11}^1 & a_{12}^1 \\ a_{11}^2 & a_{12}^2 \end{vmatrix}(x^1)^2 + \begin{vmatrix} a_{11}^1 & a_{22}^1 \\ a_{11}^2 & a_{22}^2 \end{vmatrix}x^1x^2 + \begin{vmatrix} a_{12}^1 & a_{22}^1 \\ a_{12}^2 & a_{22}^2 \end{vmatrix}(x^2)^2 = 0.$$

Consequently, there exists two real constants $k_1$ and $k_2$ such that $k_1^2 + k_2^2 = 1$ and $k_1 a^1_{\alpha\beta} x^\alpha x^\beta + k_2 a^2_{\alpha\beta} x^\alpha x^\beta = 0$. Then the rotation $X^1 := k_1 x^1 + k_2 x^2$, $X^2 := -k_2 x^1 + k_1 x^2$ leads the initial system to the sought form.

From this lemma it follows the proposition:

**Proposition 1** *If the system (2) has four isolated critical points (real or complex), then $K_1^2 - K_7 \neq 0$.*

## 4  Geometric Properties of the Saddle Conic

Suppose that(2) has four isolated critical points. By [8], any three of these points are never into the same straight line. Then it is possible to find an affine transformation of the plane, denoted $\Phi$ such that the points $(0,0)^T = O$, $(0,1)^T = A$, $(1,0) = B$, and $(c,d) = D$ become critical for the transformed system

$$\frac{dy^i}{dt} = b^i + b^i_\alpha y^\alpha + b^i_{\alpha\beta} y^\alpha y^\beta \quad (i, \alpha, \beta = 1, 2) \tag{6}$$

whose coefficients verify the relations:

$$b^i = 0, \quad b^i_1 = -b^i_{11}, \quad b^i_2 = -b^i_{22}, \quad (i = 1, 2) \tag{7}$$
$$b^i_{11} c(c-1) + 2b^i_{12} cd + b^i_{22} d(d-1) = 0, \quad (i = 1, 2).$$

Moreover, $cd \neq 0$ and $c + d - 1 \neq 0$.

Let $B_{ij}$, $(i, j = 1, 2)$ be the transformed quantities of $A_{ij}$ and $\tilde{\delta}_1, \tilde{\delta}_2$ the expressions of $\delta_1$ and $\delta_2$ where the $A_{ij}$ are replaced by $B_{ij}$. Since the affine invariants $\delta_1$ and $\delta_2$ are relative and of weight 2, we have

$$\tilde{\delta}_1 = \Delta^{-2} \delta_1 \quad \text{and} \quad \tilde{\delta}_2 = \Delta^{-2} \delta_2,$$

where $\Delta$ is the determinant of the linear part of $\Phi$.

**Remark**   The signs of the affine invariants $\delta_1$ and $\delta_2$ do not change under the affine transformation of the plane.

We have arrived at the interesting geometrical fact.

**Lemma 3** *The quadrilateral whose vertices are the four isolated singular points of the quadratic system (2) is convex (resp. not convex) if and only if $\delta_1 = 2J_7 - J_8 - J_9 < 0$ (resp. $\delta_1 > 0$).*

**Proof.**   Note that the quadrilateral is not convex if, and only if, one vertex lies in the triangle formed by other vertices. For systems (6 -7), the vertices $O$, $A$, $B$ being fixed, the quadrilateral is convex if and only if $(c + d - 1)cd > 0$. Taking into account the relations (7) we obtain

$$\tilde{\delta}_1 = -2 \frac{(c + d - 1)(b^1_{11} b^2_{22} - b^2_{11} b^1_{22})^2}{cd}.$$

Moreover, $B_{12} = b^1_{11}b^2_{22} - b^2_{11}b^1_{22} \neq 0$, because $K^2_1 - K_7 \neq 0$. This completes the proof.

A saddle point is an elementary critical point whose corresponding linearized system admits real eigenvalues of opposite signs. Its geometrical index is equal to $-1$. All other elementary critical points (nodes, center and foci) of geometrical index $+1$ are called anti-saddles.

To know whether a given critical point $x_0$ is a saddle or not we have to compute the determinant of the linearized part around the considered point, i.e., $q(x_0)$: $x_0$ is a saddle if and only if $q(x_0) < 0$. In the case of four isolated critical points we get the following result which was established the first time by Berlinski [2].

**Theorem 1** *Suppose that there are four real critical points. If the quadrilateral with vertices at the points is convex then two opposite critical points are saddles and the other two are antisaddles. But if the quadrilateral is not convex then either the three exterior vertices are saddles and the interior antisaddle or the exterior vertices are antisaddles and the interior vertex a saddle.*
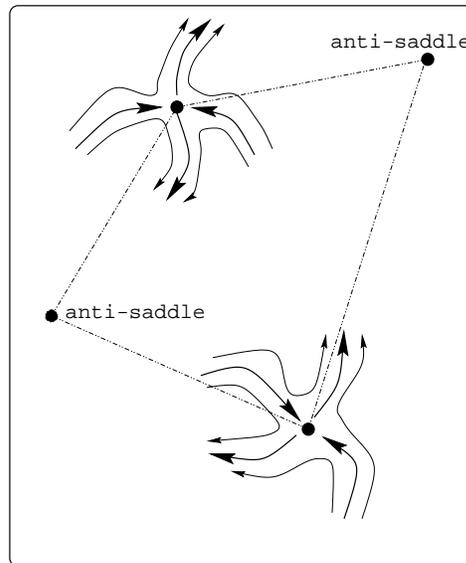


Figure 1: Case with two anti-saddle points

**Proof [8].**   After substitution $x_0$ by critical points $O$, $A$, $B$ and $D$ in (6 - 7), we obtain:

$$q(0) = B_{12}, \quad q(A) = -\frac{(c+d-1)B_{12}}{d},$$

$$q(B) = -\frac{(c+d-1)B_{12}}{c}, \quad q(D) = (c+d-1)B_{12}.$$
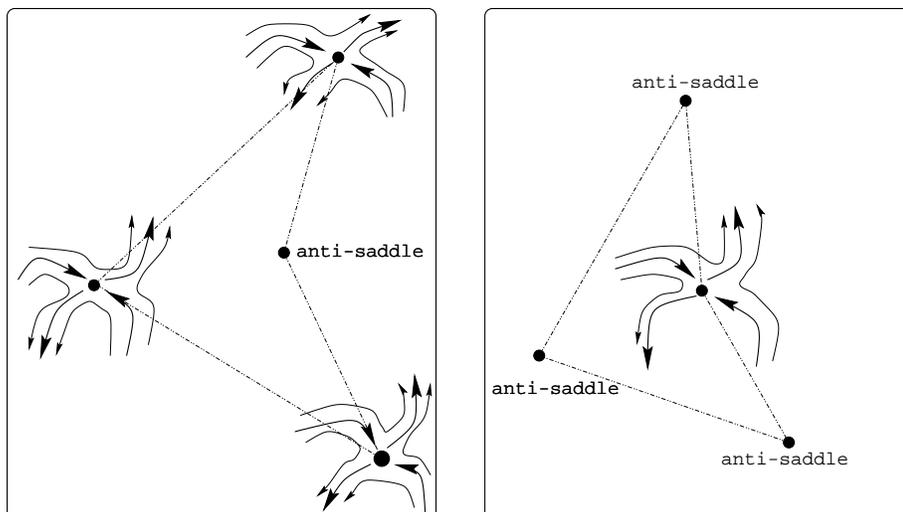
Figure 2: Case with one and with three saddle points

Consequently,

$$q(0)q(A)q(B)q(D) = \frac{(c+d-1)^3 B_{12}^4}{cd}.$$

If $\tilde{\delta}_1 < 0$, the quadrilateral OABD is convex and $q(0)q(A)q(B)q(D) > 0$. There are three possibilities: zero, two, or four saddles. We shall show that the first and third cases cannot hold.

   If $c + d - 1 < 0$, $q(0)$ and $q(D)$ have opposite sign. Then, there exists at least one saddle point and one anti-saddle point. If $c + d - 1 > 0$ and taking into account the inequality $(c + d - 1)/(cd) > 0$, we have necessarily $cd > 0$. Because $c + d - 1 > 0$, this implies that $c > 0$ and $d > 0$. Thus, the quantities $q(0)$ and $q(A)$ are of opposite signs. If $\tilde{\delta}_1 > 0$, then the quadrilateral OABD is not convex and $q(0)q(A)q(B)q(D) < 0$. This implies that there exists either one or three saddle points.

Using the Poincaré's index of vectors fields around critical points another simplified proof of this theorem was proposed in [13].
Actually, the second discriminant of the saddle conic may distinguish between the cases of one and three saddle points.

**Theorem 2** *Suppose that the differential system (2) admits four real isolated critical points. Then*

- *(2) has one saddle point if, and only if, $\delta_1 > 0$ and $\delta_2 < 0$,*

- *(2) has two saddle points if, and only if, $\delta_1 < 0$,*

- *(2) has three saddle points if, and only if, $\delta_1 > 0$ and $\delta_2 > 0$.*

**Proof.**   For the systems (6), (7), we have

$$\widetilde{\delta}_2 = \frac{(c+d-1)(c+d)(c-1)(d-1)(b_{11}^1 b_{22}^2 - b_{11}^2 b_{22}^1)^3}{c^2 d^2}\,.$$

Suppose that $\tilde{\delta}_1 > 0$; i.e., $(c+d-1)/(cd) < 0$ and $0$ is a saddle point: if $q(D)$ is of negative sign, then $c+d-1 > 0$ and $c$ and $d$ are of opposite sign. Thus, one of the points $A$ or $B$ is of saddle type. Without loss of generality, we can suppose that $A$ is a saddle point. Then $c < 0$, $d > 1$ and $c + d > 1 > 0$. Necessarily, $\widetilde{\delta}_2 > 0$.

Suppose that $q(D)$ is of positive sign, then $c + d - 1 < 0$ and $c$ and $d$ have the same sign. If $c$ and $d$ are of negative sign, then there are three antisaddle points and $\widetilde{\delta}_2 < 0$. If $c$ and $d$ are of positive sign, then $A$ and $B$ are of saddle type and $0 < c < 1$, $0 < d < 1$. There are three saddle points and $\widetilde{\delta}_2 > 0$. This result is partially obtained in [9, 10].

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