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A classification scheme for positive solutions of second order nonlinear iterative differential equations *

Xianling Fan, Wan-Tong Li, & Chengkui Zhong

Abstract

This article presents a classification scheme for eventually-positive solutions of second-order nonlinear iterative differential equations, in terms of their asymptotic magnitudes. Necessary and sufficient conditions for the existence of solutions are also provided.

1 Introduction

A systematic study of oscillatory properties and asymptotic behavior of solutions of functional differential equations began with the works [4, 11, 12]. However, a considerable number of papers dealing with these problems are from the last two decades. In 1987, the monograph [5] presented a systematic investigation of the oscillatory properties of solutions to ordinary differential equations with deviating arguments. Recently, Bainov, Markova and Simeonov [3] studied the equation

$$(r(t)x'(t))' + f(t, x(t), x(\Delta(t, x(t)))) = 0 \tag{1}$$

with the condition

$$\int_0^\infty \frac{ds}{r(s)} = \infty\,.$$

They provide a classification scheme for non-oscillatory solutions, and provide necessary and sufficient conditions for the existence of solutions. Such schemes are important since further investigations of qualitative behaviors of solutions can then be reduced to only a number of cases. However, a more difficult problem [9] is to characterize the case when

$$\int_0^\infty \frac{ds}{r(s)} < \infty\,.$$

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This paper concerns with the general class of second order nonlinear differential equations

$$(r(t)(x'(t))^\sigma)' + f(t, x(t), x(\Delta(t, x(t)))) = 0 \tag{2}$$

with the conditions $\int_0^\infty ds/r(s)^{1/\sigma} = \infty$ and $\int_0^\infty ds/r(s)^{1/\sigma} < \infty$. We give a classification scheme for eventually-positive solutions of this equation in terms of their asymptotic magnitude, and provide necessary and/or sufficient conditions for the existence of solutions. Our results extend and improve the results in [3, 5].

When $f(t, x(t), x(\Delta(t, x(t)))) = f(t, x(t))$, the oscillation and asymptotic behavior of the solutions of (2) have been studied by Li [6]-[10], Ruan [13] and Wong and Agarwal [14].

It is known [3] that the differential equation of the from (1) with delay depending on the unknown function have been investigated only in the papers [1], [2].

Let $T \in \mathbb{R}_+ = [0, \infty)$. Define $T_{-1} = \inf\{\Delta(t, x) : t \geq T, x \in R\}$.

Definition 1. The function $x(t)$ is called a solution of the differential equation (2) in the interval $[T, +\infty)$, if $x(t)$ is defined for $t \geq T_{-1}$, it is twice differentiable and satisfies (2) for $t \geq T$.

Definition 2. The solution $x(t)$ of (2) is called regular, if it is defined on some interval $[T_x, \infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $t \geq T_x$.

Definition 3. The solution $x(t)$ of (2) is said to be:
(i) eventually positive: if there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$;
(ii) eventually negative: if there exists $T \geq 0$ such that $x(t) < 0$ for all $t \geq T$;
(iii) non-oscillatory: if it is either eventually positive or eventually negative;
(iv) oscillatory: if it is neither eventually positive nor eventually negative.

Throughout this paper, we assume that the following conditions hold:

H1) $r \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $r(t) > 0, t \in \mathbb{R}_+$.

H2) $f \in C(\mathbb{R}_+ \times R^2, \mathbb{R})$.

H3) There exists $T \in \mathbb{R}_+$ such that $uf(t, u, v) > 0$ for $t \geq T$, $uv > 0$ and $f(t, u, v)$ is non-decreasing in u and v for each fixed $t \geq T$.

H4) $\Delta \in C(\mathbb{R}_+ \times R, \mathbb{R})$.

H5) There exist a function $\Delta_*(t) \in C(\mathbb{R}_+, \mathbb{R})$ and $T \in \mathbb{R}_+$ such that $\lim_{t\to\infty} \Delta_*(t) = +\infty$ and $\Delta_*(t) \leq \Delta(t, x)$ for $t \geq T$, $x \in \mathbb{R}$.

H6) There exist a function $\Delta^*(t) \in C(\mathbb{R}_+, \mathbb{R})$ and $T \in \mathbb{R}_+$ such that $\Delta^*(t)$ is a nondecreasing function for $t \geq T$ and $\Delta(t, x) \leq \Delta^*(t) \leq t$ for $t \geq T, x \in \mathbb{R}$.

H7) σ is a quotient of odd integers.

For the sake of convenience, we will employ the following notation

$$R(t)=\int_t^\infty \frac{ds}{r(s)^{1/\sigma}},\quad R(t,T)=\int_T^t \frac{ds}{r(s)^{1/\sigma}},\quad R_0=\int_0^\infty \frac{ds}{r(s)^{1/\sigma}}\,.$$

In the following section, we give several preparatory lemmas which will be used for later results. In Section 3, we will discuss the case $R_0<\infty$. The case $R_0=\infty$ will be studied in Section 4.

2 Preparatory Lemmas

Lemma 1 *Suppose $x(t)$ is an eventually-positive solution of (2). Then $x'(t)$ is of constant sign eventually.*

Proof. Assume that there exists $t_0\geq 0$ such that $x(t)>0$, for $t\geq t_0$. It follows from (H6) that there exists $t_1\geq t_0$ such that $x(\Delta(t,x(t)))>0$ for $t\geq t_1$. From (H4) and (2) we conclude that $(r(t)(x'(t))^\sigma)'<0$ for $t\geq t_1$. If $x'(t)$ is not eventually positive, then there exists $t_2\geq t_1$ such that $x'(t_2)\leq 0$. Therefore, $r(t_2)(x'(t_2))^\sigma\leq 0$. From (2), we have

$$r(t)(x'(t))^\sigma - r(t_2)(x'(t_2))^\sigma + \int_{t_2}^t f(s,x(s),x(\Delta(s,x(s))))ds = 0.$$

Thus

$$r(t)(x'(t))^\sigma \leq -\int_{t_2}^t f(s,x(s),x(\Delta(s,x(s))))ds<0,$$

for $t\geq t_2$. This shows that $x'(t)<0$ for $t\geq t_2$. The proof is complete. ♢

As a consequence, an eventually positive solution $x(t)$ of (2) either satisfies $x(t)>0$ and $x'(t)>0$ for all large t, or, $x(t)>0$ and $x'(t)<0$ for all large t.

Lemma 2 *Suppose that*

$$R_0=\int_0^\infty \frac{ds}{r(s)^{1/\sigma}}<\infty\,, \tag{3}$$

and that $x(t)$ is an eventually positive solution of (2). Then $\lim_{t\to\infty} x(t)$ exists.

Proof. If not, then we have $\lim_{t\to\infty} x(t)=\infty$ by Lemma 1. On the other hand, we have noted that $r(t)(x'(t))^\sigma$ is monotone decreasing eventually. Therefore, there exists $t_1\geq 0$ such that

$$r(t)(x'(t))^\sigma \leq r(t_1)(x'(t_1))^\sigma,\quad \text{for } t\geq t_1\,.$$

Then

$$x'(t)\leq (r(t_1))^{1/\sigma}x'(t_1)\frac{1}{r(t)^{1/\sigma}}, \tag{4}$$

for $t \geq t_1$, and after integrating,

$$x(t) - x(t_1) \leq (r(t_1))^{1/\sigma} x'(t_1) R(t_1, t),$$

for $t \geq t_1$. But this is contrary to the fact that $\lim_{t\to\infty} x(t) = \infty$ and the assumption that $R_0 < \infty$. The proof is complete. ♢

Lemma 3 *Suppose that $R_0 < \infty$. Let $x(t)$ be an eventually positive solution of (2). Then there exist $a_1 > 0, a_2 > 0$ and $T \geq 0$ such that $a_1 R(t) \leq x(t) \leq a_2$ for $t \geq T$.*

Proof. By Lemma 2, there exists $t_0 \geq 0$ such that $x(t) \leq a_2$ for some positive number a_2. We know that $x'(t)$ is of constant sign eventually by Lemma 1. If $x'(t) > 0$ eventually, then $R(t) \leq x(t)$ eventually because $\lim_{t\to\infty} R(t) = 0$. If $x'(t) < 0$ eventually, then since $r(t)(x'(t))^\sigma$ is also eventually decreasing, we may assume that $x'(t) < 0$ and $r(t)(x'(t))^\sigma$ is monotone decreasing for $t \geq T$. By (4), we have

$$x(s) - x(t) \leq (r(T))^{1/\sigma} x'(T) R(t, s), \quad s \geq t \geq T.$$

Taking the limit as $s \to \infty$ on both sides of the above inequality,

$$x(t) \geq -(r(T))^{1/\sigma} x'(T) R(t),$$

for $t \geq T$. The proof is complete. ♢

Our next result is concerned with necessary conditions for the function f to hold in order that an eventually positive solution of (2) exist.

Lemma 4 *Suppose that $R_0 < \infty$ and $x(t)$ is an eventually positive solution of (2). Then*

$$\int_0^\infty \frac{1}{r(t)^{1/\sigma}} \left(\int_0^t f(s, x(s), x(\Delta(s, x(s))))ds \right)^{1/\sigma} dt < \infty.$$

Proof. In view of Lemma 1, we may assume without loss of generality that $x(t) > 0$, and, $x'(t) > 0$ or $x'(t) < 0$ for $t \geq 0$. From (2), we have

$$r(t)(x'(t))^\sigma - r(0)(x'(0))^\sigma + \int_0^t f(s, x(s), x(\Delta(s, x(s))))ds = 0\,.$$

Thus, if $x'(t) > 0$ for $t \geq 0$, we have

$$\begin{aligned} &\int_0^u \frac{1}{r(t)^{1/\sigma}} \left(\int_0^t f(s, x(s), x(\Delta(s, .x(s))))ds \right)^{1/\sigma} dt \\ &\qquad \leq (r(0))^{1/\sigma} x'(0) \int_0^u \frac{1}{r(t)^{1/\sigma}} dt\,, \end{aligned}$$

for $u \geq 0$, and

$$\int_0^u \frac{1}{r(t)^{1/\sigma}} \left(\int_0^t f(s, x(s), x(\Delta(s, x(s))))ds \right)^{1/\sigma} dt \leq (r(0))^{1/\sigma} x'(0) R_0 < \infty \,.$$

If $x'(t) < 0$ for $t \geq 0$, we have

$$\int_0^u \frac{1}{r(t)^{1/\sigma}} \left(\int_0^t f(s, x(s), x(\Delta(s, x(s))))ds \right)^{1/\sigma} dt \leq -\int_0^\infty x'(s)ds \leq x(0) < \infty \,.$$

The proof is complete. ♢

We now consider the case where $R_0 = \infty$.

Lemma 5 *Suppose that*

$$R_0 = \int_0^\infty \frac{ds}{r(s)^{1/\sigma}} = \infty \,. \tag{5}$$

Let $x(t)$ be an eventually positive solution of (2). Then $x'(t)$ is eventually positive and there exist $c_1 > 0$, $c_2 > 0$ and $T \geq 0$ such that $c_1 \leq x(t) \leq c_2 R(t, T)$ for $t \geq T$.

Proof. In view of Lemma 1, $x'(t)$ is of constant sign eventually. If $x(t) > 0$ and $x'(t) < 0$ for $t \geq T$, then we have

$$r(t)(x'(t)^\sigma) \leq r(T)(x'(T)^\sigma) < 0 \,.$$

Thus

$$x'(t) \leq r(T)^{1/\sigma} x'(T) \frac{1}{r(t)^{1/\sigma}}, \quad t \geq T,$$

which after integrating yields

$$x(t) - x(T) \leq r(T)^{1/\sigma} x'(T) \int_T^t \frac{ds}{r(s)^{1/\sigma}}.$$

The left hand side tends to $-\infty$ in view of (5), which is a contradiction. Thus $x'(t)$ is eventually positive, and thus $x(t) \geq c_1$ eventually for some positive constant c_1. Furthermore, the same reasoning just used also leads to

$$x(t) \leq x(T_0) + r(T_0)^{1/\sigma} x'(T_0) \int_{T_0}^t \frac{ds}{r(s)^{1/\sigma}},$$

for $t \geq T_0$, where T_0 is a number such that $x(t) > 0$ and $x'(t) > 0$ for $t \geq T_0$. Since $R_0 = \infty$, thus there is $c_2 > 0$ such that $x(t) \leq c_2 R(T, t)$ for all large t. The proof is complete. ♢

3 The case $R_0 < \infty$

We have shown in the previous section that when $x(t)$ is an eventually positive solution of (2), then $(r(t)(x'(t))^\sigma)'$ is eventually decreasing and $x'(t)$ is eventually of constant sign. We have also shown that under the assumption that $R_0 < \infty$, $x(t)$ must converge to some (nonnegative) constant. As a consequence, under the condition $R_0 < \infty$, we may now classify an eventually positive solution $x(t)$ of (2) according to the limits of the sequences $x(t)$ and $r(t)(x'(t))^\sigma$. For this purpose, we first denote the set of eventually-positive solutions of (2) by P. We then single out eventually-positive solutions of (2) which converge to zero or to positive constants, and denote the corresponding subsets by P_0 and P_α respectively. But for any $x(t)$ in P_α, since $r(t)(x'(t))^\sigma$ either tends to a finite limit or to $-\infty$, we can further partition P_+ into P_α^β and $P_\alpha^{-\infty}$.

Theorem 1 *Suppose $R_0 < \infty$. Then any eventually positive solutions of (2) must belong to one of the following classes:*

$$P_0 = \left\{ x(t) \in P | \lim_{t\to\infty} x(t) = 0 \right\},$$

$$P_\alpha^\beta = \left\{ x(t) \in P | \lim_{t\to\infty} x(t) = \alpha > 0, \quad \lim_{t\to\infty} r(t)(x'(t)^\sigma) = \beta \right\},$$

$$P_\alpha^{-\infty} = \left\{ x(t) \in P | \lim_{t\to\infty} x(t) = \alpha > 0, \quad \lim_{t\to\infty} r(t)(x'(t)^\sigma) = -\infty \right\}.$$

To justify the above classification scheme, we will derive several existence theorems.

Theorem 2 *Suppose $R_0 < \infty$. Then a necessary and sufficient condition for (2) to have an eventually positive solution $x(t)$ which belong to P_α is that for some $C > 0$,*

$$\int_0^\infty \left(\frac{1}{r(t)} \int_0^t f(s, C, C) ds \right)^{1/\sigma} dt < \infty \,. \tag{6}$$

Proof. Let $x(t)$ be any eventually positive solution of (2) such that $\lim_{t\to\infty} x(t) = c > 0$. Thus, in view of (H6), there exist $C_1 > 0$, $C_2 > 0$ and $T \geq 0$ such that $C_1 \leq x(t) \leq C_2$, $C_1 \leq x(\Delta(t, x(t))) \leq C_2$ for $t \geq T$. On the other hand, using Lemma 4 we have

$$\int_T^\infty \left(\frac{1}{r(t)} \int_0^t f(s, x(s), x(\Delta(s, x(s)))) ds \right)^{1/\sigma} dt < \infty.$$

Since $f(t, u, v)$ is nondecreasing in u and v for each fixed t, thus we have

$$\int_T^\infty \left(\frac{1}{r(t)} \int_0^t f(s, C_1, C_1) ds \right)^{1/\sigma} dt < \infty.$$

Conversely, let $a = C/2$. In view of (6), we may choose a $T \geq 0$ so large that

$$\int_T^\infty \left(\frac{1}{r(t)} \int_0^t f(s, C, C) ds \right)^{1/\sigma} dt < a. \tag{7}$$

Define the set

$$\Omega = \left\{ \begin{array}{l} x \in C([T_{-1}, +\infty), \mathbb{R}) : x(t) = a, \text{ for } T_{-1} \le t < T, \\ \text{and } a \le x(t) \le 2a, \text{ for } t \ge T \end{array} \right\}.$$

Then Ω is a bounded, convex and closed subset of $C([T_{-1}, +\infty), \mathbb{R})$. Let us further define an operator $F : \Omega \to C([T_{-1}, +\infty), \mathbb{R})$ by

$$Fx(t) = \begin{cases} a + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, x(u), x(\Delta(u, x(u))))du \right)^{1/\sigma} ds, & t \ge T, \\ Fx(T), & T_{-1} \le t \ge T. \end{cases} \tag{8}$$

The mapping F have the following properties. F maps Ω into Ω. Indeed, if $x(t) \in \Omega$, then

$$\begin{aligned} a \le Fx(t) = a + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, x(u), x(\Delta(u, x(u))))du \right)^{1/\sigma} ds \\ \le a + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds \le 2a. \end{aligned}$$

Next, we show that F is continuous. To see this, let $\epsilon > 0$. Choose $M \ge T$ so large that

$$\int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds < \frac{\epsilon}{2}\,. \tag{9}$$

Let $\{x^{(n)}\}$ be a sequence in Ω such that $x^{(n)} \to x$. Since Ω is closed, $x \in \Omega$. Furthermore, for any $s \ge t \ge M$,

$$\begin{aligned} &\left| Fx^{(n)}(t) - Fx(t) \right| \\ &\le \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds \\ &\le 2 \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds < \epsilon\,. \end{aligned}$$

For $T \le t \le s \le M$,

$$\begin{aligned} &\left| Fx^{(n)}(t) - Fx(t) \right| \\ &\le \int_M^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds + \int_M^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds \\ &\quad + \int_t^M \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds - \int_s^M \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds \\ &\le \epsilon + \int_t^s \left(\frac{1}{r(s)} \int_0^s f(u, C, C)du \right)^{1/\sigma} ds \\ &\le \epsilon + \max_{T \le u \le M} \frac{1}{r(u)} \int_0^u f(v, C, C)dv \, |s - t| \\ &\le \epsilon + C_0 \, |s - t| < 2\epsilon, \quad \text{if} \, |s - t| < \frac{\epsilon}{C_0}\,, \end{aligned}$$

where $C_0 = \max_{T\le u\le M}\int_0^u f(v,C,C)\,dv/r(u)$. And for $T_{-1}\le t\le s<T$,

$$|Fx^{(n)}(t) - Fx(t)| = 0\,.$$

These statements show that $\|Fx^{(v)} - Fx\|$ tends to zero, i.e., F is continuous.

When $s, t \ge M$, by (9) we have

$$\begin{aligned}|Fx(s) - Fx(t)| &\le \int_t^\infty \left(\frac{1}{r(s)}\int_0^s f(u,C,C)du\right)^{1/\sigma} ds\\ &\quad + \int_t^\infty \left(\frac{1}{r(s)}\int_0^s f(u,C,C)du\right)^{1/\sigma} ds < \epsilon\,,\end{aligned}$$

which holds for any $x\in\Omega$. Therefore, $F\Omega$ is precompact. In view of Schauder's fixed point theorem, we see that there is an $x^*\in\Omega$ such that F$x^* = x^*$. It is easy to check that x^* is an eventually positive solution of (2). The proof is complete. $\diamondsuit$

Theorem 3 *Suppose $R_0<\infty$. A necessary and sufficient condition for (2) to have an eventually-positive solution $x(t)$ which belongs to P_α^β is that (6) holds for some $C>0$ and that for some $D>0$,*

$$\int_0^\infty f(t,D,D)dt < \infty\,. \tag{10}$$

Proof. If $x(t)$ is an eventually-positive solution in P_α^β, then, in view of Theorem 2, we see that (6) holds. Furthermore, as in the proof of Theorem 2, $0<C_1\le x(t)\le C_2, C_1\le x(\Delta(t,x(t)))\le C_2$ for $t\ge T$. In view of (2), we see that

$$\begin{aligned}\int_T^\infty f(s,C_1,C_1)ds &\le \int_T^\infty f(s,x(s),x(\Delta(s,x(s))))ds\\ &= r(T)(x'(T))^\sigma - \lim_{t\to\infty} r_m(t)(x'(t))^\sigma < \infty\,.\end{aligned}$$

Conversely, in view of (10), we can choose a $T\ge 0$ such that

$$\int_T^\infty f(t,D,D)dt < \left(\frac{D}{2R_0}\right)^\sigma .$$

We define the subset Ω of $C([T_{-1},+\infty),\mathbb{R})$ as follows

$$\Omega = \left\{\begin{array}{l} x\in C([T_{-1},+\infty),\mathbb{R}) : x(t) = D/2 \text{ for } T_{-1}\le t<T,\\ \text{ and } D/2\le x(t)\le D, \text{ for } t\ge T\end{array}\right\}.$$

Then Ω is a bounded, convex and closed subset of $C([T_{-1},+\infty),\mathbb{R})$. In view of R_0 and (10), we can further define an operator $F:\Omega\to C([T_{-1},+\infty),\mathbb{R})$ as

$$Fx(t) = \begin{cases} D - \int_t^\infty \left(\frac{1}{r(s)}\int_s^\infty f(u,x(u),x(\Delta(u,x(u))))du\right)^{1/\sigma} ds & t\ge T,\\ Fx(T) & T_{-1}\le t\ge T\,.\end{cases}$$

Then, arguments similar to those in the proof of Theorem 2 show that F has a fixed point u which satisfies

$$r(t)(u'(t))^\sigma = \int_t^\infty f(s,u(s),u(\Delta(s,u(s))))ds, \quad t \ge T\,.$$

Hence $\lim_{t\to\infty} r(t)(u'(t))^\sigma = 0$ as required. Choose a $T \ge 0$ such that

$$\int_T^\infty f(t,D,D)dt < \left(\frac{D}{4R_0}\right)^\sigma \quad \text{and} \quad R(t) < \left(\frac{D}{4R_0}\right)^\sigma$$

for $t \ge T$, and let

$$Fx(t) = \begin{cases} D - \int_t^\infty \left(\frac{1}{r(s)} + \frac{1}{r(s)}\int_0^s f(u,x(u),x(\Delta(u,x(u))))du\right)^{1/\sigma} ds, & t \ge T,\\ Fx(T), & T_{-1} \le t < T\,. \end{cases}$$

Then under the same conditions (6) and (10), we can shows that F has a fixed point u which satisfies $\lim_{t\to\infty} u(t) = D > 0$ and

$$r(t)(u'(t))^\sigma = 1 + \int_t^\infty f(s,u(s),u(\Delta(s,u(s))))ds, \quad t \ge T\,.$$

Therefore, $\lim_{t\to\infty} r(t)(u'(t))^\sigma = 1 > 0$, and the present proof is complete. ♢

In view of Theorem 3, the following result is obvious.

Theorem 4 *Suppose $R_0 < \infty$. A necessary and sufficient condition for (2) to have an eventually-positive solution $x(t)$ which belongs to $P_\alpha^{-\infty}$ is that (6) holds for some $C > 0$ and that for any $D > 0$,*

$$\int_0^\infty f(t,D,D)dt = \infty \tag{11}$$

Our final result concerns with the existence of eventually-positive solutions in P_0 .

Theorem 5 *Suppose $R_0 < \infty$ and $\sigma = 1$. If for some $C > 0$,*

$$\int_0^\infty f(t,CR(t),CR(\Delta_*(t)))dt < \infty, \tag{12}$$

then (2) has an eventually-positive solution in P_0. Conversely, if (2) has an eventually-positive solution $x(t)$ such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} r(t)(x'(t))^\sigma = d \ne 0$, then for some $C > 0$,

$$\int_0^\infty f(t,CR(t),CR(\Delta_*(t)))dt < \infty\,.$$

Proof. Suppose (12) holds. Then there exists a $T \geq 0$ such that

$$\int_t^\infty f(s, CR(s), CR(\Delta_*(s)))ds < \frac{C}{2} \quad \text{for } t \geq T\,.$$

Consider the equation

$$x(t) = \begin{cases} R(t)\left(\frac{C}{2} + \int_T^t f(s, x(s), x(\Delta(s, x(s))))ds\right) & \\ + \int_t^\infty R(s) f(s, x(s), x(\Delta(s, x(s))))ds & t \geq T, \\ Fx(T) & T_{-1} \leq t < T\,. \end{cases} \tag{13}$$

It is easy to check that a solution of (13) must be a solution of (2). We shall show that (13) has a positive solution $x(t)$ which belongs to P_0 by means of the method of successive approximations. Consider the sequence $\{x_k(t)\}$ of successive approximating sequences defined as follows.

$$\begin{gathered} x_1(t) = 0 \quad \text{for } t \geq T_{-1}, \\ x_{k+1}(t) = Fx_k(t), \quad \text{for } t \geq T_{-1},\ k = 1, 2, \dots, \end{gathered}$$

where F is defined by

$$Fx(t) = \begin{cases} R(t)\left(\frac{C}{2} + \int_T^t f(s, x(s), x(\Delta(s, x(s))))ds\right) & \\ + \int_t^\infty R(s) f(s, x(s), x(\Delta(s, x(s))))ds & t \geq T, \\ Fx(T) & T_{-1} \leq t < T\,. \end{cases}$$

In view of (H3), it is easy to see that $0 \leq x_k(t) \leq x_{k+1}(t)$ for $t \geq T$ and $k = 1, 2, \dots$. On the other hand,

$$x_2(t) = Fx_1(t) = \frac{C}{2} R(t) \leq CR(t), \quad t \geq T\,,$$

and inductively,

$$\begin{aligned} Fx_k(t) &\leq \frac{C}{2}R(t) + R(t)\int_T^t f(s, CR(s), CR(\Delta^*(s)))ds \\ &\quad + R(t)\int_t^\infty f(s, CR(s), CR(\Delta^*(s)))ds \\ &\leq \frac{C}{2}R(t) + R(t)\int_T^\infty f(s, CR(s), CR(\Delta^*(s)))ds \\ &\leq CR(t)\,, \end{aligned}$$

for $k \geq 2$. Therefore, by means of Lebesgue's dominated convergence theorem, we see that $Tx^* = x^*$. Furthermore, it is clear that $x(t)$ converges to zero as $t \to \infty$.

Let $x(t)$ be an eventually positive solution of (2) such that $x(t) \to 0$ and $r(t)(x'(t))^\sigma \to d < 0$ (the proof of the case $d > 0$ being similar). Then there exist $C_1 > 0$, $C_2 > 0$ and $T \geq 0$ such that $-C_1 < r(t)(x'(t))^\sigma < -C_2$, for $t \geq T$. Hence,

$$-C_1^{1/\sigma}\frac{1}{r(t)^{1/\sigma}} < x'(t) < -C_2^{1/\sigma}\frac{1}{r(t)^{1/\sigma}},$$

and, after integrating,

$$-C_1^{1/\sigma}R(s,t) < x(s) - x(t) < -C_2^{1/\sigma}R(s,t),$$

for $s > t \geq T$. Let $s \to \infty$, then $-C_1^{1/\sigma}R(t) < -x(t) < -C_2^{1/\sigma}R(t)$. That is, $C_2^{1/\sigma}R(t) < x(t) < C_1^{1/\sigma}R(t)$. On the other hand, by (2),

$$r(t)(x'(t))^\sigma = r(T)(x'(T))^\sigma + \int_T^t f(s, x(s), x(\Delta(s, x(s))))ds, \quad t \geq T\,.$$

Since $\lim_{t\to\infty} r(t)(x'(t))^\sigma = d < 0$, we have

$$\int_T^\infty f(s, x(s), x(\Delta(s, x(s))))ds = r(T)(x'(T))^\sigma - d < \infty\,.$$

Thus,

$$\int_T^\infty f(s, C_1^{1/\sigma}R(s), C_1^{1/\sigma}R(\Delta_*(s)))ds \leq \int_T^\infty f(s, x(s), x(\Delta(s, x(s))))ds < \infty\,.$$

The proof is complete. ♢

4 The case $R_0 = \infty$

In this section, we assume that $R_0 = \infty$. Let P denotes the set of all eventually-positive solutions of (2). Recall that if $x(t)$ belongs to P, then $r(t)(x'(t))^\sigma$ is eventually decreasing. Furthermore, in view of Lemma 5, we see that $x'(t)$, and hence $r(t)(x'(t))^\sigma$, are eventually positive. Hence $x(t)$ either tends to a positive constant or to positive infinity, and $r(t)(x'(t))^\sigma$ tends to a nonnegative constant. Note that if $x(t)$ tends to a positive constant, then $r(t)(x'(t))^\sigma$ must tend to zero. Otherwise $r(t)(x'(t))^\sigma \geq d > 0$ for t larger than or equal to T, so that

$$x'(t) \geq d^{1/\sigma}\frac{1}{r^{1/\sigma}(t)},$$

and

$$x(t) \geq x(T)d^{1/\sigma}\int_T^t \frac{1}{r^{1/\sigma}(s)}ds \to \infty\,,\ \text{as } t \to \infty\,,$$

which is a contradiction.

Theorem 6 *Suppose that $R_0 = \infty$. Then any eventually-positive solution $x(t)$ of (2) must belong to one of the following classes:*

$$P^0_\alpha = \left\{x(t) \in P | \lim_{t\to\infty} x(t) \in (0,\infty), \quad \lim_{t\to\infty} r(t)(x'(t))^\sigma = 0\right\},$$
$$P^0_\infty = \left\{x(t) \in P | \lim_{t\to\infty} x(t) = +\infty, \quad \lim_{t\to\infty} r(t)(x'(t))^\sigma = 0\right\},$$
$$P^\beta_\infty = \left\{x(t) \in P | \lim_{t\to\infty} x(t) = +\infty, \quad \lim_{t\to\infty} r(t)(x'(t))^\sigma = \beta \neq 0\right\}.$$

In order to justify our classification scheme, we present the following two results.

Theorem 7 *Suppose that $R_0 = \infty$. A necessary and sufficient condition for (2) to have an eventually-positive solution $x(t)$ which belongs to P^0_α is that for some $C > 0$,*

$$\int_0^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s,C,C)ds\right)^{1/\sigma} dt < \infty\,. \tag{14}$$

Proof. Let $x(t)$ be an eventually-positive solution of (2) which belong to P^0_α, i.e., $\lim_{t\to\infty} x(t) = \alpha > 0$ and $\lim_{t\to\infty} r(t)(x'(t))^\sigma = 0$. Then there exist two positive constants C_1, C_2 and $T \geq 0$ such that $C_1 \leq x(t) \leq C_2$, $C_1 \leq x(\Delta(t,x(t)) \leq C_2$ for $t \geq T$. On the other hand, in view of (2) we have

$$r(t)(x'(t))^\sigma = \int_t^\infty f(s,x(s),x(\Delta(s,x(s))))ds\,,$$

for $t \geq T$. After integrating, we see that

$$\begin{aligned}
&\int_T^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s,C,C)ds\right)^{1/\sigma} dt \\
&\leq \int_0^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s,x(s),x(\Delta(s,x(s))))ds\right)^{1/\sigma} dt \\
&\leq \alpha - x(T).
\end{aligned}$$

The proof of the converse is similar to that of Theorem 1 and hence is sketched. In view of (14), we may choose a $T \geq 0$ so large that

$$\int_T^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s,C,C)ds\right)^{1/\sigma} < \frac{C}{2}\,. \tag{15}$$

Define a bounded, convex, and closed subset Ω of $C([T_{-1},\infty),\mathbb{R})$ and an operator $F : \Omega \to \Omega$ as

$$\Omega = \left\{ \begin{array}{l} x \in C([T_{-1},+\infty),\mathbb{R}) : x(t) = \frac{C}{2} \text{ for } T_{-1} \leq t < T, \\ \text{ and } \frac{C}{2} \leq x(t) \leq C, \text{ for } t \geq T\,, \end{array} \right\}$$

and

$$Fx(t) = \begin{cases} \frac{C}{2} + \int_t^\infty \left(\frac{1}{r(s)} \int_s^\infty f(u, x(u), x(\Delta(u, x(u))))du \right)^{1/\sigma} ds & t \geq T, \\ Fx(T) & T_{-1} \leq t < T\,, \end{cases}$$

respectively. As in the proof of Theorem 3, we prove that F maps Ω into Ω, that F is continuous, and that FΩ is precompact. The fixed point $x^*(t)$ of F will converge to $C/2$ and satisfies (2). The proof is complete. ♢

We remark that Theorem 7 extends Theorem 6 of Bainov, Markova and Simeonov [3]. The proof of the following result is again similar to that of Theorem 3 and hence is omitted.

Theorem 8 *Suppose $R_0 = \infty$. If for a positive constant C,*

$$\int_0^\infty f(t, CR(t,0), CR(\Delta^*(t), 0))dt < \infty \tag{16}$$

then (2) has a solution in P_∞^β. Conversely, if (2) has a solution $x(t)$ in P_∞^β, then for some positive constant C,

$$\int_0^\infty f(t, CR(t,0), CR(\Delta_*(t), 0))dt < \infty\,.$$

We remark that our Theorem 8 extends Theorem 5 of Bainov, Markova and Simeonov [3]. In view of Theorems 7 and 8, the following result is clear.

Theorem 9 *Suppose $R_0 = \infty$. If for any positive constant C and for some positive constant D such that*

$$\begin{gathered} \int_0^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s, C, C)ds \right)^{1/\sigma} dt = \infty\,, \\ \int_0^\infty f(t, DR(t,0), DR(\Delta^*(t), 0))dt < \infty\,, \end{gathered}$$

then (2) has a solution in P_∞^0.

We remark that our Theorem 9 extends Theorem 7 in [3], and that several oscillation statements for (2) can be proven. Since the method is similar to that of [3], we omit them here.

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Xianling Fan, Wan-Tong Li, & Chengkui Zhong
Department of Mathematics, Lanzhou University
Lanzhou, Gansu, 730000, People's Republic of China
e-mail: liwt@gsut.edu.cn