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Basis properties of eigenfunctions of nonlinear Sturm-Liouville problems *

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Abstract

We consider three nonlinear eigenvalue problems that consist of

$$-y'' + f(y^2)y = \lambda y$$

with one of the following boundary conditions:

$$y(0) = y(1) = 0 \quad y'(0) = p\,,$$
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where p is a positive constant. Under smoothness and monotonicity conditions on f, we show the existence and uniqueness of a sequence of eigenvalues $\{\lambda_n\}$ and corresponding eigenfunctions $\{y_n\}$ such that $y_n(x)$ has precisely n roots in the interval $(0,1)$, where $n = 0, 1, 2, \ldots$. For the first boundary condition, we show that $\{y_n\}$ is a basis and that $\{y_n/\|y_n\|\}$ is a Riesz basis in the space $L_2(0,1)$. For the second and third boundary conditions, we show that $\{y_n\}$ is a Riesz basis.

1 Introduction

We consider three eigenvalue problems which consist of the nonlinear equation

$$-y'' + f(y^2)y = \lambda y \quad \text{on } (0,1) \tag{1}$$

with one of the following three boundary conditions:

$$y(0) = y(1) = 0 \quad y'(0) = p\,, \tag{2a}$$
$$y'(0) = y(1) = 0 \quad y(0) = p\,, \tag{2b}$$
$$y'(0) = y'(1) = 0 \quad y(0) = p\,. \tag{2c}$$

Hereafter all quantities are real, including λ which is a spectral parameter and p which is an arbitrary constant. However, we consider only $p > 0$, because when

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$p < 0$ the substitution of $y(x)$ by $-y(x)$ leads to a positive p. We assume that f is a continuously differentiable function on $\mathbb{R}$; so that the standard theorems, such as local existence, uniqueness, and continuous dependence on parameters are valid for (1).

A pair (λ, y) with λ real and $y = y(x)$ twice continuously differentiable for all $x \in [0, 1]$, that satisfies (1) and one of the boundary conditions (2) is called an *eigenvalue* and corresponding *eigenfunction.* Since each of the boundary conditions (2) contains Cauchy data and an additional condition, by the standard uniqueness theorem to each value λ there corresponds at most one eigenfunction.

By the *spectrum* we mean the set of all eigenvalues for each problem defined by (1)-(2). For any $g \in L_2 = L_2(0, 1)$, by the *normalized function* g we mean the normalization of this function to 1 in the above space.

In the present paper, we investigate the properties of the eigenvalues and eigenfunctions of each problem defined by (1)-(2). In particular, we prove that for problems (1)-(2b) and (1)-(2c) the eigenvectors form Riesz bases, and that the normalized eigenvectors of problem (1)-(2a) also form a Riesz basis. For problem (1)-(2b), these questions have already been considered in [8]. The proof that eigenfunctions form a Riesz basis was done using Bary's theorem. In the present paper the author establishes a direct proof (not based on Bary's theorem) of the result in [8], and then studies problems (1)-(2a) and (1)-(2c).

According to Bary's theorem [1, 2, 3, 7], a linearly independent system of functions $\{h_n\}$ which is $L_2(a, b)$ quadratically close to a Riesz basis in this space is a Riesz basis in $L_2(a, b)$. This result in [1, 2] is presented in a weaker form. However the proof in [2] applies without modifications to the above statement.

Note that an orthonormal basis in $L_2(a, b)$ is also a Riesz basis in L_2 (see definitions in Section 2). A Riesz basis that is quadratically close to an orthonormal basis is called Bary basis [3]. Important properties of Riesz and Bary bases are presented in [1, 2, 3].

In the present paper, we shall prove linear independence and quadratic closeness of the normalized eigenfunctions of all problems (1)-(2) to orthonormal bases in the space L_2. We also obtain estimates of the form $\overline{c} < \|y_n\|_{L_2} < \overline{C}$ for the eigenfunctions of problems (1)-(2b) and (1)-(2c) with constants $0 < \overline{c} < \overline{C}$ independent of n. Thus, eigenfunctions of problems (1)-(2b) and (1)-(2c) and normalized eigenfunctions of the problem (1)-(2a) are Riesz bases. Moreover, the latter system is a Bary basis in L_2. We present a direct simple and short proof, not based on Bary's theorem, of these systems being Riesz bases in L_2.

Several papers have presented proofs of completeness, of being a basis, and of other properties for eigenfunctions of nonlinear boundary-value problems; see for example [7]. In [9], the main result consists of proving that the eigenfunctions of a nonlinear Sturm-Liouville-type problem form a basis in L_2. These results are also announced (without proofs) in [10]. However, [9] contains errors which have been corrected in [11], where also an analog of the Fourier transform over eigenfunctions of nonlinear Sturm-Liouville-type problems on a half-line is considered. An independent (and shorter) proof based on the Bary's theorem of the result from [9] is done in [12]. Lately, in [13] it is shown that the eigenfunctions form a basis in H^s, where s is a negative constant. This is done for a boundary-value similar to (1), without the spectral parameter, and with zero Dirichlet boundary conditions.

Concerning the applications of our results, the author hopes that once developed our results be used in Galerkin and Fourier methods. In addition, our results could be applied in those numerous areas of quantum physics where various modifications of the nonlinear Schrödinger equation arise.

2 Main results

First we introduce some notation. Let $L_2 = L_2(0,1)$ (or $L_2(a,b)$ where $a < b$) be the standard Lebesgue space consisting of real-valued functions that are square integrable over the interval $(0,1)$ (resp. over (a,b)). In this space let the scalar product be $\langle u, v\rangle = \int_0^1 u(x)v(x)\,dx$ and let the corresponding norm be defined by $\|u\|^2 = \langle u, u\rangle$. For an arbitrary Banach space B, with a norm $\|\cdot\|_B$, let $\mathcal{L}(B;B)$ denote the Banach space of linear bounded operators defined from B to B. In this space, the standard norm is

$$\|A\| = \sup\{\|Au\| : u \in B,\ \|u\|_B = 1\}\,.$$

Definition. A family of functions $\{h_n\}$ in $L_2(a,b)$ is called a basis of the space $L_2(a,b)$ if for an arbitrary function $h \in L_2(a,b)$ there exists a unique sequence of real numbers $\{a_n\}$ such that $\sum_{n=0}^{\infty} a_n h_n = h$.

Definition. (see [1, 2]) A basis $\{h_n\}$ in $L_2(a,b)$ is called the Riesz basis of this space if the series $h = \sum_{n=0}^{\infty} a_n h_n$ converges in $L_2(a,b)$ when and only when $\sum_{n=0}^{\infty} a_n^2 < \infty$. Here $\{a_n\}$ is a sequence of real numbers.

Definition. A system of functions $\{h_n\}$ in $L_2(a,b)$ is called linearly independent in the space $L_2(a,b)$ if $\sum_{n=0}^{\infty} a_n h_n = 0$ in $L_2(a,b)$ only when $0 = a_0 = a_1 = a_2 \ldots$.

Definition. Two systems of functions $\{e_n\}$ and $\{h_n\}$ in L_2 are called quadratically close in L_2 if $\sum_{n=0}^{\infty} \|e_n - h_n\|^2 < \infty$.

In what follows, we assume f satisfies the condition

(F) The function $f(y^2)y$ is continuously differentiable for all $y \in \mathbb{R}$, and f is monotonically nondecreasing and continuous on $[0, +\infty)$.

As an example of a function that satisfies this assumption, we have $f(r) = |r|^q$ for positive q.

The first result of the present paper is technical, and is stated as follows.

Theorem 1 *Assume that* (F) *holds. Then*

(a) For each of the problems in (1)-(2) there exist sequences of eigenvalues $\{\lambda_n\}$ and corresponding eigenfunctions $\{y_n\}$ such that $y_n(x)$ has precisely n roots in $(0,1)$.

(b) For each integer $n \geq 0$ the eigenfunction with n roots in $(0,1)$ is unique.

(c) Every eigenfunction as a solution of (1) can be uniquely continued on the whole line. In what follows, by eigenfunctions of problems (1)-(2) we mean eigenfunctions continued on the whole real line $\mathbb{R}$.

(d) For each problem in (1)-(2), the system of eigenfunctions $\{y_n\}$ from the above statements (a) and (c) is uniformly bounded in $\mathbb{R}$.

(e) For each integer $n \geq 0$, the roots of y_n are the points $k/(n+1)$ for problem (1)-(2a), and the points $(2k+1)/(2n+1)$ for problem (1)-(2b). For problem (1)-(2c), the function y_n with $n > 0$ has the points $1/(2n) + k/n$ as roots; while y_0 is identically equal to p, thus it has no roots. In each of these three cases $k = 0, \pm 1, \pm 2, \ldots$.

(f) For any eigenfunction y of problems (1)-(2) and for all $x \in \mathbb{R}$, $y(c+x) = -y(c-x)$ if $y(c) = 0$, and $y(d-x) = y(d+x)$ if $y'(d) = 0$.

(g) The eigenvalues of problems (1)-(2) satisfy $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$.

Parts of Theorem 1 are known (see, for example, [4]) but the author does not know a reference for all of this theorem. The author does not know of any function g that is an eigenfunction of a problem (1)-(2) and is representable as a superposition of elementary functions.

The main result of the paper is stated as follows.

Theorem 2 *Under Assumption* (F) *the eigenfunctions of problems (1)-(2b) and (1)-(2c) form Riesz bases in L_2. Also under Assumption* (F)*, the eigenfunctions of problem (1)-(2a) form a basis in L_2 and they form a Riesz basis after being normalized.*

Remark 1. It follows from (17) in the proof of Theorem 2 that for problem (1)-(2a), $\|y_n\| \to 0$ as $n \to \infty$. Therefore, this sequence of eigenfunctions is not a Riesz basis in L_2.

3 Proofs

First we prove the statement of Theorem 1 for problem (1)-(2a). Consider the Cauchy problem

$$-y'' + f(y^2)y = \lambda y \tag{3}$$

$$y(0) = 0, \quad y'(0) = p\,. \tag{4}$$

Since $p > 0$, for any $\lambda \in \mathbb{R}$ the solution y satisfies $y'(x) > 0$ in a neighborhood of zero. It is well known that equation (3) can be solved by quadratures and our proof of Theorem 1 is based on this fact (see below). Clearly, if a solution $y(x)$

of the problem (3)-(4) can be continued onto a segment $[0,d]$, and if $y'(x)>0$ for $x\in[0,d)$, then the inverse function $x(y)$ with domain $[0,y(d)]$ has the form

$$x(y)=\int_0^y \frac{dr}{\sqrt{U(\lambda,r^2)}}\,, \tag{5}$$

where $U(\lambda,r^2)=p^2+F(r^2)-\lambda r^2$, $F(r)=\int_0^r f(t)\,dt$, $y\in[0,y(d)]$, and $x(y(d))=d$. Note that if $y'(d)=0$, then $x'(y(d)-0)=+\infty$. If there exists $\overline{\lambda}$ such that for $\lambda>\overline{\lambda}$ the function $U(\lambda,r^2)$ is positive in a half-open interval $r\in[0,b)$, where $b>0$, then the solution y is defined on the closed interval $[0,x(b)]$. In this case $y(x(b))=b$, $y'(x)>0$ for $x\in[0,x(b))$ and the function $x(y)$ with the domain $[0,b]$ is inverse of $y(x)$ on $[0,x(b)]$. In addition, $y'_x(x(b))=0$ if $x(b)<+\infty$ and $x'(b-0)=+\infty$.

We denote by Λ the set of values λ for which there exists $r>0$ such that $U(\lambda,r^2)=0$. In view of condition (F), $\inf\Lambda\geq f(0)$. It is clear that if $\lambda\in\Lambda$ and $\tilde{\lambda}>\lambda$ then $\tilde{\lambda}$ also belongs to Λ. Therefore, either $\Lambda=(\overline{\lambda},+\infty)$ (the case A) or $\Lambda=[\overline{\lambda},+\infty)$ (the case B) for some $\overline{\lambda}\geq f(0)$.

For each $\lambda\in\Lambda$ we denote by $z(\lambda)$ the greatest lower bound of the set of values $r>0$ for which $U(\lambda,r^2)=0$. Clearly, $z(\lambda)>0$ and $U(\lambda,z^2(\lambda))=0$. Furthermore, $z(\lambda)$ is a monotonically decreasing function. Therefore, if $\lambda>\overline{\lambda}$, then

$$[U(\lambda,r^2)]'_r\big|_{r=z(\lambda)}=2z(\lambda)f(z^2(\lambda))-2\lambda z(\lambda)<2z(\lambda)(f(z^2(\lambda'))-\lambda')\leq 0$$

for any $\lambda'\in(\overline{\lambda},\lambda)$ (because $0\geq U'_r(\lambda',r^2)\big|_{r=z(\lambda')}=2z(\lambda')(f(z^2(\lambda'))-\lambda')$, where $z(\lambda')>0$); hence by the implicit function theorem $z(\lambda)$ is a continuously differentiable function and $z'(\lambda)<0$ for $\lambda>\overline{\lambda}$. Also, in the case B $\lim_{\lambda\to\overline{\lambda}+0}z(\lambda)=z(\overline{\lambda})$.

Let us prove now that for any $d>0$ there exists $\lambda>\overline{\lambda}$ such that

$$J(\lambda)=\int_0^{z(\lambda)} \frac{dr}{\sqrt{U(\lambda,r^2)}}=d\,, \tag{6}$$

where the integral in the right-hand side is understood as improper in a neighborhood of the point $r=z(\lambda)$. Note that as indicated above, $[U(\lambda,r^2)]'_r\big|_{r=z(\lambda)}<0$ for $\lambda>\overline{\lambda}$; thus for $\lambda>\overline{\lambda}$ the improper integral in the right-hand side of (6) converges, and $J(\lambda)$ is a continuous function.

Let us consider the case A: when $\Lambda=(\overline{\lambda},+\infty)$. It is clear that $z(\overline{\lambda}+0)=+\infty$, because otherwise in view of the continuity of the function $U(\lambda,r^2)$ of the argument (λ,r) we should get that $\overline{\lambda}\in\Lambda$, and that $z(+\infty)=0$. This easily implies that $\overline{\lambda}=\lim_{r\to\infty}f(r^2)<+\infty$ and the latter fact yields that

$$\lim_{\lambda\to\overline{\lambda}+0}J(\lambda)=+\infty \quad\text{and}\quad \lim_{\lambda\to+\infty}J(\lambda)=0\,. \tag{7}$$

Therefore, in case A the existence of $\lambda>\overline{\lambda}$ such that (6) holds is proved.

Let us consider the case B: when $\Lambda = [\overline{\lambda}, +\infty)$. We have as in the case A $\lim_{\lambda\to+\infty} J(\lambda) = 0$. Hence, in view of the continuity of the function $J(\lambda)$, we have only to prove that $\lim_{\lambda\to\overline{\lambda}+0} J(\lambda) = +\infty$. It follows from the implicit function theorem that $[U(\overline{\lambda}, r^2)]'_r\big|_{r=z(\overline{\lambda})} = 0$ (because otherwise by this theorem there exist values of λ smaller than $\overline{\lambda}$ and belonging to Λ). This fact implies the required property because $\lim_{\lambda\to\overline{\lambda}+0} J(\lambda) \geq J(\overline{\lambda})$ (indeed, $U(\overline{\lambda}, r^2) > 0$ for $r \in (0, z(\overline{\lambda}))$ and it is clear that for any $y \in (0, z(\overline{\lambda}))$

$$\int_0^y \frac{dr}{\sqrt{U(\lambda, r^2)}} \to \int_0^y \frac{dr}{\sqrt{U(\overline{\lambda}, r^2)}}$$

as $\lambda \to \overline{\lambda} + 0$ and also, $U(\overline{\lambda}, r^2) = \mathrm{O}((z(\overline{\lambda}) - r)^2)$ as $r \to z(\overline{\lambda}) - 0$).

In view of the above arguments, it is proved that for any $d > 0$ there exists $\lambda \in \mathbb{R}$ such that the corresponding solution $y(x)$ of the problem (3)-(4) can be continued on the segment $[0, d]$, $y'(x) > 0$ for $x \in [0, d)$ and $y'(d) = 0$ where $y(d) = z(\lambda)$. Let us prove that for any $d > 0$ there exists precisely one value λ possessing this property. On the contrary, suppose that for some $d > 0$ there exist two values $\lambda_1 < \lambda_2$ such that the corresponding solutions $y_1(x)$ and $y_2(x)$ of the problem (3)-(4) can be continued on the segment $[0, d]$, $y_i'(x) > 0$ for $x \in [0, d)$ and $y_i'(d) = 0$ $(i = 1, 2)$. In view of the above arguments $\lambda_1 > \overline{\lambda}$. Multiply equation (3), with $y(x) = y_1(x)$, by $y_2(x)$. Multiply equation (3), with $y(x) = y_2(x)$, by $y_1(x)$. Then subtract these equalities from each other and integrate over the segment $[0, d]$. In view of the initial data (4), applying the integration by parts, we get

$$0 = \int_0^d y_1(x)y_2(x)[\lambda_1 - \lambda_2 - f(y_1^2(x)) + f(y_2^2(x))]dx\,. \tag{8}$$

By (5) the inverse functions $x_1(y)$ and $x_2(y)$ of $y_1(x)$ and $y_2(x)$ respectively, satisfy $x_1(y) < x_2(y)$ for all $y \in (0, \min\{y_1(d); y_2(d)\}]$. Hence $y_1(x) > y_2(x)$ for all $x \in (0, d]$. Then, in view of the assumption (F), the right-hand side in (8) is negative, which is a contradiction. Thus, it is proved that for any $d > 0$ there exists a unique λ such that the corresponding solution $y(x)$ of the problem (3)-(4) satisfies $y'(x) > 0$ if $x \in [0, d)$ and $y'(d) = 0$.

Finally, in view of the autonomy of the equation (3) and its invariance with respect to the changes of variables $y(x) \to -y(x)$ and $x \to c - x$, where c is an arbitrary constant, we obtain that if a solution $y(x)$ of the problem (3)-(4) can be continued onto the segment $[0, d]$ and satisfies the conditions $y'(x) > 0$ for $x \in [0, d)$ and $y'(d) = 0$, then it can be continued onto the whole real line, and $y(d + x) = y(d - x)$ (in particular $y(2d) = 0$) and $y(2kd + x) = -y(2kd - x)$ for all $x \in \mathbb{R}$ and for $k = 0, \pm1, \pm2, \ldots$. Therefore, Theorem 1, except the statements (d) and (g), for the problem (1)-(2a) is proved. The statement (d) follows from the fact that the function $z(\lambda)$ for $\lambda > \overline{\lambda}$, coinciding with $y(d)$, decreases monotonically. Since, as it is proved earlier, for any $d > 0$ there exists a unique $\lambda \in \mathbb{R}$ (where $\lambda > \overline{\lambda}$) such that $y'(x) > 0$ for $x \in [0, d)$ and

$y'(d) = 0$, where $y(x)$ is the solution of the problem (3)-(4) corresponding to this value of the parameter λ, and since $d > 0$ continuously depends on λ (because $d = J(\lambda)$), by the properties (7), which, as it is proved earlier, hold in each of the cases A and B, the value $d > 0$ is a monotonically decreasing function of the argument $\lambda \in (\overline{\lambda}, +\infty)$. The statement (g) of Theorem 1 for the problem (1)-(2a) immediately follows from this fact and the above arguments.

For problems (1)-(2b) and (1)-(2c) the proof of the statements of Theorem 1 can be made by analogy. Thus, Theorem 1 is proved. $\diamondsuit$

Next, we turn to the proof of Theorem 2. We associate with three problems (1)-(2), respectively, the following three linear eigenvalue problems:

$$-u'' = \mu u, \quad x \in (0,1), \quad u = u(x), \quad u(0) = u(1) = 0, \tag{9a}$$

$$-u'' = \mu u, \quad x \in (0,1), \quad u = u(x), \quad u'(0) = u(1) = 0, \tag{9b}$$

$$-u'' = \mu u, \quad x \in (0,1), \quad u = u(x), \quad u'(0) = u'(1) = 0, \tag{9c}$$

where all quantities are real and μ is the spectral parameter. For each of the problems (9) by $\{e_n\}$ $(n = 0, 1, 2, \ldots)$ we denote the orthonormal basis consisting of its normalized eigenfunctions. More precisely, $e_n(x) = \sqrt{2}\sin \pi(n+1)x$ for problem (1)-(2a), $e_n(x) = \sqrt{2}\cos\frac{\pi(2n+1)x}{2}$ for problem (1)-(2b) (with $n = 0, 1, 2, \ldots$), and $e_0(x) \equiv 1$, $e_n(x) = \sqrt{2}\cos \pi n x$ (with $n = 1, 2, 3, \ldots$) for the problem (1)-(2c). The corresponding eigenvalues are the numbers $\mu_n = (\pi(n+1))^2$ for problem (9a), the numbers $\mu_n = (\pi(2n+1)/2)^2$ for problem (9b), and the numbers $\mu_n = (\pi n)^2$ for problem (9c) $(n = 0, 1, 2, \ldots)$. For each of the problems (1)-(2) we set $v_n(x) = y_n(x)/\|y_n\|$.

Lemma 1 *For each of the problems (1)-(2) and an arbitrary integer $n \geq 0$ the following expansion in the Fourier series, understood in the sense of the space L_2, takes place*

$$v_n(x) = \sum_{k=0}^{\infty} d_{n,k} e_k,$$

where coefficients $d_{n,k}$ are real, $d_{n,0} = \ldots = d_{n,n-1} = 0$ and $d_{n,n} > 0$.

Proof. For arbitrary integers $n \geq 0$ consider the spaces $L_2(0, I_n)$, where $I_n = 1/(n+1)$ for problem (1)-(2a), and $I_n = 1/(2n+1)$ for problem (1)-(2b) (i. e., I_n is the smallest positive root of $v_n(x)$, see Theorem 1(e)). It is clear that in each of these two cases functions from the system $\{e_k\}$ equal to zero at the point $x = I_n$ $(k = 0, 1, \ldots)$ form an orthogonal basis of the space $L_2(0, I_n)$. We also note that for any integer $n \geq 0$ the minimal integer $k \geq 0$, for which $e_k(I_n) = 0$, is $k = n$. Hence,

$$v_n(x) = \sum_{k=0}^{\infty} d_{n,k} e_k, \quad n = 0, 1, 2, \ldots \tag{10}$$

in $L_2(0, I_n)$, where $d_{n,k} = 0$ if $e_k(I_n) \neq 0$, and in view of the above arguments $d_{n,0} = \ldots = d_{n,n-1} = 0$. In addition, $d_{n,n} > 0$ because $v_n(x) > 0$ and $e_n(x) > 0$ for $x \in (0, I_n)$. We also note that in the case of the problem (1)-(2b) the functions $v_n(x)$ and $e_k(x)$ are even with respect to the point $x = 0$, therefore the expansions (10) also hold in the space $L_2(-I_n, I_n)$. Further, one can easily verify that if $e_k(I_n) = 0$ for some value of the index k, then this function $e_k(x)$, continued for all $x \in \mathbb{R}$, is equal to zero at all points of the real line which are roots of the function $v_n(x)$. Therefore, since clearly any function $e_k(x)$ is odd with respect to an arbitrary its root, by Theorem 1(f) we easily get that expansions (10) of the functions $v_n(x)$ $(n = 0, 1, 2, \ldots)$ take place in each space $L_2(I)$, too, where I is the interval between two arbitrary nearest roots of the function $v_n(x)$. Hence, the expansions (10) are also valid in the space L_2.

For problem (1)-(2c) using the same procedure, one can obtain $v_n(x) = \sum_{k=0}^{\infty} d_{n,k} e_k(x)$, $n = 1, 2, 3, \ldots$, in the space L_2, where $d_{n,0} = \ldots = d_{n,n-1} = 0$ and $d_{n,n} > 0$, and, in addition, that $v_0(x) = pe_0(x)$. Thus, Lemma 1 is proved. $\diamondsuit$

Remark 2. An example of expansions of the form as in Lemma 1 is given in [13]. In that example the matrix $D = (d_{n,k})$ is upper-triangular with positive elements on the main diagonal, which shows that, generally speaking, the completeness (in particular, the property of being a basis) of the system of functions $\{v_n\}$ does not follow from the indicated properties of the matrix D.

Lemma 2 *There exists a positive constant C such that*

$$\|v_n - e_n\| \le C(n+1)^{-1}, \quad n = 0, 1, 2, \ldots$$

for each of the problems (1)-(2).

Proof. Let $t_n(x) = pe_n(x)/(\sqrt{2}\pi(n+1))$ for problem (1)-(2a), $t_n(x) = pe_n(x)/\sqrt{2}$ for the problem (1)-(2b) (with $n = 0, 1, 2, \ldots$), and $t_0(x) = pe_0(x)$, $t_n(x) = pe_n(x)/\sqrt{2}$ (with $n = 1, 2, 3, \ldots$) for problem (1)-(2c). We note that by the standard comparison theorem [5], Theorem 1(d) immediately implies the existence of $C_1 > 0$ such that

$$|\lambda_n - \mu_n| \le C_1 \tag{11}$$

for $n = 0, 1, 2, \ldots$ for each of three problems (1)-(2). Therefore, by Theorem 1(d), for each of three problems (1)-(2) we get for $u_n(x) = y_n(x) - t_n(x)$

$$-u_n'' = W_n(x) + \mu_n u_n, \quad x \in (0, 1)\,. \tag{12}$$

In addition, for problems (1)-(2a) and (1)-(2c) by Theorem 1(f),

$$u_n(0) = u_n'(0) = u_n(1) = u_n'(1) = 0 \tag{13}$$

where $W_n(x) = (\lambda_n - \mu_n)y_n(x) - f(y_n^2(x))y_n(x)$ is a sequence of continuous functions uniformly bounded with respect to $x \in [0, 1]$.

For problem (1)-(2a) or (1)-(2c) for each number n we multiply the equality (12) by $2xu_n'(x)$ and integrate over the segment $[0, 1]$ with the application of the integration by parts. Then, in view of (13) and the obvious inequality $2ab \le a^2 + b^2$ we get

$$\mu_n \|u_n\|^2 = 2\int_0^1 xW_n(x)u_n'(x)dx - \int_0^1 [u_n'(x)]^2 dx \le \int_0^1 W_n^2(x)dx. \qquad (14)$$

Hence, there exists $C_2 > 0$ such that

$$\mu_n \|u_n\|^2 \le C_2 \qquad (15)$$

for all integer $n \ge 0$. Since in problem (1)-(2c) we have $\|t_0\| = p, \quad \|t_n\| = \frac{p}{\sqrt{2}}$, where $n = 1, 2, 3, \ldots$, it follows from (15) that $\left| \quad \|y_n\| - \frac{p}{\sqrt{2}} \right| \le C_2'(n+1)^{-1}$ for some $C_2' > 0$ independent of n. Therefore, taking into account (15), we get the statement of Lemma 2 for problem (1)-(2c).

For problem (1)-(2a) we have

$$\|t_n\| = \frac{p}{\sqrt{2}\pi(n+1)} \quad (n = 0, 1, 2, \ldots); \qquad (16)$$

therefore, (15) implies the existence of $C_3 > 0$ such that

$$\|y_n\| \le C_3(n+1)^{-1} \qquad (17)$$

for all integer $n \ge 0$. In view of Theorem 1(d), (17) yields the existence of $C_4 > 0$ such that $\|W_n\| \le C_4(n+1)^{-1}$ for all $n = 0, 1, 2, \ldots$. So, in problem (1)-(2a) we get from the latter estimate and (14) that

$$\|u_n\| \le C_5(n+1)^{-2}$$

with a constant $C_5 > 0$ independent of the number $n = 0, 1, 2, \ldots$. The statement of Lemma 2 for problem (1)-(2a) follows from this estimate together with (16) as for problem (1)-(2c).

For problem (1)-(2b) the proof of the statement in Lemma 2 can be done as for problem (1)-(2c). One should only take into account the estimate for the eigenfunctions of the problem (1)-(2b)

$$|y_n'(1) - t_n'(1)| \le C_6$$

with a constant $C_6 > 0$ independent of $n = 0, 1, 2, \ldots$ (this estimate follows from (11), the properties of the function $t_n(x)$, Theorem 1(e), and the identities

$$-[y_n'(x)]^2 + F(y_n^2(x)) - F(p^2) = \lambda_n y_n^2(x) - \lambda_n p^2 \quad (n = 0, 1, 2, \ldots).$$

These identities can be obtained by multiplying (1), with $y(x) = y_n(x)$, by $2y_n'(x)$, and then integrating from 0 to x). Thus, Lemma 2 is proved. $\diamondsuit$

Lemma 3 *For each of the problems (1)-(2) the system of functions $\{v_n\}$ is linearly independent in the space L_2.*

Proof. On the contrary suppose that

$$\sum_{n=0}^{\infty} c_n v_n = 0 \tag{18}$$

in the space L_2 where c_n are real coefficients, $c_0 = \ldots = c_{l-1} = 0$ and $c_l \neq 0$ for a number l. Multiply (18) in the space L_2 by the function $e_l(x)$. Then, in view of Lemma 1, we get $d_{l,l} c_l = 0$. But according to Lemma 1 $d_{l,l} > 0$, which is a contradiction that proves the present lemma. $\diamondsuit$

Remark 3. In view of Lemmas 2 and 3, it follows from Bary's theorem that for each of the problems (1)-(2) the system of functions $\{v_n\}$ is a Riesz basis in the space L_2. Moreover, since $\{e_n\}$ is an orthonormal basis in L_2, the functions $\{v_n\}$ form a Bary basis in L_2 for each case. As stated in the introduction, we shall establish a direct proof of this fact without using Bary's theorem.

In what follows, unless otherwise stated, all three problems (1)-(2) are considered simultaneously. For an arbitrary number $N \geq 0$, denote by L^N the closure in L_2 of the set of all finite linear combinations of the functions $\{e_n\}_{n \geq N}$. Denote by A the linear operator defined by the rule $Ae_n = (d_{n,n})^{-1} v_n$. Then A is defined for the linear combinations of $\{e_n\}$. In view of Lemma 1, it is clear that A maps the subspace L^N into itself, and that A is defined on dense subsets of L_2 and of the subspaces L^N. Let A_N denote the restriction of the operator A to the subspace L^N ($N = 0, 1, 2, \ldots$). Then, in view of Lemma 1 $A_N = \Lambda_N + G_N$ where Λ_N is the identity operator in L^N and G_N is an operator defined on all finite linear combinations of $\{e_n\}_{n \geq N}$. In addition, $G_N e_n = (d_{n,n})^{-1} \sum_{k>n} d_{n,k} e_k$ so that $A_N e_n = (d_{n,n})^{-1} v_n$ for $n = N, N+1, N+2, \ldots$. In view of Lemma 2, the inequality $d_{n,n} > 0$, and the condition $\|v_n\| = 1$ for $n = 0, 1, 2, \ldots$, there exists $\overline{d} \in (0, 1)$ such that for $n \geq 0$,

$$\overline{d} \leq d_{n,n} \leq 1\,. \tag{19}$$

Let $\|G_N\| = \sup\{\|G_N g\| : g = \sum_{n=N}^{M} c_n e_n \text{and } \|g\| = 1\}$, where supremum is taken over all numbers $M = N, N+1, N+2, \ldots$ and all real coefficients c_n.

Lemma 4 *For each of three problems (1)-(2), $\|G_N\| < \infty$ for all numbers $N \geq 0$ and $\lim_{N \to \infty} \|G_N\| = 0$.*

Proof. Let $g = \sum_{n=N}^{\infty} c_n e_n$ where only finitely many coefficients c_n are nonzero, and $\|g\| = 1$. It follows from Lemma 2 and (19) that $\sum_{\substack{n,k=0 \\ n \neq k}}^{\infty} (d_{n,n})^{-2} d_{n,k}^2 < \infty$. Therefore,

$$\begin{aligned} \|G_N g\|^2 &= \sum_{k=N+1}^{\infty} \Big[\sum_{n=N}^{k-1} c_n (d_{n,n})^{-1} d_{n,k} \Big]^2 \\ &\leq \sum_{k=N+1}^{\infty} \Big[\sum_{n=N}^{k-1} c_n^2 \Big] \times \Big[\sum_{n=N}^{k-1} (d_{n,n})^{-2} d_{n,k}^2 \Big] \end{aligned}$$

$$\leq \quad \|g\|^2 \sum_{\substack{n,k=N \\ n\neq k}}^{\infty} (d_{n,n})^{-2} d_{n,k}^2 \to 0 \quad \text{as } N \to \infty,$$

and the proof of Lemma 4 is complete. $\diamondsuit$

Let N be a positive integer such that $\|G_N\| < 1/2$. As shown in [6], the linear operator G_N can be uniquely extended to the whole space L^N. For this extension we keep the same symbol G_N, and we have $\|G_N\| < 1/2$. In view of the latter fact, the linear operator $A_N = \Lambda_N + G_N$, with $G_N \in \mathcal{L}(L^N; L^N))$ belongs to $\mathcal{L}(L^N; L^N)$ and possesses a bounded inverse:

$$A_N^{-1} = \Lambda_N + \sum_{m=1}^{\infty} (-1)^m (G_N)^m .$$

The proof of this statement can be found in [6].

The following statement is in fact proved in [2, 3]. However, since its proof is simple and short, we present it here.

Lemma 5 *Let N be the number fixed above. Then for each of the problems (1)-(2), the functions $\{v_n\}_{n\geq N}$ form a Riesz basis in L^N.*

Proof. Obviously, the functions $\{e_n\}_{n\geq N}$ form an orthonormal basis in L^N. Take an arbitrary function $u \in L^N$ and let $v = A_N^{-1}u = \sum_{n=N}^{\infty} c_n e_n$ in the space L^N where c_n are real coefficients. We have $u = A_N v = \sum_{n=N}^{\infty} c_n A_N e_n = \sum_{n=N}^{\infty} c_n (d_{n,n})^{-1} v_n$ where all series converge in L^N. Therefore, in view of Lemma 3, the system of functions $\{v_n\}_{n\geq N}$ is a basis in the space L^N. Further, with the same notation, $\sum_{n=N}^{\infty} c_n^2 < \infty$ because the series $\sum_{n=N}^{\infty} c_n e_n$ converges in the space L^N. Hence, if $\sum_{n=N}^{\infty} c_n^2 < \infty$, then the series $\sum_{n=N}^{\infty} c_n (d_{n,n})^{-1} v_n$ converges in the space L^N. Conversely, let the series $\sum_{n=N}^{\infty} c_n v_n$ converge in L^N. Then, $A_N^{-1} \sum_{n=N}^{\infty} c_n v_n = \sum_{n=N}^{\infty} c_n A_N^{-1} v_n = \sum_{n=N}^{\infty} c_n d_{n,n} e_n$ in the space L^N, therefore, $\sum_{n=N}^{\infty} c_n^2 d_{n,n}^2 < \infty$. So, in view of the estimates (19), Lemma 5 is proved. $\diamondsuit$

Lemma 6 *For each of the three problems (1)-(2) the functions $\{v_n\}_{n=0,1,2,\ldots}$ form a Riesz basis in L_2.*

Proof. Let N be the number fixed above, and $L_\perp^N$ be the finite-dimensional subspace in L_2 spanned by $e_0, \ldots, e_{N-1}$, and P_N be the orthogonal projector in the space L_2 onto the subspace $L_\perp^N$. We set $w_n = P_N v_n$, with $0 \leq n \leq N-1$. Then

$$w_n = v_n - \sum_{k=N}^{\infty} d_{n,k} e_k . \tag{20}$$

Further, the dimension of the subspace $L_\perp^N$ is obviously equal to N and, in view of Lemma 1, the system of functions $\{w_n\}_{0\leq n\leq N-1}$ is linearly independent in

L_2, therefore, it is a basis in $L_\perp^N$. Hence, in view of Lemma 5, the system of functions $\{w_n\}_{0\le n\le N-1} \cup \{v_n\}_{n\ge N}$ is a basis in L_2.

For an arbitrary $u \in L_2$,

$$u = \sum_{n=0}^{N-1} c_n w_n + \sum_{n=N}^{\infty} c_n v_n \tag{21}$$

in L_2 where c_n are real coefficients. Substituting the expressions for functions w_n from (20) into (21), we get

$$u = \sum_{n=0}^{N-1} c_n v_n + \sum_{n=N}^{\infty} \left(c_n - \sum_{m=0}^{N-1} c_m d_{m,n} \right) e_n$$

in L_2, hence the functions $\{v_n\}_{n=0,1,2,\dots}$ form a basis for L_2. Finally, that these functions form a Riesz basis in L_2 follows from Lemma 5, and concludes the present proof. $\diamondsuit$

For problem (1)-(2c), the estimate (15) and the arguments following the estimate imply the existence of constants $0 < c < C$ such that $c \le \|y_n\| \le C$ for all $n = 0, 1, 2, \dots$. For problem (1)-(2b), a similar statement can be proved by analogy (see the proof of Lemma 2). The statement of Theorem 2 follows ¿from Lemma 6 and these estimates.

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