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Global minimizing domains for the first eigenvalue of an elliptic operator with non-constant coefficients *

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Abstract

We consider an elliptic operator, in divergence form, that is a uniformly elliptic matrix. We describe the behavior of every sequence of domains which minimizes the first Dirichlet eigenvalue over a family of fixed measure domains of $\mathbb{R}^N$. The existence of minimizers is proved in some particular situations, for example when the operator is periodic.

1 Introduction

Let $A = (a_{i,j})_{1\le i,j\le N}$ be a symmetric matrix function in $L^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$, with $a_{ij} = a_{ji}$ for $1 \le i, j \le N$. Suppose that there exists $\alpha > 0$ such that a.e. $x \in \mathbb{R}^N$ and for every $\zeta \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{n} a_{ij}\zeta_i\zeta_j \ge \alpha|\zeta|^2 \quad \forall \zeta \in \mathbb{R}^N.$$

By $\mathcal{A} = -\operatorname{div}(A(x)\nabla)$ we denote the elliptic operator associated to $A(\cdot)$. For every open set $\Omega \subseteq \mathbb{R}^N$ with finite measure, we denote by $\lambda_1(\Omega)$ the first eigenvalue in $H^1_0(\Omega)$; i.e.,

$$\lambda_1(\Omega) = \min_{u\in H^1_0(\Omega)} \frac{\int_\Omega A\nabla u\cdot\nabla u dx}{\int_\Omega u^2 dx}.$$

We are concerned with the minimization of $\lambda_1(\Omega)$ in the class of domains of $\mathbb{R}^N$ with fixed Lebesgue measure. More precisely, given $c > 0$, find

$$\min\big\{\lambda_1(\Omega) : \Omega \subseteq \mathbb{R}^N,\ \Omega \text{ is open},\ |\Omega| = c\big\}. \tag{1}$$

When A is the identity matrix Id, this problem was raised by Rayleigh and solved by Faber and Krahn [6, 10]; the solution to (1), in this case, is the

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ball of measure equal to c. There are many proofs of this result, for example: based on symmetrization, shape derivative techniques, moving plane method, etc. When A has constant coefficients, by a suitable change of variables, one can prove that a minimizer exists and is an ellipse. When $A(x) = a(|x|)Id$, under some additional conditions on $a(\cdot)$, rearrangement techniques [1] can be used for solving the minimization problem.

Nevertheless, when $A(\cdot)$ is not constant, each one of these techniques fails because they depend strongly on the symmetry of the Laplace operator. This is the reason for which we raise the question about the existence of a solution to problem (1) in this paper.

If instead of looking for the minimizer in the class of all open sets of $\mathbb{R}^N$ of measure c, the minimizer is searched in the family of quasi open sets of fixed measure contained in a bounded design region, the proof of existence is given by Buttazzo and Dal Maso in [4] (see the exact definition of quasi open in [4] and in section 2). The technique of the proof is based on relaxation and γ-convergence arguments. However, the extension of this result in $\mathbb{R}^N$ fails because of the lack of compactness of the injection $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$. An idea how to overcome this difficulty of lack of compactness is given in [2] and in [3], but works well only for elliptic operators with constant coefficients.

In this paper, we study the behavior of a minimizing sequence for problem (1) when $A(\cdot)$ has non constant coefficients. We prove that two situations may occur: either a minimizing domain exists (and is γ-limit of a minimizing sequence) or the minimizing sequence has a particular behavior called concentrative, i.e., up to some translation the sequence of the first eigenvectors converges strongly in $L^2(\mathbb{R}^N)$, but the sequence of domains "goes" to infinity. In some particular situations, as for example if the coefficients of $A(\cdot)$ satisfy some coerciveness like property, or if the coefficients are periodic in space, the second situation can not occur. As a consequence, we conclude with the existence of a minimizer.

2 Notation and preliminary results

The Lebesgue measure of a set $E \subseteq \mathbb{R}^N$ is denoted by $|E|$. The capacity of a the set E is defined by

$$C(E) = \inf \Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx,\ u \in \mathcal{U}_E \Big\}$$

where $\mathcal{U}_E$ is the class of all functions $u \in H^1(\mathbb{R}^N)$ such that $u \geq 1$ a.e. in a neighborhood of E. It is said that a property holds quasi everywhere on E (shortly q.e. on E) if the set of all points $x \in E$ for which the property does not hold has capacity zero. We refer to [8, 14] for details on capacity.

A set $A \subseteq \mathbb{R}^N$ is called quasi open if for every $\epsilon > 0$ there exists an open set G_ϵ such that $A \cup G_\epsilon$ is open and $C(G_\epsilon) < \epsilon$. It can be easily seen that for any quasi open set there exists a decreasing sequence $\{\Omega_n\}_{n\in N}$ of open sets, containing A, such that $C(\Omega_n \setminus A) \to 0$. Of course, any open set is also quasi open. A quasi open set is called quasi connected, if it can not be written as the union of two quasi open sets of positive capacity having the intersection of zero capacity. A function $f : \mathbb{R}^N \mapsto \mathbb{R}$ is said to be quasi continuous if for all $\epsilon > 0$ there exists an open set G_ϵ with $C(G_\epsilon) < \epsilon$ such that $f_{|\mathbb{R}^N \setminus G_\epsilon}$ is continuous on $\mathbb{R}^N \setminus G_\epsilon$ (see [8], [14]). Any function $u \in H^1(\mathbb{R}^N)$ has a quasi continuous representatives, $\tilde{u}$, such that $\tilde{u}(x) = u(x)$ a.e. All quasi continuous representative of u, are equal q.e. For a quasi open set A, the Sobolev space $H_0^1(A)$ is defined as follows:

$$H_0^1(A) = \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ q.e. on } \mathbb{R}^N \setminus A\}.$$

In this definition, the function u is supposed to be quasi continuous. If A is open, then $H_0^1(A)$ defined as above is the usual H_0^1-space (see [8]).

Let A be a quasi open set of finite measure. Then the injection $H_0^1(A) \hookrightarrow L^2(A)$ is compact and the constant of the Poincaré inequality depends only on the measure of A and the dimension of the space (see [13]).

The support of a function u is denoted by $\text{supp}(u)$. The ball of $\mathbb{R}^N$ centered in x of radius r is denoted by $B(x, r)$. If $r = 1$ we simply write $B(x)$.

We recall the concentration-compactness principle from [5, 12].

Lemma 2.1 *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u_n^2 dx = 1$. Then there exists a subsequence $\{u_{n_k}\}$ such that one of the following situations holds*

1. **Vanishing** *For every $0 < R < \infty$*

$$\lim_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_{n_k}^2 dx = 0\,.$$

2. **Compactness** *There exists a sequence $\{y_k\} \subseteq \mathbb{R}^N$ such that for every $\varepsilon > 0$ there exists $R < +\infty$ for which*

$$\int_{B(y_k,R)} u_{n_k}^2 dx \geq 1 - \varepsilon \quad \forall k \geq 1\,.$$

3. **Dichotomy** *There exists $\alpha \in (0, 1)$ such that for every $\varepsilon > 0$ there exist two sequences $\{u_k^1\}$ and $\{u_k^2\}$ and $k_0 \geq 1$ such that for every $k \geq k_0$,*

$$\|u_{n_k} - u_k^1 - u_k^2\|_{L^2(\mathbb{R}^N)} \leq \varepsilon\,,$$
$$|\int_{\mathbb{R}^N} (u_k^1)^2 dx - \alpha| \leq \varepsilon \quad |\int_{\mathbb{R}^N} (u_k^2)^2 dx - (1-\alpha)| \leq \varepsilon\,,$$
$$\text{dist}(\text{supp}(u_k^1), \text{supp}(u_k^2)) \to_{k\to+\infty} +\infty\,,$$

and

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 - |\nabla u_k^1|^2 - |\nabla u_k^2|^2 dx \geq 0\,. \tag{2}$$

In (2), one can easily replace the norm of the gradient by the norm given by the operator $\mathcal{A}$, i.e. to obtain

$$\liminf_{k\to\infty}\int_{\mathbb{R}^N}\{A\nabla u_{n_k}\cdot\nabla u_{n_k}-A\nabla u_k^1\cdot\nabla u_k^1-A\nabla u_k^2\cdot\nabla u_k^2\}dx\geq 0 \qquad (3)$$

We recall the following two lemmas from [9] (see also [11]).

Lemma 2.2 *There exists $C>0$ such that*

$$\inf_{\substack{u\in H^1(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla u\|_{L^2(\mathbb{R}^N)}\leq 1}}\ \sup_{y\in\mathbb{R}^N}(2+\|u\|^{-2}_{L^2(\mathbb{R}^N)})|B(y)\cap\operatorname{supp}(u)|\geq C.$$

Lemma 2.3 *Let ε, δ and M be positive constants, and $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^N)$ such that for all $n\in\mathbb{N}$,*

$$\|\nabla u_n\|_{L^2(\mathbb{R}^N)}\leq M \quad \textit{and} \quad |\{|u_n|>\varepsilon\}|\geq\delta\,.$$

There exists a sequence of vectors $\{y_n\}$ in $\mathbb{R}^N$ such that the sequence $u_n(\cdot+y_n)$ does not possess a weakly convergent subsequence to zero in $H^1(\mathbb{R}^N)$.

3 Behavior of a minimizing sequence

This section contains our main result which describes the behavior of a minimizing sequence to problem (1). We remark that

$$\begin{aligned}&\inf\{\lambda_1(\Omega):\Omega\subseteq\mathbb{R}^N,\Omega\quad\text{is open},\quad|\Omega|=c\}\\ =&\quad\inf\{\lambda_1(\Omega):\Omega\subseteq\mathbb{R}^N,\Omega\quad\text{is quasi open},\quad|\Omega|\leq c\}.\end{aligned}$$

For this reason we set the problem

$$\min\{\lambda_1(\Omega):\Omega\subseteq\mathbb{R}^N,\Omega\quad\text{is quasi open},\quad|\Omega|\leq c\}. \qquad (4)$$

Observe that is possible that problem (4) has a solution, while problem (1) does not. Nevertheless, we are not able to give an example of matrix $A(\cdot)$ such that the optimum is quasi open, and not open.

Suppose that $\{\Omega_n\}$ is a minimizing sequence of open sets for problem (4). By u_n we denote a first eigenvector of $\mathcal{A}$ on Ω_n, such that $|u_n|_{L^2(\mathbb{R}^N)}=1$. As Ω_n is minimizing, it is easy to see that $\lambda_1(\Omega_n)$ is simple from rank on, and that Ω_n can be replaced by the open connected set $\{u_n\neq 0\}$, in the sense that the sequence $\{u_n\neq 0\}$ is also minimizing for the same problem. For connected open sets Ω_n, we therefore denote by u_n the unique positive normalized first eigenvector.

In order to describe the behavior of a minimizing sequence for problem (4), we define what is a concentrative sequence.

Definition 3.1 *A sequence of connected open sets $\{\Omega_n\}$ is called $\mathcal{A}$-concentrative, if there exists a sequence of vectors $\{y_n\} \in \mathbb{R}^N$ such that $u_n(\cdot + y_n)$ strongly converges in $L^2(\mathbb{R}^N)$.*

The following theorem contains the main result of the paper.

Theorem 3.2 *One of the two following situations occurs:*
1. There exists a quasi open, quasi connected set A solution of problem (4).
2. For any minimizing sequence of open sets $\{\Omega_n\}$ for problem (4), there exists $\{n_k\}$ and a concentrative sequence of open connected sets $\tilde{\Omega}_k \subseteq \Omega_{n_k}$ which is also minimizing, such that $\inf_{x\in\tilde{\Omega}_k} |x| \to \infty$ when $k \to \infty$.

Proof Let us suppose that the first situation does not occur. Then, we consider a minimizing sequence of open sets for problem (4), say $\{\Omega_n\}$. We denote by $\{u_n\}$ the sequence of positive, normalized first eigenvectors on Ω_n. It is obvious that this sequence is bounded in $H^1(\mathbb{R}^N)$ since $\{\Omega_n\}$ is minimizing and $A(\cdot)$ is uniformly elliptic. Therefore, the concentration-compactness lemma can be applied for the sequence $\{u_n\}$. We treat separately each situation.
Vanishing Let us suppose that up to a change of the indices and extraction of a subsequence, for every $R > 0$ we have

$$\lim_{k\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} u_n^2 dx = 0.$$

In the sequel, we prove that this situation can not occur, since this would imply that $\lambda_1(\Omega_n) \to \infty$.

Lemma 3.3 *Let $\{u_n\}$ a bounded sequence in $H^1(\mathbb{R}^N)$ such that $\|u_n\|_{L^2(\mathbb{R}^N)} = 1$ and $u_n \in H^1_0(\Omega_n)$ with $|\Omega_n| \le c$. There exists a sequence of vectors $\{y_n\} \subset \mathbb{R}^N$ such that the sequence $\{u_n(.+y_n)\}$ does not possess a weakly convergent sequence in $H^1(\mathbb{R}^N)$.*

Proof Assume that $\|u_n\|_{H^1(\mathbb{R}^N)} \le M$. For every $\varepsilon > 0$, we have

$$1 = \int_{\Omega_n} u_n^2 dx \le \int_{\{u_n\le\varepsilon\}\cap\Omega_n} \varepsilon^2 + \int_{\{u_n>\varepsilon\}} ((u_n - \varepsilon)^+ + \varepsilon)^2 \, dx\,.$$

Using the Cauchy-Schwarz inequality, a simple computation leads to

$$1 \le (\sqrt{c}\varepsilon + \|(u_n - \varepsilon)^+\|_{L^2(\mathbb{R}^N)})^2.$$

Taking $\varepsilon_0 = 1/(2\sqrt{c})$, we obtain

$$1 \ge \|(u_n - \varepsilon_0)^+\|_{L^2(\mathbb{R}^N)} \ge \frac{1}{2}\,. \tag{5}$$

Since $\|\nabla(u_n - \varepsilon_0)^+\|_{L^2(\mathbb{R}^N)} \le M$, we can apply lemma 2.2. Therefore, there exists a constant $C > 0$, a sequence $\{y_n\}$ of $\mathbb{R}^N$ such that for every $n \in \mathbb{N}$

$$(2 + M^2\|(u_n - \varepsilon_0)^+\|^{-2}_{L^2(\mathbb{R}^N)})\, |B(y_n) \cap \ \mathrm{supp}(u_n - \varepsilon_0)^+| \ge C\,.$$

Using (5) and $|\{u_n > \varepsilon_0\}| \geq |B(y_n) \cap \operatorname{supp}(u_n - \varepsilon_0)^+|$, we get

$$|\{u_n > \varepsilon_0\}| \geq \frac{C}{4M^2+2}.$$

Applying lemma 2.3, the conclusion follows. $\diamondsuit$

We prove now that if the vanishing occurs for the sequence $\{u_n\}$, then we get $\limsup_{n\to\infty} \lambda_1(\Omega_n) = \infty$, which will contradict the choice of $\{\Omega_n\}$ as minimizing sequence of the first eigenvalues. Indeed, suppose that $\limsup_{n\to\infty} \lambda_1(\Omega_n) < \infty$. The sequence $\{u_n\}$ is therefore bounded in $H^1(\mathbb{R}^N)$, hence lemma 3.3 applies. One can therefore subtract a subsequence (still denoted with the same index) and a sequence of vectors $y_n \in \mathbb{R}^N$ such that $u_n(\cdot + y_n)$ weakly converges to a non zero element in $H^1(\mathbb{R}^N)$. This contradicts the vanishing of $\{u_n\}$.

Compactness Let us suppose that when applying the concentration-compactness lemma for $\{u_n\}$, the compactness situation holds. For a subsequence (still denoted with the same index) and for a sequence of vectors $\{y_n\} \in \mathbb{R}^N$ we have

$$u_n(\cdot + y_n) \overset{L^2(\mathbb{R}^N)}{\longrightarrow} u\,.$$

Two situations may occur: either $\{y_n\}$ has a bounded subsequence, or $|y_n|_{\mathbb{R}^N} \to \infty$. In the first case, up to subtraction of a subsequence and a renotation of the indices, we can suppose that $y_n \to y$. It is easy to see that

$$u_n \overset{L^2(\mathbb{R}^N)}{\longrightarrow} u(\cdot - y)\,.$$

In the second case

$$\liminf_{n\to\infty} \int_{\Omega_n} A\nabla u_n \cdot \nabla u_n \, dx \geq \int_{\mathbb{R}^N} A\nabla u(x-y) \cdot \nabla u(x-y)\, dx\,.$$

Taking a quasi continuous representative for u, we set $\Omega = \{u(\cdot - y) > 0\}$. Then Ω is a solution for problem (4).

Assuming that $|y_n|_{\mathbb{R}^N} \to \infty$ we get that the sequence $\{\Omega_n\}$ is concentrative. Indeed, one can construct two sequences $\{u_k^1\} \subseteq H^1(\mathbb{R}^N)$ and $\{R_k\} \subseteq \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} |u_{n_k} - u_k^1|^2 dx \to 0, \qquad R_k \to +\infty$$
$$\operatorname{supp} u_k^1 \subseteq \operatorname{supp} u_{n_k} \cap B_{y_{n_k}, R_k}, \quad |y_{n_k}|_{\mathbb{R}^N} \geq 2R_k,$$
$$\liminf_{k\to\infty} [\int_{\Omega_{n_k}} A\nabla u_{n_k} \cdot \nabla u_{n_k}\, dx - \int_{\Omega_{n_k} \cap B_{y_{n_k}, R_k}} A\nabla u_k^1 \cdot \nabla u_k^1 \, dx] \geq 0\,.$$

The construction of $\{u_k^1\} \subseteq H^1(\mathbb{R}^N)$ and $\{R_k\} \subseteq \mathbb{R}^N$ can be done using the concentration function $Q_n(R) = \sup_{y\in\mathbb{R}^N} \int_{B_{y,R}} u_{n_k}^2 dx$. For every $\varepsilon > 0$, there exists R_ε and n_ε such that for every $R \geq R_\varepsilon$ and $n \geq n_\varepsilon$ we have $Q_n(R) \geq 1 - \varepsilon$ (see [12]). We consider a function $\xi \in C_0^\infty(\mathbb{R}^N)$ with $0 \leq \xi \leq 1$, $\xi = 1$ on $B(0,1)$ $\xi = 0$ on $\mathbb{R}^N \setminus B(0,2)$. We define $\phi = 1 - \xi$, and $\phi_R(x) = \phi(\frac{x}{R})$.

We chose n_k such that $|y_{n_k}| \geq 4R_\varepsilon$ and define $u^1_k = u_{n_k}\phi_{R_\varepsilon}$. For a sequence $\varepsilon \to 0$ we chose $R_k = 2R_\varepsilon$, and like in [12] it can be proved that all requirements are satisfied.

As a consequence we have that $\tilde{\Omega}_k = \Omega_{n_k} \cap B_{y_{n_k}, R_k}$ is also a minimizing sequence, but concentrative. Moreover $\inf_{x \in \tilde{\Omega}_k} |x| \to +\infty$.

Dichotomy Let us suppose that the sequence $\{u_n\}$ is in the dichotomy situation. There exists $\alpha > 0$ such that by a diagonal subtraction procedure, one can extract a subsequence (still denoted with the same index) and construct two sequences $\{u^1_n\}$ and $\{u^2_n\}$ with disjoint support such that $\|u_n - (u^1_n + u^2_n)\|_{L^2(\mathbb{R}^N)} \to 0$, $\|u^1_n\|_{L^2(\mathbb{R}^N)} \to \alpha > 0$, $\|u^2_n\|_{L^2(\mathbb{R}^N)} \to 1 - \alpha > 0$ and

$$\liminf_{n \to \infty} \int_{\Omega_n} [A\nabla u_n \cdot \nabla u_n - A\nabla u^1_n \cdot \nabla u^1_n - A\nabla u^2_n \cdot \nabla u^2_n] dx \geq 0\,.$$

The construction of these two sequences follows the same ideas as for compactness (see [12]). In this case, the concentration functions are upper bounded by α, and for suitables R_n we have that $u^1_n = u_n \xi_{R_n}$ $u^2_n = u_n \phi_{k_n R_n}$, with $k_n \to \infty$.

It is immediate that the sequences of quasi open sets $\{u^1_n > 0\}$ and $\{u^2_n > 0\}$ are also minimizing for problem (4) (these sets are defined for quasi continuous representatives of u^1_n and u^2_n). It is obvious that $|\{u^1_n > 0\}| + |\{u^2_n > 0\}| \leq c$.

For each $n \in \mathbb{N}$, at least one of the sets $\{u^1_n > 0\}, \{u^2_n > 0\}$ has the Lebesgue measure, less than or equal to $\frac{c}{2}$. Renoting this sequence $\{\Omega^1_n\}$ we obtain in such a way a minimizing sequence for problem (4), such that $|\Omega^1_n| \leq \frac{c}{2}$. For this minimizing sequence we start again the algorithm, i.e. we apply the concentration-compactness lemma to the sequence of normalized first eigenvectors. If this sequence vanishes or leads to compactness, the answer of theorem 3.2 follows. If not, using the previous procedure given by the dichotomy, we construct a minimizing sequence $\{\Omega^2_n\}$ with $|\Omega^2_n| \leq \frac{c}{4}$.

Continuing this procedure, two possibilities occur: either we stop at some moment since vanishing or compactness occurs, or by a diagonal procedure we can subtract a minimizing sequence $\{\Omega^{k_n}_n\}$ with the property that $|\Omega^{k_n}_n| \to 0$ for $n \to \infty$. This last situation is in fact impossible, since if $|\Omega^{k_n}_n| \to 0$, then $\lambda_1(\Omega^{k_n}_n) \to \infty$. The proof of this assertion is an easy consequence of the ellipticity condition of $\mathcal{A}$ and the Poincaré inequality. ♡

If instead of a general non constant operator $\mathcal{A}$ we would have an elliptic operator in divergence form with constant coefficients, the dichotomy could have been solved directly since any minimizing sequence of sets must have the measure converging to c. This assertion comes by an homothethy argument, which is not anymore valid for operators with non constant coefficients.

If problem (4) has a solution, then this solution has the Lebesgue measure of the optimal set is equal to c. Indeed, if Ω is a solution such that $|\Omega| < c$, then there exits a connected open set containing strictly Ω with measure equal to c, denoted Ω^*. From the optimality of Ω, we get that the first eigenvector on Ω is also first eigenvector on Ω^*, but this contradicts the strong maximum principle, since the first eigenvector on an open connected set can not vanish on a set of positive measure.

If problem (4) has a solution, then this solution is quasi connected. Indeed, if Ω is solution, and $\Omega = \Omega_1 \cup \Omega_2$, such that Ω_1, Ω_2 are nonempty quasi open with $C(\Omega_1 \cap \Omega_2) = 0$, we immediately have that Ω_1 and Ω_2 are also minimizers. This is not possible, since $|\Omega_1| < c$ and contradicts the previous assertion.

4 Further remarks and applications

In this section we give some applications to Theorem 3.2 and formulate some open questions. Assume that A is a periodic matrix in $\mathbb{R}^N$; i.e., suppose the existence of $l \in \mathbb{R}^N$ such that for all $k_1, \dots, k_N \in \mathbb{N}^N$ we have

$$A(x) = A(x_1 + k_1 l_1, \dots, x_N + k_N l_N).$$

Proposition 4.1 *Under the previous hypotheses on A, problem (4) has at least one solution.*

Proof We apply Theorem 3.2. If we are in the first situation, problem (4) has at least a solution. In the sequel, we prove that the second situation also leads to the existence of a solution. Suppose that $\{\Omega_n\}$ is a minimizing sequence, which is concentrative. There exists $\{y_n\}$ such that $u_n(\cdot - y_n)$ strongly converges to u in $L^2(\mathbb{R}^N)$. There exits $l_n = (k_n^1 l_1, \dots, k_n^N l_N)$ with $k_n^1, .., k_n^N \in \mathbb{N}$ such that $|y_n - l_n|_{\mathbb{R}^N} \leq |l|_{\mathbb{R}^N}$.

The sequence $v_n := u_n(\cdot - l_n)$ has a subsequence strongly convergent in $L^2(\mathbb{R}^N)$. Indeed, for a subsequence still denoted with the same index, we have $y_n - l_n \to x \in \mathbb{R}^N$. Then, we get $u_n(\cdot - l_n)$ strongly converges to $u(\cdot + x)$ in $L^2(\mathbb{R}^N)$.

From the periodicity of A, we have that $\Omega_n - l_n$ is also minimizing, hence by the same construction as in theorem 3.2 we get that $\{u(\cdot + x) > 0\}$ is a solution for problem (4). $\diamondsuit$

Example For the operator $A(x) = (1 + \exp(-|x|))Id$, there is no solution to problem (4). Let

$$\lambda_c = \inf\{\lambda_1(\Omega) : \Omega \quad \text{is quasi open}, \quad |\Omega| \leq c\}$$

be the first eigenvalue of the Laplace operator on a ball of measure equal to c. This infimum is obtained by a ball (of measure c) "going" to infinity. However, the infimum is not attained.

Given a general operator with non constant coefficients, it is difficult to establish whether problem (4) has a solution or not. If by any mean, one can prove a "coerciveness" result asserting that the minimum should be searched in a bounded region of $\mathbb{R}^N$, then the results of [4] would immediately give the positive answer. Nevertheless, this type of result is, in general, very difficult to obtain. The minimizer itself might be unbounded. It would be interesting to find an example of a matrix function $A(\cdot)$ for which problem (4) has an unbounded solution.

In some particular cases, as for example if the matrix A satisfies some coerciveness hypotheses which forbids to the minimizing sequence of domains to be concentrative, then the existence of a solution for problem (4) can still be proved. Let us denote by $c(x)$ the ellipticity coefficient of the matrix $A(x)$. If there exist a quasi open set Ω of measure c such that $\lambda_1(\Omega) \leq \lambda_c \liminf_{|x|\to+\infty} c(x)$ then it is easy to see that the second situation of theorem 3.2 can not occur.

If the same problem is raised for another eigenvalue, or for a function of eigenvalues, it seems to be rather difficult to describe precisely the behavior of a minimizing sequence. The main reason is because one should relay on a general concentration-compactness like result for the resolvent operators associated to $A(\cdot)$. An extension of the general result of [4] in $\mathbb{R}^N$ would be of great interest.

Open questions

1. If A has periodic coefficients, is the minimizer domain bounded?
2. If A has periodic and smooth coefficients, is the minimizer domain smooth?

For the second question, we refer the reader to [7], where a penalty technique is used to prove openness of some optimal domains in a class of shape optimization problem.

Another interesting question is to decide whether the dichotomy might occur or not when solving problem (4) for a general operator. If this would be the case, then

$$\inf\{\lambda_1(\Omega) : |\Omega| \leq c\} = \inf\{\lambda_1(\Omega) : |\Omega| \leq c - \varepsilon\}.$$

for some $\varepsilon > 0$. We do not have an example of elliptic operator for which this equality occurs.

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