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SOME CONSTANCY RESULTS FOR HARMONIC MAPS FROM NON-CONTRACTABLE DOMAINS INTO SPHERES

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Abstract. We use the Pohozaev identity on sub-domains of a Euclidean r-neighbourhood for a closed or broken curve to show that harmonic maps from such domains into spheres with constant boundary value remain constant.

§1. Introduction

In this paper we generalize a constancy result for harmonic maps from a non-star shaped domain in $\mathbb{R}^3$ to the sphere S^2 obtained by Chou and Zhu [CZ]. In [CZ] a special class of non-star shaped domains was constructed by rotating a curve which is carefully designed by using inversions in Euclidean spaces. The first result of the present paper is to generalize this result to domains including all smooth rotational ones (Theorem 1). For domains in $\mathbb{R}^m$ with $m \geq 3$, we can show that the same result holds on a tubular neighbourhood (see e.g. [S, I. Cha.9]) of a closed planar curve under a nondegeneracy condition for closed geodesic in planar domains (Theorem 3). One such example is the tubular neighbourhood of a closed convex curve such as the solid torus. When $m \geq 4$, we can show that the same claim is true for a thin tubular neighbourhood of any smooth embedded curve with an orthogonal moving frame. We state the results only for $u : \Omega \subset \mathbb{R}^3 \to S^2$ although they can be easily proved for higher dimensional cases. The only exception is Theorem 4 where we can only prove the result for domains at least in $\mathbb{R}^4$.

It is well known that if either $\Omega \subset \mathbb{R}^2$ is contractable [L] or $\Omega \subset \mathbb{R}^m$ is star-shaped with $m \geq 3$ [W], the constancy result holds. If one perturb a star-shaped domain in a C^2 manner, one expect to have the so-called 'nearly star-shaped' domains and the constancy result is still true [DZ]. It is also known that if the boundary of the domain $\partial\Omega$ is disconnected, the constancy result fails [BBC].

The method we use is the Pohozaev identity (see [P,PS,CZ]). We carefully divide the original domain into sub-domains which are thin slices of the original domain such that each sub-domain is star-shaped with respect to some specific point on a curve. We apply Pohozaev identity on each of these sub-domains and use the the constacy condition $u = u_0$ only on part of its boundary. We then obtain an inequality on each sub-domains. We sum up the resulting terms and use the definition of Riemann integral. In the limit, we obtain an inequality connecting two volume integrals. We reach our conclusions by comparing quatities on both sides

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of the inequality. Some results on shortest path in an Euclidean domain in $\mathbb{R}^2$ (or geodesics in such a domain) are used.

A smooth mapping $u : \Omega \subset \mathbb{R}^m \to S^n$ is harmonic if

$$-\Delta u = u|Du|^2 \quad \text{in} \quad \Omega \tag{1}$$

and u is a critical point of the total energy $E(u) = \int_\Omega |Du|^2 dx$.

Let $\Omega \subset \mathbb{R}^m$ be piecewise smooth and $u \in C^2(\Omega, S^n) \cap C^1(\bar{\Omega}, S^n)$ be a smooth solution of (1). Let $\nu(x) = (\nu_1(x), \cdots, \nu_m(x))$ be the outward normal vector at $x \in \partial\Omega$, and let $h = (h_1, h_2, \cdots, h_m)$ be a smooth vector field on $\bar{\Omega}$. Then (see [CZ])

$$\frac{\partial}{\partial x_\alpha}\left(h_\alpha |Du|^2 - 2h_\beta \frac{\partial u_k}{\partial x_\alpha}\frac{\partial u_k}{\partial x_\beta}\right) = \frac{\partial h_\alpha}{\partial x_\alpha}|Du|^2 - 2\frac{\partial h_\beta}{\partial x_\alpha}\frac{\partial u_k}{\partial x_\alpha}\frac{\partial u_k}{\partial x_\beta}, \tag{2}$$

where the summation convention is assumed with $1 \le \alpha$, $\beta \le m$ and $1 \le k \le n+1$. Recall that a domain Ω is star-shaped if there is a point $x_0 \in \Omega$ such that the line segment $\overline{x_0 x}$ is contained in Ω. For convenience, we call x_0 the *central point* of Ω if Ω is star-shaped with respect to x_0.

§2. Main Results

Theorem 3 below covers Theorems 1 and 2. However, since the proofs of both theorems are needed for establishing Theorem 3, we prove them separately.

Theorem 1. *Suppose $\Omega \subset R^3$ is a smooth domain and the orthogonal projection of the domain to the first component is an interval $[a, b]$. We assume that there is a $\delta > 0$, such that for all $a \le t_1 < t_2 \le b$, $|t_2 - t_1| \le \delta$, the set*

$$\Omega_{t_1,t_2} = \{x = (x_1, x_2, x_3) \in \Omega,\ t_1 \le x_1 \le t_2\}$$

is star shaped and there is some $t_0 \in [t_1, t_2]$ such that $x_0 = (t_0, 0, \cdots, 0)$ is a central point. Let $u : \bar{\Omega} \to S^2 \subset R^3$ be a smooth harmonic map such that $u(x) = u_0 \in S^2$ on $\partial\Omega$ for some constant u_0. Then $u \equiv u_0$ in $\bar{\Omega}$.

Remark 1. The rotational domains are a special case of those defined in Theorem 1. More precisely, suppose $x_2 = f(x_1) > 0$ is a smooth function defined in $[a, b]$, then the rotation of the the two dimensional region bounded by f and x_1 axis around $\mathbb{R}^{m-2}$ defines the domain. In particular, the domain we defined is much more general that the one given by [CZ].

Let $\gamma : [0, l] \to \mathbb{R}^m$ be an simple, smooth and convex curve with bounded curvatures. Then it is easy to see that the r-neighbourhood

$$\Omega_r = \{x \in \mathbb{R}^m,\ \text{dist}(x, \gamma) < r\}$$

is a tubular neighbourhood of γ with $(m-1)$-dimensional open balls of radius r as its fibres. If γ is a broken curve, Ω_r is the union of a tubular neighbourhood $\cup_{0<s<l} B_s$ and two half-balls at each end of the curve, where B_s is an $m-1$-dimensional open ball lying in the normal hyperplane of $\gamma(s)$ and is centred at $\gamma(s)$.

Theorem 2. *Let $\gamma \subset R^2$ be a smooth, closed convex curve with maximal curvature $k_0 > 0$. Let Ω_r be the r-neighbourhood of γ in $\mathbb{R}^3$ with $0 < rk_0 < 1$. Suppose $u : \bar{\Omega}_r \to S^2$ is a smooth harmonic map such that $u = u_0$ on the boundary. Then $u \equiv u_0$ in $\bar{\Omega}_r$.*

Remark 2. Let $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, and Ω_r be the tubular domain of γ defined in Theorem 2. We see that the boundary of the two-dimensional domain $\Omega_r \cap \mathbb{R}^2 \times \{0\}$ consists of two disconnected convex curves. The inner curve is a closed geodesic of $\Omega_r \cap \mathbb{R}^2 \times \{0\}$ with curvature $1 - rk(s)$. The intersection of the normal plane in $\mathbb{R}^3$ of the inner curve and Ω_r is a disc with radius r, so it is a convex two dimensional set. In fact, use these two properties of Ω_r to prove Theorem 1. In the proof we shall see that if the intersection is a copy of the same convex set up to a rigid motion, the proof can still go through. We have

Corollary 1. *Suppose γ is defined as in Theorem 2. Let $[a, b] \times \gamma$ be a closed cylinder and G_r be the r-neighbourhood of $\gamma \times [a, b]$ with $rk_0 < 1$. Then any smooth harmonic map from $\bar{G}_r$ to S^2 with constant boundary value remains a constant in $\bar{G}_r$.*

The following result is more general than Theorem 1 and 2. We need some technical assumptions on the geodesic (locally shortest path) in the tubular neigbourhood Ω_r of the curve. It is known that for a smooth, compact and connected manifold M with boundary, there is a geodesic of class $C^{1,1}$ connecting any two points in M. If the fundamental group of M is nontrivial, that is $\pi_1(M) \neq \{0\}$, then there is a nontrivial geodesic of class $C^{1,1}$ ([A,S,ABB,ABL,AB,C1,C2] and the references therein). In a closed Jordan region on the plane, the geodesics are known as the locally shortest paths [BR, BH].

We are interested in tubular neighbourhoods of an embedded planar curve γ. (closed or broken). If γ is closed, it is not hard to show that there is a closed geodesic (or locally shortest curve) which is homotopic to γ, following the arguments in [BR] or [S]. If $\gamma : [0, l] \to \mathbb{R}^2$ is broken, the r-neighbourhood $\Omega_r \cap \mathbb{R}^2 \times \{0\}$ of γ is a Jordan region for small $r > 0$ and the shortest path connecting $\gamma(0)$ and $\gamma(l)$ is unique [BR]. Furthermore, in both cases, the geodesics are of class $C^{1,1}$ (see, for example [C1,C2]).

The geometric descriptions for geodesics in domains of $\mathbb{R}^n$ can be found, for example, in [AB]: A geodesic contacting the boundary in a segment is a geodesic of the boundary (in $\mathbb{R}^2$, it is part of the boundary); a geodesic segment not touching the boundary is a straight line segment. A segment on the boundary joins a segment in the ambient space in a differentiable join. An endpoint on the boundary of a segment not touching the boundary is called a *switching point*. The accumulation points of switching points are called *intermittent points*.

We need some technical conditions on the tubular domains which exclude the intermittent points. The reason for such assumptions is purely for avoiding technical complications.

Hypothesis (H1). *If $\gamma \subset \mathbb{R}^2 \times \{0\}$ is closed and γ_0 is a closed geodesic in $\bar{\Omega}_r \cap \mathbb{R}^2 \times \{0\}$ which is homotopic to γ, where $\Omega_r \subset \mathbb{R}^3$ is a tubular neighbourhood of γ. Then*

(i) *γ_0 has finite number of switching points, hence it does not have intermittent points;*

(ii) *there is a $\delta > 0$, such that for every straight line segment $\mu \subset \gamma_0$ lying inside $\Omega_r \cap \mathbb{R}^2 \times \{0\}$ and any p, $q \in \mu$ with $|p-q| \le \delta$, the sub-domain of Ω_r bounded by normal planes of μ passing through p and q respectively is a star-shaped domain with any point on μ between p and q a central point.*

Hypothesis (H2). *If $\gamma : [0,l] \to \mathbb{R}^2$ is broken with $\gamma(0) = p \neq q = \gamma(l)$. Let $p' = p - \dot\gamma(0)r$, $q' = q + \dot\gamma(l)r$. Then for $r > 0$ sufficiently small, p', $q' \in \partial(\Omega_r \cap \mathbb{R}^2 \times \{0\})$. Let γ_0 be the geodesic in $\overline{\Omega_r \cap \mathbb{R}^2 \times \{0\}}$ connecting p' and q'. Then*

(i) *γ_0 has finite number of switching points, hence it does not have intermittent points;*

(ii) *there is a $\delta > 0$, such that for every straight line segment $\mu \subset \gamma_0$ lying inside $\Omega_r \cap \mathbb{R}^2 \times \{0\}$ and any a, $b \in \mu$ with $|a-b| \le \delta$, the sub-domain of Ω_r bounded by normal planes of μ passing through p and q respectively is a star-shaped domain with any point on μ between p and q a central point.*

Theorem 3. *$\gamma \subset R^2$ be a smooth closed or broken curve with maximal curvature $k_0 > 0$. Let Ω_r be the r-neighbourhood of γ in $\mathbb{R}^3$ with $0 < rk_0 < 1$ satisfying (H1) (or (H2) respectively). Then, if $u : \bar\Omega_r \to S^2$ is a smooth harmonic map such that u is a constant u_0 on the boundary, Then $u \equiv u_0$ in $\bar\Omega_r$.*

Notice that if γ is not a closed curve, Ω_r is not a C^2 domain. However, we can perturb it slightly near both ends of Ω_r to make it C^2. For simplicity, we just prove the result on Ω_r. Theorem 4 below deals with the constancy problem in general tubular neighbourhoods of embedded curves under a technical condition. We assume that there is a smooth orthogonal moving frame along the curve [S, Ch 1]. Suppose that $\gamma : [0,l] \to \mathbb{R}^m$ is a smooth curve parameterized by its arc-length $s \in [0,l]$. We also assume that there is a smooth orthogonal basis $e_2(s), \cdots, e_n(s)$ on the normal hyperplane of $\gamma(s)$. Let $\dot\gamma(s) = e_1(s)$. Then

$$\begin{aligned}
&\dot e_1(s) = -k_1(s)e_2,\\
&\dot e_j(s) = k_{j-1}(s)e_{j-1} - k_j(s)e_{j+1}, \quad 2 \le j \le m-1,\\
&\dot e_m(s) = k_{m-1}e_{m-1}.
\end{aligned}$$

We call $k_1(s)$ the first curvature of γ and $E(s)$ a moving orthogonal frame along γ. We have

Theorem 4. *Suppose $n \ge 2$ and $m \ge 4$. Suppose that γ is an embedded smooth curve (closed or not closed) in $\mathbb{R}^m$ with a smooth orthogonal moving frame. Let $k_1(s)$ be the first curvature of γ and Ω_r be its r-neighbourhood in $\mathbb{R}^m$. Then for sufficiently small $r > 0$, the only smooth harmonic map u from $\bar\Omega_r$ to S^n with constant boundary value u_0 is $u \equiv u_0$.*

§3. Proofs of the main results

Proof of Theorem 1. We divide $[a,b]$ evenly as $a = t_0 < t_1 < \cdots < t_N = b$, with $t_{k+1} - t_k = (b-a)/N$, $i = 0, 1, 2, \ldots, N$ such that $(b-a)/N < \delta$. Let

$$\Omega_i = \{x \in \Omega,\ t_i \le x_1 \le t_{i+1}\}$$

for $i = 0, 1, \cdots, N-1$. From the property of Ω, we see that Ω_i is star-shaped and there is some $t'_i \in [t_k, t_{k+1}]$ such that $x^i = (t'_i, 0, \cdots, 0)$ is a central point of Ω_i. We divide the boundary of Ω_i into three parts:

$$\partial\Omega_i = \Gamma_i \cup \Gamma_{i+1} \cup S_i,$$

where $\Gamma_i = \{x \in \bar{\Omega},\ x_1 = t_i\}$, and $S_i = \partial\Omega \cup \bar{\Omega}_i$. Notice that both Γ_0 and Γ_N are contained in $\partial\Omega$. Now, we apply (2) with $m = 3$ to u over the domain Ω_i for each fixed i with $h = x - x^i$. Since $\frac{\partial h_\alpha}{\partial x_\beta} = \delta_{\alpha,\beta}$ and $\frac{\partial h_\alpha}{\partial x_\alpha} = 3,$, we have,

$$\frac{\partial}{\partial x_\alpha}\left(h_\alpha|Du|^2 - 2h_\beta \frac{\partial u_k}{\partial x_\alpha}\frac{\partial u_k}{\partial x_\beta}\right) = |Du|^2 \tag{3}$$

for $x \in \Omega_i$. Integrating both sides of (3) and applying the divergence theorem, we obtain

$$\int_{\partial\Omega_i}\left((x_\alpha - x^i_\alpha)|Du|^2 - 2(x_\beta - x^i_\beta)\frac{\partial u_k}{\partial x_\beta}\frac{\partial u_k}{\partial x_\alpha}\right)\nu_\alpha dS = \int_{\Omega_i}|Du|^2 dx. \tag{4}$$

Since on S_k, Γ_0 and Γ_N, we have assumed that $u = u_0$, we have

$$(x_\beta - x^i_\beta)\frac{\partial u_k}{\partial x_\beta}\frac{\partial u_k}{\partial x_\alpha}\nu_\alpha = |Du|^2\langle(x - x^i), \nu\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^3$. Therefore (4) can be rewritten as

$$\int_{\partial\Omega_i}\big(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\big) = \int_{\Omega_i}|Du|^2 dx. \tag{4'}$$

Now, if $0 < i < N - 1$, we have

$$\begin{aligned}
&\int_{\partial\Omega_i}\big(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\big)\, dS \\
&= -\int_{S_i}|Du|^2\langle x - x^i, \nu\rangle dS \\
&\quad + \int_{\Gamma_{i+1}}\big(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\big)\, dS \\
&\quad - \int_{\Gamma_i}\big(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\big)\, dS \\
&\le \int_{\Gamma_{i+1}}\big(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\big)\, dS \\
&\quad - \int_{\Gamma_i}\big(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\big)\, dS,
\end{aligned} \tag{5}$$

where we choose the normal direction of Γ_i as towards the positive side of the x_1-axis. If $i = 0$, we have

$$\begin{aligned}
&\int_{\partial\Omega_0}\big(|Du|^2\langle x - x^0, \nu\rangle - 2\langle Du_k, x - x^0\rangle\langle Du_k, \nu\rangle\big)\, dS \\
&= -\int_{S_0}|Du|^2\langle x - x^0, \nu\rangle dS \\
&\quad + \int_{\Gamma_1}\big(|Du|^2\langle x - x^0, \nu\rangle - 2\langle Du_k, x - x^0\rangle\langle Du_k, \nu\rangle\big)\, dS \\
&\quad - \int_{\Gamma_0}|Du|^2\langle x - x^0, \nu\rangle dS \\
&\le \int_{\Gamma_1}\big(|Du|^2\langle x - x^0, \nu\rangle - 2\langle Du_k, x - x^0\rangle\langle Du_k, \nu\rangle\big)\, dS,
\end{aligned} \tag{6}$$

where the normal direction of Γ_0 is the outward normal direction of Ω_0. Similarly, When $i = N-1$, we have,

$$\begin{aligned}&\int_{\partial\Omega_{N-1}} \big(|Du|^2\langle x - x^{N-1}, \nu\rangle - 2\langle Du_k, x - x^{N-1}\rangle\langle Du_k, \nu\rangle\big)\, dS \\ &\le -\int_{\Gamma_{N-1}} \big(|Du|^2\langle x - x^{N-1}, \nu\rangle - 2\langle Du_k, x - x^{N-1}\rangle\langle Du_k, \nu\rangle\big)\, dS.\end{aligned} \tag{7}$$

Now we sum up (5), (6) and (7) for $i = 0, 1, \cdots, N-1$ and compare the resulting inequality with (4'), we have

$$\begin{aligned}&\sum_{i=0}^{N-1}\int_{\Omega_i} |Du|^2 dx = \int_\Omega |Du|^2 dx \\ &\le \sum_{i=0}^{N-1}\int_{\Gamma_{i+1}} \big(|Du|^2\langle x^{i+1} - x^i, \nu\rangle - 2\langle Du_k, x^{i+1} - x^i\rangle\langle Du_k, \nu\rangle\big)\, dS.\end{aligned} \tag{8}$$

Since $x^{i+1} - x^i = (t'_{i+1} - t'_i, 0, 0)$ and the normal vector ν on Γ_{i+1} is $(1, 0, 0)$, we have, from (8) that

$$\int_\Omega |Du|^2 dx \le \sum_{i=0}^{N-1}\left[\int_{\Gamma_{i+1}} \left(|Du|^2 - 2\left|\frac{\partial u}{\partial x_1}\right|^2\right) dS\right](t'_{i+1} - t'_i).$$

Now we let $N \to \infty$ and use the definition of Riemann integral to obtain

$$\int_\Omega Du|^2 dx \le \int_\Omega |Du|^2 dx - 2\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 dx.$$

Hence

$$\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 dx = 0, \qquad \text{so that} \qquad \frac{\partial u}{\partial x_1} = 0$$

in Ω. Thus $u \equiv u_0$. □

Proof of Theorem 2. Let $\gamma : [0, l] \to \mathbb{R}^2$ be a C^2 closed embedded convex curve parameterized by its arc-length, $\gamma(0) = \gamma(l)$. We take the curve as along the counter-clockwise direction so that when travelling along the curve, the region bounded by it is on the left side. We assume that the curvature bound $0 \le k(s) \le k_0$. We write

$$\gamma = (x_1(s), x_2(s)) \qquad \text{and} \qquad \beta(s) = (-\dot{x}_2(s), \dot{x}_1(s)),$$

where $\beta(s)$ is the unit normal vector pointing towards the interior of the region. Let $\bar{\Omega}_r$ be the closed r-neighbourhood in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, where $0 < rk_0 < 1$ so that the mapping

$$F : (s, t, z) \to \gamma(s) + t\beta(s) + ze_3^2$$

is periodic in s with period l. F is smooth and is both one-to-one and onto from $[0, l) \times \bar{B}_r(0)$ to $\bar{\Omega}_r$, with $F(0, \cdot) = F(l, \cdot)$ where e_3 is the unit vector normal to the plane and

$$\bar{B}_r(0) = \{(t, z) \in \mathbb{R}^2,\ t^2 + z^2 \le r^2\}$$

is the closed disc in $\mathbb{R}^2$. The Jacobian of this mapping is $1 - tk(s)$, where $k(s)$ is the curvature of γ. Now we take the inner curve of the tubular domain defined as $\gamma_r(s) = \gamma(s) + r\beta(s)$ and divide $[0, l]$ evenly as

$$0 = s_0 < s_1 < \cdots s_{N-1} < s_N = l, \qquad s_{k+1} - s_k = \frac{l}{N},\ k = 0, 1, \cdots N-1$$

and let s'_k be the mid-point of $[s_k, s_{k+1}]$. We let Γ_i be the intersection of the normal plane of γ_r at $s = s_i$ and Ω_r. We then define $\bar{\Omega}_i$ be the closed sub-domain of Ω_r bounded by Γ_i and Γ_{i+1}. Notice that γ_r is a closed convex curve so that $\Gamma_N = \Gamma_0$ and $\Omega_N = \Omega_0$. Similar to the proof of Theorem 1, we apply (4') to each Ω_i with $h^i(x) = x - \gamma_r(s'_i)$. We have

$$\int_{\partial\omega_i} \big(|Du|^2 \langle h^i(x), \nu\rangle - 2\langle Du_k, h^i(x)\rangle\langle Du_k, \nu\rangle\big)\, dS = \int_{\omega_i} |Du|^2 dx. \tag{9}$$

As in the proof of Theorem 1, we let I_i and J_i be the left and right hand sides of (9) and let $\partial\Omega_i = \Gamma_i \cup \Gamma_{i+1} \cup S_i$, where $S_i = \partial\Omega_i \cap \partial\Omega_r$. Let us first consider the surface integral over $S_i \subset \partial\Omega_r$ and notice that $u = u_0$ which is a constant on S_i, so (5) gives

$$\int_{S_i} \big(|Du|^2 \langle h^i(x), \nu\rangle - 2\langle Du_k, h^i(x)\rangle\langle Du_k, \nu\rangle\big)\, dS = -\int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 \langle h^i, \nu\rangle dS. \tag{10}$$

Since $\langle h^i, \nu\rangle$ is not necessarily positive on S_i from our choice of the centre $\gamma_r(s'_i)$, we need to find a bound. A general point $x \in S_i$ can be written as

$$x = \gamma(s) + t\beta(s) + ze_3$$

with $t^2 + z^2 = r^2$, for some $s \in [s_i, s_{i+1}]$ and the outward normal vector at x is

$$\nu = t\beta(s) + ze_3.$$

Recall that $\gamma_r(s'_i) = \gamma(s'_i) + r\beta(s'_i)$, we then have

$$\begin{aligned}
\langle h^i, \nu\rangle &= \langle x - \gamma_r(s'_i), \nu\rangle \\
&= \langle \gamma(s) + t\beta(s) + ze_3 - [\gamma(s'_i) + r\beta(s'_i)],\ t\beta(s) + ze_3\rangle \\
&= \langle \gamma(s) - [\gamma(s'_i) + r\beta(s'_i)], t\beta(s)\rangle + t^2 + z^2 \\
&= \langle \gamma(s) - \gamma(s'_i), t\beta(s)\rangle + r^2 - rt\langle \beta(s), \beta(s'_i)\rangle \\
&\geq t\langle \gamma(s) - \gamma(s'_i), \beta(s)\rangle.
\end{aligned}$$

Here we have used the fact that $|t| \leq r$ and $|\langle \beta(s), \beta(s'_i)\rangle| \leq 1$ because $|\beta| = 1$. Now, by using Taylor's expansion we have,

$$\begin{aligned}
& t\langle \gamma(s) - \gamma(s'_i), \beta(s)\rangle \\
&= t\langle \gamma(s) - \gamma(s'_i), \beta(s) - \beta(s'_i)\rangle + t\langle \gamma(s) - \gamma(s'_i), \beta(s'_i)\rangle \\
&= t\langle \int_{s'_i}^{s} \dot\gamma(\tau)d\tau, \int_{s'_i}^{s} \dot\beta(\eta)d\eta\rangle + t\langle \dot\gamma(s'_i)(s - s'_i) + \frac{1}{2}\ddot\gamma(\xi_i)(s - s'_i)^2, \beta(s'_i)\rangle \qquad (11) \\
&\geq -tk_0(s - s'_i)^2 - \frac{k_0}{2}(s - s'_i)^2 \geq -\frac{3rk_0}{2}\left(\frac{l}{2N}\right)^2,
\end{aligned}$$

where we have used the facts that $\dot{\gamma}(s_i') \perp \beta(s_i')$, $|\dot{\beta}(\eta)| = k(\eta) \le k_0$, and $|\ddot{\gamma}(\xi_i)| = |k(x_i)| \le k_0$, with x_i a point between s and s_i' given by the Taylor expansion. Thus the right hand side of (10) has the following upper bound

$$-\int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 \langle h^i, \nu\rangle dS \le \frac{3rk_0 l^2}{16N^2} \int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 dS.$$

Now we sum up I_i's as in the proof of Theorem 1 to obtain

$$\begin{aligned}
&\sum_{i=0}^{N-1} I_i \le \frac{3rk_0 l^2}{8N^2} \sum_{i=0}^{N-1} \int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \\
&+ \sum_{i=0}^{N-1} \int_{\Gamma_{i+1}} \big(|Du|^2 \langle \gamma_r(s_{i+1}') - \gamma_r(s_i'),\, \nu\rangle - 2\langle Du_k, \gamma_r(s_{i+1}') - \gamma_r(s_i')\rangle \langle Du_k, \nu\rangle\big)\, dS \\
&= A_1 + A_2,
\end{aligned} \tag{12}$$

where

$$A_1 = \frac{3rk_0 l^2}{8N^2} \sum_{i=0}^{N-1} \int_{S_i} \left|\frac{\partial u}{\partial \nu}\right|^2 dS = \frac{3rk_0 l^2}{16N^2} \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 dS \to 0 \tag{13}$$

as $N \to \infty$;

$$A_2 = \sum_{i=0}^{N-1} \int_{\Gamma_{i+1}} \Big(|Du|^2 \langle \gamma_r(s_{i+1}') - \gamma_r(s_i'),\, \nu\rangle - 2\langle Du_k, \gamma_r(s_{i+1}') - \gamma_r(s_i')\rangle \langle Du_k, \nu\rangle\Big) dS.$$

Notice that $\Gamma_N = \Gamma_0$, $\nu = \dot{\gamma}(s_{i+1})$,

$$\begin{aligned}
\langle \gamma_r(s_{i+1}') - \gamma_r(s_i'), \nu\rangle =& \langle \dot{\gamma}_r(s_{i+1})(s_{i+1}' - s_i'), \dot{\gamma}(s_{i+1})\rangle + \langle \frac{1}{2}\ddot{\gamma}_r(x_{i+1})(s_{i+1}' - s_{i+1})^2 \\
&- \frac{1}{2}\ddot{\gamma}_r(\eta_{i+1})(s_{i+1} - s_i')^2, \dot{\gamma}(s_{i+1})\rangle,
\end{aligned}$$

where ξ_{i+1} and η_{i+1} are two points in (s_{i+1}, s_{i+1}') and (s_i', s_{i+1}) respectively. Now we have

$$\begin{aligned}
\langle \dot{\gamma}_r(s_{i+1})(s_{i+1}' - s_i'), \dot{\gamma}(s_{i+1})\rangle &= \langle \dot{\gamma}(s_{i+1}) + r\dot{\beta}(s_{i+1}), \dot{\gamma}(s_{i+1})\rangle (s_{i+1}' - s_i') \\
&= [1 - rk(s_{i+1})](s_{i+1}' - s_i').
\end{aligned} \tag{14}$$

Since γ is of C^2, there is a constant $C_0 > 0$ such that $|\ddot{\gamma}(s)| \le C_0$ for all $s \in [0, l]$. Therefore we also have

$$\begin{aligned}
&\left|\langle \frac{1}{2}\ddot{\gamma}_r(x_{i+1})(s_{i+1}' - s_{i+1})^2 - \frac{1}{2}\ddot{\gamma}_r(\eta_{i+1})(s_{i+1} - s_i')^2, \rangle\right| \\
&\le \frac{1}{2} C_0 \left[(s_{i+1}' - s_{i+1})^2 + (s_{i+1} - s_i')^2\right] \\
&\le C_0 (s_{i+1}' - s_i')^2.
\end{aligned} \tag{15}$$

Similarly, we have

$$
\begin{aligned}
&\langle \gamma_r(s'_{i+1}) - \gamma_r(s'_i), Du_k\rangle \\
&= \langle \dot\gamma(s_{i+1}), Du_k\rangle (1 - rk(s_{i+1})(s'_{i+1} - s'_i) \\
&+ \langle \frac{1}{2}\ddot\gamma_r(x'_{i+1})(s'_{i+1} - s_{i+1})^2 - \frac{1}{2}\ddot\gamma_r(\eta'_{i+1})(s_{i+1} - s'_i)^2, Du_k\rangle,
\end{aligned}
\tag{16}
$$

with

$$
\begin{aligned}
&\left|\langle \frac{1}{2}\ddot\gamma_r(x_{i+1})(s'_{i+1} - s_{i+1})^2 - \frac{1}{2}\ddot\gamma_r(\eta_{i+1})(s_{i+1} - s'_i)^2, Du_k\rangle\right| \\
&\le C_0|Du_k|(s'_{i+1} - s'_i)^2.
\end{aligned}
\tag{17}
$$

Now we can estimate the second sum in (12) as follows

$$
\begin{aligned}
&A_2 = \\
&\sum_{i=0}^{N-1}\int_{\Gamma_{i+1}} \left(|Du|^2\langle \gamma_r(s'_{i+1}) - \gamma_r(s'_i),\, \nu\rangle - 2\langle Du_k, \gamma_r(s'_{i+1}) - \gamma_r(s'_i)\rangle\langle Du_k, \nu\rangle\right) dS \\
&\le \int_{\Gamma_{i+1}} \left(|Du|^2 - 2\langle Du_k, \dot\gamma(s_{i+1})\rangle^2\right) dS[1 - rk(s_{i+1})](s'_{i+1} - s'_i) \\
&+ C_0(s'_{i+1} - s'_i)\frac{l}{N}\int_{\Gamma_{i+1}} 3|Du|^2 dS,
\end{aligned}
\tag{18}
$$

where we have used the fact that $|\langle Du_k, \nu\rangle| \le |Du_k|$. Now we can estimate the two sums A_1 and A_2 in (12).

$$
\begin{aligned}
\sum_{i=0}^{N-1} I_i \le& A_1 + A_2 \\
\le& \frac{3rk_0l^2}{16N^2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 dS \\
&+ \int_{\Gamma_{i+1}}\left(|Du|^2 - 2\sum_{k=1}^{3}\langle Du_k, \dot\gamma(s_{i+1})\rangle^2\right) dS[1 - rk(s_{i+1})](s'_{i+1} - s'_i) \\
&+ C_0(s'_{i+1} - s'_i)\frac{l}{N}\int_{\Gamma_{i+1}} 3|Du|^2 dS.
\end{aligned}
\tag{19}
$$

Passing to the limit $N \to \infty$ in (19) and noticing that

$$
\lim_{N\to\infty} A_1 \to 0, \quad \lim_{N\to\infty} C_0\sum_{i=0}^{N-1}(s'_{i+1} - s'_i)\frac{l}{N}\int_{\Gamma_{i+1}} 3|Du|^2 dS = 0
$$

because

$$
\sum_{i=0}^{N-1}(s'_{i+1} - s'_i)\int_{\Gamma_{i+1}} 3|Du|^2 dS
$$

converges to an integral while $l/N \to 0$, we have

$$\limsup_{N\to\infty} \sum_{i=0}^{N-1} I_i \le \int_0^l \int_{\Gamma_s} |Du|^2(1-rk(s))dS\,ds - 2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{3} \langle Du_k, \dot\gamma\rangle^2 (1-rk(s))dS\,ds, \tag{20}$$

where

$$\Gamma_s = \{\gamma(s) + t\beta(s) + ze_3,\ t^2+z^2 \le r^2\}.$$

Now we sum up the right hand side of (10):

$$\sum_{i=0}^{N-1} J_i = \int_{\omega_i} |Du|^2 dx = \int_{\Omega_r} |Du|^2 dx. \tag{21}$$

We now change variables

$$x = \gamma(s) + t\beta(s) + ze_3,$$

to obtain

$$\int_{\Omega_r} |Du|^2 dx = \int_0^l \int_{\Gamma_s} |Du|^2 (1-tk(s))dS\,ds. \tag{22}$$

Finally we obtain, from (20) and (22),

$$\int_0^l \int_{\Gamma_s} |Du|^2(1-tk(s))dS\,ds \le \int_0^l \int_{\Gamma_s} |Du|^2(1-rk(s))dS\,ds - 2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{3} \langle Du_k, \dot\gamma\rangle^2 (1-rk(s))dS\,ds,$$

so that

$$\int_0^l \int_{\Gamma_s} |Du|^2(r-t)k(s)dS\,ds \le -2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{3} \langle Du_k, \dot\gamma\rangle^2 (1-rk(s))dS\,ds. \tag{23}$$

The first consequence of (23) is

$$\int_0^l \int_{\Gamma_s} \sum_{k=1}^{3} \langle Du_k, \dot\gamma\rangle^2 (1-rk(s))dS\,ds = 0,$$

so that $\langle Du_k, \dot\gamma\rangle = 0$ hence for each fixed (t,z), $u(\gamma(s)+t\beta(s)+ze_3)$ is independent of s. Now, at least in an interval $[a,b] \subset [0,l]$ with $a<b$, $k(s)>0$ hence the left hand side of (23) gives $|Du|^2 = 0$ for $s \in [a,b]$. Therefore in

$$\{\gamma(s) + t\beta(s) + ze_3,\ s \in [a,b], t^2+z^2 \le r\},$$

we have $u = u_0$. Since u is independent of s, we see that $u \equiv u_0$. $\square$

Proof of Theorem 3. If the curve γ is closed and $\Omega_r \subset \mathbb{R}^3$ is its open r-neighbourhood with $rk_0 < 1$, we let γ_0 be a closed geodesic homotopic to γ. If γ_0 does not have switching point, γ_0 must be the inner curve γ_r of Ω_r defined in the proof of Theorem 2 and it must be convex. Otherwise it is not the locally shortest geodesic. Therefore, from Theorem 2, $u \equiv u_0$. It is also obvious that if γ_0 has switching points, it must have at least two such points. Since we assumed that γ_0 has finitely many switching points, we denote them by $p_1, p_2, \cdots p_m, p_{m+1}$ with $p_1 = p_{m+1}$ such that p_k and p_{k+1} are two consequent switching points along γ_0. We parameterize γ_0 by its arc-length $\gamma_0 : [0, l] \to \bar{\Omega}_r$ with $\gamma_0(0) = p_1$. let $0 = s_1 \le s_2 < \cdots < s_m < s_{m+1} = l$ be such that $\gamma_0(s_k) = p_k$, $k = 1, 2, \cdots m + 1$. Let A_k be the sub-domain of Ω_r between the normal plane of γ_0 at p_k and p_{k+1}. The part of γ_0 in A_k is either a line segment or the shorter part of the boundary of $\mathbb{R}^2 \cap \bar{\Omega}_r \cap \bar{A}_k$ and is convex. We let $\gamma_0 \cap \bar{D}_k = \gamma_k$ with $k = 1, 2, \cdots, m$. If γ_k is a line segment, we set $B_k = A_k$. Otherwise we simply set $B_k = \emptyset$. On B_k we may use the proof of Theorem 1. If it is part of the boundary of $\Omega_r \cap \mathbb{R}^2$ in the plane, we let $C_k = A_k$. Otherwise, we let $C_k = 0$. We can use the method for proving Theorem 2. If we further divide each A_k along γ_0 and pass to the limit as in the proof of Theorem 1 or Theorem 2, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{m} \mu(k) \int_{B_k} |Du|^2 dx + \sum_{k=1}^{m} (1 - \mu(k) \int_{s_k}^{s_{k+1}} \int_{\Gamma_s} |Du|^2 (1 - tk(s)) dS\, ds \\
&\le \sum_{k=1}^{m} \mu(k) \int_{B_k} |Du|^2 dx + \sum_{k=1}^{m} (1 - \mu(k)) \int_{s_k}^{s_{k+1}} \int_{\Gamma_s} |Du|^2 (1 - rk(s)) dS\, ds \qquad (24) \\
&\quad - 2(1 - rk_0) \sum_{k=1}^{m} \int_{s_k}^{s_{k+1}} \int_{\Gamma_s} \sum_{j=1}^{3} \langle Du_j, \dot{\gamma} \rangle^2 dS\, ds,
\end{aligned}
$$

where $\mu(k) = 1$ if $A_k = B_k$, $\mu(k) = 0$ otherwise. We have also used the estimate $1 - tk(s) \ge 1 - rk_0$ and $1 \ge 1 - rk_0$ for the last sum on the right hand side of (24). A similar argument as in the proof of Theorems 1 and 2 can conclude the proof. If γ is a broken curve, the proof is similar. Let $\gamma(0) = p$, $\gamma(l) = q$. We define two half-balls as

$$
\begin{aligned}
B_r^- &= \{-s\dot{\gamma}(0) + t\beta(0) + ze_3,\ s \ge 0,\ s^2 + t^2 + z^2 < r^2\}, \\
B_r^+ &= \{s\dot{\gamma}(l) + t\beta(0) + ze_3,\ s \ge 0,\ s^2 + t^2 + z^2 < r^2\}
\end{aligned}
$$

and let

$$
\Omega = \{\gamma(s) + t\beta(s) + ze_3,\ t^2 + z^2 \le r^2\},
$$

we have $\bar{\Omega}_r = \bar{B}_r^- \cup \bar{\Omega} \cup \bar{B}_r^+$, $B_r^\pm \cap \Omega = \emptyset$, and $B_r^- \cap B_r^+ = \emptyset$ at least when $r > 0$ is sufficiently small. We now extend γ to $\partial\Omega$ smoothly (C^1) by defining

$$
\begin{aligned}
\gamma_-(s) &= p - s\dot{\gamma}(0), \quad 0 \le s <\le r, \\
\gamma_+(s) &= q + s\dot{\gamma}(l), \quad 0 \le s \le r.
\end{aligned}
$$

Then $\gamma_- \subset \bar{B}_r^-$, and $\gamma_+ \subset \bar{B}_r^+$. We see that $\gamma_-(r), \gamma_+(r) \in \partial\Omega_r$. If we divide B_r^- along γ_- by using normal planes of γ_-, each sub-domain between two planes is star-shaped with respect to points on γ_- in the sub-domain. Similarly, we can do the same for B_r^+. Let $\gamma_0 = \gamma_- \cup \gamma \cup \gamma_+$. Then if we divide Ω_r along γ_0 by using normal planes of γ_0 and use the Pohozaev identity on each sub-domain and follow the argument for the case of closed curves, we may conclude the proof. □

Proof of Theorem 4. We use a similar idea as that in the proof of theorem 2 and theorem 3. If γ is closed, we divide Ω_r along γ itself instead of γ_r. if $\Omega \subset \mathbb{R}^m$ with $m \geq 4$, formula (4') should be changed into

$$\int_{\partial\Omega_i} \left(|Du|^2\langle x - x^i, \nu\rangle - 2\langle Du_k, x - x^i\rangle\langle Du_k, \nu\rangle\right) = (m-2)\int_{\Omega_i} |Du|^2 dx.$$

Recall that the Jacobian of the mapping

$$(s, x_2, x_3, \cdots, x_m) \to \gamma(s) + x_2 e_2(s) + \cdots, x_m e_m(s)$$

is $1 - x_2 k_1(s)$, we may follow the arguments similar to the proof of Theorem 2 by using γ as the central curve of Ω_r to obtain

$$\begin{aligned}(m-2)\int_{\Omega_r} |Du|^2 dx =&(m-2)\int_0^l \int_{\Gamma_s} |Du|^2 (1 - x_2 k_1(s)) dS\, ds \\ &\leq \int_0^l \int_{\Gamma_s} |Du|^2 dS\, ds - 2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{n+1} \langle Du_k, \dot\gamma\rangle^2 dS\, ds,\end{aligned} \tag{25}$$

where

$$\Gamma_s = \{\gamma_s + x_2 e_2(s) + \cdots, x_m e_m(s),\ x_2^2 + \cdots + x_m^2 \leq r\}.$$

Since $1 - rk_0 \leq 1 - x_2 k_1(s) \leq 1$, we estimate the right hand side of (25) as follows:

$$\begin{aligned}&\int_0^l \int_{\Gamma_s} |Du|^2 dS\, ds - 2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{n+1} \langle Du_k, \dot\gamma\rangle^2 dS\, ds \\ &\leq \frac{1}{1 - rk_0}\int_0^l \int_{\Gamma_s} |Du|^2 (1 - x_2 k_1(s)) dS\, ds - 2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{n+1} \langle Du_k, \dot\gamma\rangle^2 dS\, ds \qquad (26) \\ &= \frac{1}{1 - rk_0}\int_{\Omega_r} |Du|^2 dx - 2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{n+1} \langle Du_k, \dot\gamma\rangle^2 dS\, ds,\end{aligned}$$

Combining (25) and (26) we obtain

$$\left(m - 2 - \frac{1}{1 - rk_0}\right)\int_{\Omega_r} |Du|^2 dx \leq -2\int_0^l \int_{\Gamma_s} \sum_{k=1}^{n+1} \langle Du_k, \dot\gamma\rangle^2 dS\, ds. \tag{27}$$

Now, since $m \geq 4$, $m - 2 \geq 2$, we may have

$$m - 2 - \frac{1}{1 - rk_0} > 0, \quad \text{if } 0 < r < \frac{m-3}{k_0(m-2)}.$$

This is possible if $r > 0$ is sufficiently small, so that $|Du|^2 = 0$ in Ω_r hence $u = u_0$ in $\bar\Omega_r$. □

Remark 3. The methods for proving Theorem 4 can be used to establish similar uniqueness results for the Dirichlet problem $-\Delta u + |u|^{p-1}u = 0$ with $u = 0$ on $\partial\Omega$ [Z] at least for $p > (n+1)/(n-3)$ in a tubular neighbourhood of a closed or broken curve in $\mathbb{R}^n$ with $n \geq 4$. One can only divide the domain along the central curve because the corresponding energy density is $F(u, Du) = |Du|^2/2 - |u|^{p+1}/(p+1)$ which is not necessarily positive. Therefore the approach in Theorem 2 and Theorem 3 of using the shortest path does not improve the result.

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