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Multiplicity results for classes of one-dimensional p-Laplacian boundary-value problems with cubic-like nonlinearities *

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Abstract

We study boundary-value problems of the type

$$\begin{gathered} -(\varphi_p(u'))' = \lambda f(u), \text{ in } (0,1) \\ u(0) = u(1) = 0, \end{gathered}$$

where $p > 1$, $\varphi_p(x) = |x|^{p-2}\, x$, and $\lambda > 0$. We provide multiplicity results when f behaves like a cubic with three distinct roots, at which it satisfies Lipschitz-type conditions involving a parameter $q > 1$. We shall show how changes in the position of q with respect to p lead to different behavior of the solution set. When dealing with sign-changing solutions, we assume that f is *half-odd*; a condition generalizing the usual oddness. We use a quadrature method.

1 Introduction

We consider a quasilinear Dirichlet boundary-value problem of the type

$$\begin{gathered} -(\varphi_p(u'))' = \lambda f(u), \text{in } (0,1) \\ u(0) = u(1) = 0, \end{gathered} \tag{1}$$

where $p > 1$, $\varphi_p(x) = |x|^{p-2}\, x$, $\lambda > 0$, and $f \in C(\mathbb{R}, \mathbb{R})$ is a cubic-like nonlinearity to be specified below. We study non-existence, existence, and multiplicity results. In some cases, the exact number of solutions to (1) is given.

Several studies related to Dirichlet problems with cubic and cubic-like nonlinearities are reviewed in Section 2. The purpose of this work is to study the solution set of Problem (1) when f is cubic-like but *not necessarily an odd*

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function. We take f to be a nonlinearity satisfying:

$$f \in C(\mathbb{R},\mathbb{R}) \text{ and } f(\alpha_-) = f(\alpha_+) = f(0) = 0 \text{ for constants } \alpha_- < 0 < \alpha_+ \tag{2}$$

$$f(x) > 0 \quad \text{for } x \in (-\infty, \alpha_-) \cup (0, \alpha_+); \tag{3}$$

$$f(x) < 0 \quad \text{for } x \in (\alpha_-, 0) \cup (\alpha_+, +\infty); \tag{4}$$

$$\lim_{s\to 0} \frac{f(s)}{\varphi_q(s)} = a_0 > 0 \text{ for some } q > 1; \tag{5}$$

there exist $\delta > 0$, $m_\pm > 0$ and $M_\pm > 0$ such that

$$\begin{aligned} -m_+ &\geq \frac{f(\alpha_+) - f(\xi)}{\varphi_q(\alpha_+ - \xi)} \geq -M_+, \text{ for all } \xi \in (\alpha_+ - \delta, \alpha_+), \\ -m_- &\geq \frac{f(\alpha_-) - f(\xi)}{\varphi_q(\alpha_- - \xi)} \geq -M_-, \text{ for all } \xi \in (\alpha_-, \alpha_- + \delta). \end{aligned} \tag{6}$$

It is well known that when f is *odd*, the conditions above are sufficient for studying constant sign solutions as well as sign-changing solutions to (1). However, when f is not necessarily odd, only constant sign solutions can be handled with these conditions. In order to study sign-changing solutions with f not necessarily odd, we introduce some functions generalizing odd ones.

Let us assign to each function f defined on $\mathbb{R}$, the function h_f defined on $[0, +\infty)$ by $h_f(x) = f(x) + f(-x)$, for all $x \in [0, +\infty)$. Notice that the oddness of a function f on $\mathbb{R}$ may be characterized by the condition: $h_f \equiv 0$ on $[0, +\infty)$. Therefore, if $I \subset [0, +\infty)$ is a non-empty set, we shall say that:

- f is *positively half-odd* (*p.h.o.*, for brevity) on $I \cup (-I)$, if

$$h_f(x) \geq 0, \text{ for all } x \in I, \tag{7}$$

- f is *negatively half-odd* (*n.h.o.*, for brevity) on $I \cup (-I)$, if

$$h_f(x) \leq 0, \text{ for all } x \in I. \tag{8}$$

Also, we shall say that f is *strictly positively half-odd* (*s.p.h.o.*, for brevity) on $I \cup (-I)$ (resp. *strictly negatively half-odd* (*s.n.h.o.*, for brevity) on $I \cup (-I)$) if the strict inequality holds in (7) (resp. in (8)).

Half-even functions may be defined analogously. Assign to each function $f \colon \mathbb{R} \to \mathbb{R}$, g_f defined on $[0, +\infty)$ by $g_f(x) = f(x) - f(-x)$. Note that f is even if and only if $g_f(x) = 0$ for all $x \in [0, +\infty)$. If $I \subset [0, +\infty)$ is a non-empty set, we say that f is positively (resp., negatively) half-even on $I \cup (-I)$ if $g_f(x) \geq 0$ (resp., $g_f(x) \leq 0$) for all $x \in I$. (Note that the criteria for f to be half-odd (resp., half-even) on $I \cup (-I)$ involve only behavior of h_f (resp., g_f) on $I \subset [0, +\infty)$, not on $I \cup (-I)$.)

When dealing with sign-changing solutions to (1) we shall assume that

$$f \text{ is } p.h.o. \quad \text{on } [\alpha_-, -\alpha_-]. \tag{9}$$

Note that condition (9) implies that the function

$$u \mapsto F(u) := \int_0^u f(\xi)d\xi \quad \text{is } p.h.e. \text{ on } [\alpha_-, -\alpha_-]. \tag{10}$$

Indeed, let $h(\xi) := F(\xi) - F(-\xi)$ for all $\xi \in [0, -\alpha_-]$. One has $h(0) = 0$ and $h'(\xi) = f(\xi) + f(-\xi) \geq 0$ for all $\xi \in [0, -\alpha_-]$. Thus $F(\xi) - F(-\xi) \geq 0$ for all $\xi \in [0, -\alpha_-]$.

Also, notice that condition (9) together with (2), (4) imply that

$$0 < -\alpha_- \leq \alpha_+. \tag{11}$$

Indeed, if $\alpha_+ < -\alpha_-$, $f(-\alpha_-) < 0$ from (4). But $f(-\alpha_-) = f(-\alpha_-) + f(\alpha_-) \geq 0$. A contradiction.

On the other hand, (9) together with (3), (4) imply that

$$\begin{gathered} \text{for all } x \in [\alpha_-, 0) \text{ there exists a unique } y(x) \in (0, -x] \text{ such that} \\ \text{for all } t \in (0, -x],\ F(t) = F(x) \text{ if and only if } t = y(x). \end{gathered} \tag{12}$$

Indeed, the function $k(t) := F(t) - F(x)$ for all $t \in [0, -x]$, satisfies $k'(t) = f(t) > 0$ for all $t \in (0, -x)$, from (3), $k(0) = -F(x) < 0$, from (4) and $k(-x) = F(-x) - F(-(-x)) \geq 0$.

To prove exact multiplicity results, we shall need in the case where $1 < p < q$, the condition

$$f \in C^2(I_\pm(\alpha_\pm)) \text{ and } \pm\left((p-2)(p-1)f(x) - x^2 f''(x)\right) > 0, \tag{13}$$

for all $x \in I_\pm(\alpha_\pm)$, where, for all $z > 0$ (resp. $z < 0$), $I_+(z)$ (resp. $I_-(z)$) designates the open interval $(0, z)$ (resp. $(z, 0)$). Notice that (13) holds if, for example, f satisfies (2)-(4) and

$$f \in C^2(I_\pm(\alpha_\pm)), \quad p \geq 2 \text{ and } \pm f'' < 0 \quad \text{in } I_\pm(\alpha_\pm). \tag{14}$$

In the case where $1 < q \leq p$ we shall make some assumptions concerning the variations of the function $x \mapsto H(x) := p\int_0^x f(t)dt - xf(x)$. Namely, we use the condition,

$$\pm H(\cdot) \text{ is strictly increasing in } I_\pm(\alpha_\pm). \tag{15}$$

A convention. For all integer $k \geq 1$ and $\kappa = +, -$, we shall say that f satisfies $(\mathbf{H})_k^\kappa$ if f satisfies

- $(6)_+$ in the case where $k = 1$ and $\kappa = +$, or,
- $(6)_-$ in the case where $k = 1$ and $\kappa = -$, or,
- (9) and $(6)_-$ in the case where $k \geq 2$ and $\kappa \in \{+, -\}$.

Also, we shall say that f satisfies $(\mathbf{K})_k^\kappa$ if f satisfies

- $(15)_+$ in the case where $k=1$ and $\kappa=+$, or,
- $(15)_-$ in the case where $k=1$ and $\kappa=-$, or,
- $(15)_+$ and $(15)_-$ in the case where $k\geq 2$ and $\kappa\in\{+,-\}$.

Recall [20] that the first eigenvalue of

$$\begin{aligned} -(|u'|^{p-2}\,u')' &= \lambda\,|u|^{p-2}\,u \ \text{ in } (0,1) \\ u(0) &= u(1)=0, \end{aligned}$$

is given by $\lambda_1(p)=(p-1)[2\int_0^1\{1-\xi^p\}^{-\frac{1}{p}}\,d\xi]^p=(p-1)(2\pi/p\sin(\frac{\pi}{p}))^p$ and the other eigenvalues constitute the sequence $\lambda_1(p)<\lambda_2(p)<\cdots<\lambda_k(p)<\cdots$, $\lambda_n(p)=n^p\lambda_1(p)$.

The method we use is the quadrature method. This one enable us to look for solutions of (1) in some prescribed subsets of $C^1([0,1])$. For any $k\in\mathbb{N}^*$, let

$$S_k^+=\left\{\begin{array}{r} u\in C^1([\alpha,\beta]): u \text{ admits exactly } (k-1) \text{ zeros in } (\alpha,\beta) \\ \text{all simple, } u(\alpha)=u(\beta)=0 \text{ and } u'(\alpha)>0 \end{array}\right\},$$

$S_k^-=-S_k^+$ and $S_k=S_k^+\cup S_k^-$.

Definition. *Let $u\in C([\alpha,\beta])$ be a function with two consecutive zeros $x_1<x_2$. We call the I-hump of u the restriction of u to the open interval $I=(x_1,x_2)$. When there is no confusion we refer to a hump of u.*

Observe that each function in S_k^+ has exactly k humps such that the first one is positive, the second is negative, and so on with alternations. Let A_k^+ $(k\geq 1)$ be the subset of S_k^+ consisting of the functions u satisfying:

- Every hump of u is symmetrical about the center of the interval of its definition.
- Every positive (resp. negative) hump of u can be obtained by translating the first positive (resp. negative) hump.
- The derivative of each hump of u vanishes once and only once.

Let $A_k^-=-A_k^+$ and $A_k=A_k^+\cup A_k^-$. Now we are ready to state the main results. The first one concerns the case where $1<q<p$.

Theorem 1.1 *Let $1<q<p$.*

(i) *If (2)-(5) and $(6)_\pm$ hold, there exists $J_\pm>0$, such that Problem (1) admits at least a solution in $A_1^\pm$ for all $\lambda\in(0,J_\pm]$. Moreover, if $(15)_\pm$ holds then,*

- *If $0<\lambda\leq J_\pm$, Problem (1) admits a unique solution in $A_1^\pm$.*
- *If $\lambda>J_\pm$, Problem (1) admits no solution in $A_1^\pm$.*

(ii) *If (2)-(5), $(6)_-$, and (9) hold, there exists (beside J_-) a positive number $J_* > 0$ such that, for all integer $n \in \mathbb{N}^*$,*

(a) *If $0 < \lambda \leq (nJ_- + nJ_*)^p$, Problem (1) admits at least a solution in A_{2n}^{κ}, for all $\kappa \in \{-, +\}$.*

(b) *If $0 < \lambda \leq (nJ_- + (n+1)J_*)^p$, Problem (1) admits at least a solution in A_{2n+1}^{+}.*

(c) *If $0 < \lambda \leq ((n+1)J_- + nJ_*)^p$, Problem (1) admits at least a solution in A_{2n+1}^{-}.*

Moreover, if both $(15)_-$ and $(15)_+$ hold, it follows that

(a1) *If $0 < \lambda \leq (nJ_- + nJ_*)^p$, Problem (1) admits a unique solution in A_{2n}^{κ}, for all $\kappa \in \{-, +\}$.*

(a2) *If $\lambda > (nJ_- + nJ_*)^p$, Problem (1) admits no solution in A_{2n}^{κ}, for all $\kappa \in \{-, +\}$.*

(b1) *If $0 < \lambda \leq (nJ_- + (n+1)J_*)^p$, Problem (1) admits a unique solution in A_{2n+1}^{+}.*

(b2) *If $\lambda > (nJ_- + (n+1)J_*)^p$, Problem (1) admits no solution in A_{2n+1}^{+}.*

(c1) *If $0 < \lambda \leq ((n+1)J_- + nJ_*)^p$, Problem (1) admits a unique solution in A_{2n+1}^{-}.*

(c2) *If $\lambda > ((n+1)J_- + nJ_*)^p$, Problem (1) admits no solution in A_{2n+1}^{-}.*

Theorem 1.2 *Let $1 < p = q$. For all $k \geq 1$ and $\kappa = +, -$, assume that (2)-(5), and $(\mathbf{H})_k^{\kappa}$ hold. Then, Problem (1) admits at least a solution in A_k^{κ} for all $\lambda > \lambda_k / a_0$. Moreover, if $(\mathbf{K})_k^{\kappa}$ holds, it follows that*

(a) *If $\lambda > \lambda_k / a_0$, Problem (1) admits a unique solution in A_k^{κ}.*

(b) *If $\lambda \leq \lambda_k / a_0$, Problem (1) admits no solution in A_k^{κ}.*

Theorem 1.3 *Let $1 < p < q$.*

(i) *If (2)-(5), and $(6)_{\pm}$ hold, then there exists a real number $\mu_1^{\pm} > 0$ such that*

- *If $\lambda < \mu_1^{\pm}$, Problem (1) admits no solution in $A_1^{\pm}$.*
- *If $\lambda = \mu_1^{\pm}$, Problem (1) admits at least a solution in $A_1^{\pm}$.*
- *If $\lambda > \mu_1^{\pm}$, Problem (1) admits at least two solutions in $A_1^{\pm}$.*

Moreover, if $(13)_{\pm}$ holds, it follows that

- *If $\lambda < \mu_1^{\pm}$, Problem (1) admits no solution in $A_1^{\pm}$.*
- *If $\lambda = \mu_1^{\pm}$, Problem (1) admits a unique solution in $A_1^{\pm}$.*
- *If $\lambda > \mu_1^{\pm}$, Problem (1) admits exactly two solutions in $A_1^{\pm}$.*

(ii) *If (2)-(5),* $(6)_-$ *and (9) hold, then there exist two strictly increasing sequences* $(\mu_k)_{k\geq 2}$ *and* $(\nu_k)_{k\geq 2}$ *such that*

$$\mu_k > \nu_k > 0 \quad \textit{for} \quad \textit{all} \quad k \geq 2, \quad \textit{and} \quad \lim_{k\to+\infty} \mu_k = \lim_{k\to+\infty} \nu_k = +\infty,$$

and such that, for all $k \geq 2$*,*

- *If* $\lambda > \mu_k$*, Problem (1) admits at least two solutions in* $A_k^{\pm}$*.*
- *If* $0 < \lambda < \nu_k$*, Problem (1) admits no solution in* $A_k^{\pm}$*.*

Regarding the results described in Section 2, Theorems 1.1, 1.2, and 1.3 seem to be new even when $p = 2$.

The paper is organized as follows. Section 2 is dedicated to some review related to boundary-value problems with cubic, and cubic-like nonlinearities. The quadrature method used for proving our results is recalled in Section 3. Some preliminary lemmas are the aim of Section 4. The main results are proved in Section 5. We close our study by listing some open questions in Section 6.

2 Some known results

In this section we shall present some results concerning boundary-value problems with cubic, and cubic-like nonlinearities. We shall not attempt to make a complete historical review.

The classical paper by Smoller and Wasserman [25] deals with semilinear problems when the nonlinearity is cubic. They consider the boundary-value problem

$$\begin{gathered} -u''(y) = f(u(y)), \quad y \in (-\lambda, \lambda) \\ u(-\lambda) = u(\lambda) = 0 \end{gathered} \tag{16}$$

with the cubic nonlinearity $f(u) = -(u-a)(u-b)(u-c)$, and $a < b < c$ are its real roots. They show that the solution set depends strongly on the position of the roots of f. Notice that the change of variable $y = \sqrt{\lambda}x$ transforms Problem (16) to

$$\begin{gathered} -u''(x) = \lambda f(u(x)), \quad x \in (-1, 1) \\ u(-1) = u(1) = 0. \end{gathered} \tag{17}$$

In the case where $0 = a < b < c$, they show the existence of a critical $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, Problem (17) admits no nontrivial positive solution, it has exactly one positive solution at $\lambda = \lambda_0$, and it has exactly two positive solutions for $\lambda > \lambda_0$.

Concerning the case $0 < a < b < c$, their study was completed by Wang [27] who showed that, under an additional condition, the behavior of the solution set of (17) is the same as that of the case $0 = a < b < c$.

Smoller and Wasserman [25] have also studied some cases when f has one or two negative roots, and they have studied the same equation with Neumann and periodic boundary conditions.

Notice that in the two papers mentioned above, autonomous boundary value problems with cubic nonlinearities were studied.

Korman and Ouyang [13] consider the non-autonomous problem

$$\begin{gathered} -u'' = \lambda f(x,u),\ x \in (-1,1), \\ u(-1) = u(1) = 0 \end{gathered} \tag{18}$$

where $\lambda > 0$ is a real parameter, and f is the cubic nonlinearity

$$f(x,u) = a(x)u^2(1 - b(x)u),\quad x \in (-1,1), u \in \mathbb{R}, \tag{19}$$

and $a(x)$, $b(x)$ are even functions, $a(x) \in C^1(-1,1) \cap C^0[-1,1]$, $b(x) \in C^2(-1,1) \cap C^0[-1,1]$ satisfying

$$\begin{gathered} a(x), b(x) > 0 \quad \text{for } -1 \le x \le 1. \\ xb'(x) > 0 \qquad xa'(x) < 0 \quad \text{for } x \in (-1,1) \setminus \{0\} \\ b''(x)b(x) - 2(b')^2(x) > 0 \quad \text{for } -1 < x < 1\,. \end{gathered}$$

They show that Problem (18) admits exactly two solutions for large λ's, admits no solution for small λ's, and admits finitely many solutions for the other values of λ. More precisely they prove:

Theorem 2.1 *[13] There exists a critical λ_1, such that for $0 < \lambda < \lambda_1$ Problem (18) has no solution; it has at least one solution at $\lambda = \lambda_1$; and it has at least two solutions for $\lambda > \lambda_1$. All solutions lie on a single curve of solutions, which is smooth in λ. For each $\lambda > \lambda_1$ there are finitely many solutions, and different solutions are strictly ordered on $(-1,1)$. Moreover, there exists $\lambda_2 \ge \lambda_1$, so that for $\lambda > \lambda_2$ Problem (18) has exactly two solutions denoted by $u^-(x,\lambda) < u^+(x,\lambda)$, with $u^+(x,\lambda)$ strictly monotone increasing in λ, $u^-(0,\lambda)$ strictly monotone decreasing in λ, and $\lim_{\lambda\to\infty} u^+(x,\lambda) = 1/b(x)$, $\lim_{\lambda\to\infty} u^-(x,\lambda) = 0$ for all $x \in (-1,1)$. (All solutions of (18) are positive by the maximum principle.)*

Remark that the nonlinearity (19) has a double root $u_0 = 0$ and a simple positive root $u_1 = 1/b(x)$. A case where the nonlinearity of the problem admits three simple roots was also studied by Korman and Ouyang [13]. Indeed, they consider the cubic nonlinearity

$$f(x,u) = u(u - a(x))(b - u),\quad x \in (-1,1),\ u \in \mathbb{R},$$

but this time, b is a positive constant, and the function $a(x) \in C^1[-1,1]$ satisfies the following conditions:

$$\begin{gathered} a(x) \ge a_0 > 0, a'(x) > 0 \text{ for } x \in (0,1) \\ a(-x) = a(x) \text{ for } x \in (-1,1). \\ a(x) < \frac{1}{2}b \text{ for all } x \in (-1,1). \end{gathered}$$

By the maximum principle every solution satisfies $0 < u < b$ in $(-1, 1)$. Thus, they prove:

Theorem 2.2 *[13] There exists a critical λ_1, such that for $0 < \lambda < \lambda_1$ Problem (18) has no solution; it has at least one solution at $\lambda = \lambda_1$; and it has at least two solutions for $\lambda > \lambda_1$. All solutions lie on a single smooth curve of solutions. For each $\lambda > \lambda_1$ there are finitely many solutions, and different solutions are strictly ordered. Moreover, there exists $\lambda_2 \geq \lambda_1$ so that for $\lambda > \lambda_2$ Problem (18) has exactly two solutions denoted by $u^-(x, \lambda) < u^+(x, \lambda)$, and $\lim_{\lambda\to\infty} u^+(x, \lambda) = b$ for all $x \in (-1, 1)$. Solution $u^-(x, \lambda)$ develops a spike layer at $x = 0$ as $\lambda \to \infty$.*

Notice that the cubic nonlinearity $f(x, u)$ in Theorems 2.1 and 2.2 is such that

$$x\frac{\partial f}{\partial x}(x, u) < 0 \quad \text{for} \quad x \neq 0. \tag{20}$$

Next, Korman and Ouyang [15] have studied Problem (18) when the condition (20) is violated. Indeed, they consider

$$f(x, u) = (u - a)(u - b(x))(c(x) - u), \ \text{ for } x \in (-1, 1), u \in \mathbb{R},$$

a is a constant, $b(x)$ and $c(x)$ are even functions and of class $C^1(-1, 1) \cap C^0[-1, 1]$, satisfy the following conditions:

$$0 < a < b(x) < c(x) \ \text{ for all } x \in (-1, 1), \tag{21}$$

$$c''(x) < 0 \ \text{ for all } x \in (-1, 1) \tag{22}$$

$$b'(x) + c'(x) \geq 0 \ \text{ for all } x \in (0, 1) \tag{23}$$

$$c'(x) < 0 \ \text{ for all } x \in (0, 1). \tag{24}$$

First, they show that any solution satisfies

$$0 < u(x) < c(x) \ \text{ for all } x \in (-1, 1),$$

and prove the following

Theorem 2.3 *[15] Assume that a, $b(x)$ and $c(x)$ satisfy (21)-(24). Assume in addition that*

$$\int_{-1}^{1} F(x, a)dx < \int_{-1}^{1} F(x, c(x))dx. \tag{25}$$

All solutions of (18) lie on at most countably many unbounded smooth solution curves. One of the curves, referred to as the lower curve, starts at $\lambda = 0$, $u = 0$, it is strictly increasing in λ, and $\lim_{\lambda\to\infty} u(x, \lambda) = a$ for all $x \in (-1, 1)$. Each upper curve has two branches $u^-(x, \lambda) < u^+(x, \lambda)$, and as $\lambda \to \infty$, $u^-(x, \lambda)$ tends to a for all $x \in (-1, 1)\setminus\{0\}$. For $u^+(x, \lambda)$ there is a $p \in (0, 1)$, such that as $\lambda \to \infty$, $u^+(x, \lambda)$ tends to $c(x)$ for $x \in (-p, p)$ and to a for $x \in (-1, 1)\setminus(-p, p)$. The number p is the same for all upper curves. Each upper curve has at most finitely many turns for λ belonging to any bounded interval.

Concerning the asymptotic behavior of solutions on the upper curve, they obtained a detailed information. They consider the condition

$$F(x,a) < F(x,c(x)) \text{ for all } x \in (-1,1). \tag{26}$$

Denoting $r_1(x) < r_2(x)$ the roots of $\frac{\partial f}{\partial x}(x,u)$, they assume that

$$r_2(x) < c(1) \text{ for all } x \in (-1,1). \tag{27}$$

They proved the following

Theorem 2.4 *[15] Assume all conditions of Theorem 2.3 hold with the condition (25) replaced by (26) and assume additionally (27). Then all of the conclusions of Theorem 2.3 hold and, in addition, the upper curve is unique and it consists for λ sufficiently large of two branches, referred to as an upper and lower branch, $u^+(x,\lambda) > u^-(x,\lambda)$ for all x, and $\lim_{\lambda\to+\infty} u^+(x,\lambda) = c(x)$ for all $x \in (-1,1)$, $\lim_{\lambda\to+\infty} u^-(x,\lambda) = a$ for all $x \in (-1,1)\setminus\{0\}$, and $u^-(0,\lambda) > b(0)$ for all λ (i.e. the lower branch approaches a spike-layer). In particular, for sufficiently large λ Problem (18) has exactly three solutions.*

Notice that Theorems 2.3 and 2.4 do not provide exact multiplicity results *for all* $\lambda > 0$. Also, in Theorems 2.1 and 2.2, the exact number of solutions for $\lambda_1 \le \lambda \le \lambda_2$ remains, in [13], an open question[1].

In this direction, Korman and Ouyang [14] have studied Problem (18) when the cubic nonlinearity is given by

$$f(x,u) = u(u-a(x))(b(x)-u), \text{for } x \in (-1,1), \text{and } u \in \mathbb{R},$$

that is, the nonlinearity has three distinct roots

$$0 < a(x) < b(x), \text{ for all } x \in (-1,1),$$

and proved a distinguished result. Indeed they find the *exact* number of solutions to (18) *for all* $\lambda > 0$. Notice that here the two positive roots may depend on x. They assume that

$$a(x) \text{ and } b(x) \text{ are even functions of class } C^2(-1,1) \cap C^0[-1,1],$$

and

$$b''(x) < 0 \quad \text{for all } x \in (-1,1). \tag{28}$$

Letting $\alpha(x) = a(x) + b(x)$ and $\beta(x) = a(x)b(x)$, for all $x \in (-1,1)$ they assume

[1] Concerning Theorem 2.1, Korman and Ouyang [13] believed, based on numerical evidence, that at $\lambda = \lambda_1$ the solution is unique, while for $\lambda > \lambda_1$ there are exactly two solutions.

that these even functions satisfy the conditions

$$\alpha'(x) < 0 \quad \text{for } x \in (0,1), \tag{29}$$

$$\beta'(x) > 0 \quad \text{for } x \in (0,1), \tag{30}$$

$$\alpha'''(x) \leq 0 \text{ for all } x \in (0,1), \tag{31}$$

$$\alpha(x) - \sqrt{\alpha^2(x) - 3\beta(x)} < \alpha(1) \text{ for all } x \in (0,1), \tag{32}$$

$$\frac{1}{2}\alpha(0) < \frac{\alpha(x) + \sqrt{\alpha^2(x) - 3\beta(x)}}{3} \quad \text{for all } x \in (0,1). \tag{33}$$

Here again they noted that any nontrivial solution of (18) is positive by the maximum principle. First they prove an alternative result:

Theorem 2.5 *[14] For Problem (18) assume that the conditions (28)-(33) are satisfied. Then only two possibilities can occur:*

(A) *Problem (18) has no nontrivial solution for any $\lambda > 0$.*

(B) *There is a $\lambda_0 > 0$ so that Problem (18) has either zero, one, or two solutions depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$, or $\lambda > \lambda_0$, respectively. Moreover, all solutions are even functions and lie on a single $\subset$-like curve. Solutions on the lower branch tend to zero over $(-1,1)\setminus\{0\}$, and moreover the maximum value of solutions on the lower branch decreases monotonously.*

Next, they give a condition ensuring existence of a positive solution of (18) for some $\lambda > 0$, thus they obtained an exact multiplicity result for all $\lambda > 0$.

Theorem 2.6 *[14] In addition to the conditions of Theorem 2.5 assume that $\int_0^1 F(x, b(x))dx > 0$, where $F(x,u) = \int_0^u f(x,t)dt$. Then, Possibility (B) of Theorem 2.5 holds. If moreover $F(x, b(x)) > 0$ for all $x \in (-1,1)$ then the upper branch tends to $b(x)$ over $(-1,1)$ as $\lambda \to \infty$.*

Observe that all the results described from the beginning of this section are concerned by polynomial cubic nonlinearities. So, it is interesting to have some description of the solution set of (18) when f behaves like a cubic nonlinearity but is not being given by formula.

In this direction Korman et al. [17] have studied the solution set of (18) when f is a cubic-like nonlinearity in u and they provide exact multiplicity results for all $\lambda > 0$.

First, they consider an autonomous case, and assume that f has three distinct roots $a < b < c$ and they provide two results. The first one concerns the case where the least root a is equal to zero. They assume that $f = f(u) \in C^2(\mathbb{R})$

has the following properties

$$f(0)=f(b)=f(c)=0 \quad \text{for some } 0<b<c, \tag{34}$$
$$f(x)>0 \quad \text{for } x\in(-\infty,0)\cup(b,c) \tag{35}$$
$$f(x)<0 \quad \text{for } x\in(0,b)\cup(c,+\infty) \tag{36}$$
$$\int_0^c f(u)du>0 \tag{37}$$
$$f''(u) \text{ changes sign exactly once when } u>0, \tag{38}$$
$$f''(u) \text{ has exactly one positive root.} \tag{39}$$

Thus, they prove

Theorem 2.7 *[17] Under the conditions (34)-(39) there is a critical $\lambda_0>0$ such that for $\lambda<\lambda_0$ Problem (18) has no nontrivial solutions, it has exactly one nontrivial solution for $\lambda=\lambda_0$, and exactly two nontrivial solutions for $\lambda>\lambda_0$. Moreover, all solutions lie on a single curve, which for $\lambda>\lambda_0$ has two branches denoted by $u^-(x,\lambda)<u^+(x,\lambda)$, with $u^+(x,\lambda)$ strictly monotone increasing in λ, $u^-(0,\lambda)$ strictly monotone decreasing in λ, and $\lim_{\lambda\to\infty}u^+(x,\lambda)=c$, $\lim_{\lambda\to\infty}u^-(x,\lambda)=0$ for $x\in(-1,1)\setminus\{0\}$, while $u^-(0,\lambda)>b$ for all $\lambda>\lambda_0$.*

The second result in the autonomous case concerns the case where the least root a is strictly positive. They consider the problem

$$\begin{gathered}-u''=\lambda f(u-a), \quad \text{in } (-1,1)\\ u(-1)=u(1)=0\end{gathered} \tag{40}$$

where a is a positive constant and f satisfies (34)-(37) and

$$\begin{gathered}\text{for } u>0,\ f''(u-a) \text{ changes sign exactly once}\\ \text{and has exactly one root.}\end{gathered} \tag{41}$$

Also, they assume an additionally condition which is

$$f(\beta)\beta-2[F(\beta)-F(-a)]\geq 0 \tag{42}$$

where β is the unique solution of $f'(\beta)=\frac{f(\beta)}{\beta}$. Thus, they prove:

Theorem 2.8 *[17] Consider Problem (40) with $f(u)$ as described by (34)-(37), (41), and (42). Then there exists a critical λ_0 such that for Problem (40) there exists exactly one positive solution for $0<\lambda<\lambda_0$, exactly two positive solutions for $\lambda=\lambda_0$, and exactly three positive solutions for $\lambda>\lambda_0$. Moreover, all solutions lie on two smooth in λ solution curves, all different solutions of (40) at the same λ are strictly ordered on $(-1,1)$. One of the curves, referred to as the lower curve, starts at $\lambda=0$, $u=0$, it is strictly increasing in λ, and $\lim_{\lambda\to\infty}u(x,\lambda)=a$. The upper curve is a parabola-like curve, consisting of two branches $u^-(x,\lambda)<u^+(x,\lambda)$. The upper branch is monotone increasing in λ and $\lim_{\lambda\to\infty}u^+(x,\lambda)=a+c$ for all $x\in(-1,1)$. The lower branch approaches a spike-layer, namely $\lim_{\lambda\to\infty}u^-(x,\lambda)=a$ for all $x\in(-1,1)\setminus\{0\}$, while $u^-(x,\lambda)>a+b$ for all $\lambda>\lambda_0$.*

In the same paper [17] they provide an exact multiplicity result for all $\lambda > 0$ in an non-autonomous case. They consider Problem (18) with

$$f(x,u) = u^2(b(x) - u), \text{ for } x \in (-1,1), u \in \mathbb{R},$$

and assume that the positive function $b(x) \in C^3[-1,1]$ satisfies the following conditions:

$$b(-x) = b(x) \quad \text{for all } x \in [-1,1], \tag{43}$$
$$b'(x) < 0 \quad \text{for all } x \in (0,1], \tag{44}$$
$$b''(x) < 0 \quad \text{for all } x \in (0,1], \tag{45}$$
$$b'''(x) \leq 0 \quad \text{for all } x \in (0,1], \tag{46}$$
$$b(1) \geq \frac{1}{2}b(0) > 0. \tag{47}$$

They provide an example of a function $b(x)$ satisfying conditions (43)-(47) by $b(x) = a - x^2$ with constant $a \geq 3$ and they show that any nontrivial solution of (18) satisfies $0 < u(x) < b(x) \quad \text{for all } x \in (-1,1)$. Next, they prove:

Theorem 2.9 *[17] Under the conditions (43)-(47) there is a critical $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ Problem (18) has no nontrivial solutions, it has exactly one nontrivial solution for $\lambda = \lambda_0$, and exactly two solutions which for $\lambda > \lambda_0$ has two branches denoted by $u^-(x,\lambda) < u^+(x,\lambda)$, with $u^+(x,\lambda)$ strictly monotone increasing in λ, $u^-(x,\lambda)$ strictly monotone decreasing in λ, and $\lim_{\lambda\to\infty} u^+(x,\lambda) = b(x)$, $\lim_{\lambda\to\infty} u^-(x,\lambda) = 0$ for all $x \in (-1,1)$.*

Korman and Shi prove an exact multiplicity result which generalize Theorem 2.7 by weakening the convexity assumptions on f.

Theorem 2.10 *[16] Suppose $f \in C^2[0,\ \infty)$, $f(0) = 0$, $f(x) < 0$ for $x \in (0,b)\cup (c,\infty)$, and $f(x) > 0$ for $x \in (b,c)$, where $c > b > 0$. Assume that for some $c > \eta > \gamma > b$ we have, $f''(u) > 0$ for $0 < u < \gamma$, $f''(u) < 0$ for $\gamma < u < \eta$, $2F(\eta) - \eta f(\eta) > 0$, $f(u) - uf'(u) > 0$ for all $u > \eta$. Then there is a critical $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ Problem (18) has no nontrivial solution, it has exactly one solution for $\lambda = \lambda_0$, and exactly two solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a unique smooth solution curve.*

In an other paper, the same authors, Korman et al. [18] have extended the previous result to the case where the dimension space is two. They consider the problem

$$\begin{gathered} \Delta u + \lambda f(u) = 0 \quad \text{in } |x| < R \\ u = 0 \;\; \text{on } |x| = R \end{gathered} \tag{48}$$

on a ball in two dimensions, i.e. $x = (x_1, x_2)$. They assume that $f \in C^2(\mathbb{R})$ has the following properties

$$f(0) = f(b) = f(c) = 0 \text{ for some constants } 0 < b < c, \tag{49}$$
$$f(u) < 0 \text{ for } u \in (0,b) \cup (c,\infty) \tag{50}$$
$$f(u) > 0 \text{ for } u \in (-\infty,0) \cup (b,c)$$
$$f'(0) < 0 \tag{51}$$
$$\int_0^c f(u)du > 0 \tag{52}$$

There exists $\alpha \in (0,c)$ such that

$$\begin{gathered} f''(u) > 0 \text{ for } u \in (0,\alpha) \text{ and} \\ f''(u) < 0 \text{ for } u \in (\alpha,c). \end{gathered} \tag{53}$$

Also, letting

$$g_\mu(u) = \mu(f'(u)u - f(u)) - 2f(u), \mu \in (0,\infty),$$

they assume that

The function $g_\mu(s)$ can have at most one sign change when $s \in (0,c)$ for any value of the parameter $\mu \in (0,\infty)$. (54)

The final condition on the function $f(u)$ is

$$(f')^2u - f'f - ff''u > 0 \text{ for } b < u < \beta \tag{55}$$

where β is the unique solution of the equation $f'(\beta) = f(\beta)/\beta$.

Condition (50) implies that $f(u) > 0$ for $u < 0$. Therefore by the maximum principle, all solutions of (48) are positive, hence by a well-known result of Gidas, Ni and Nirenberg [9] they are radially symmetric. Also, by a result of Lin and Ni [19] all solutions of the linearized equation

$$\begin{gathered} \Delta w + \lambda f'(u)w = 0 \text{ in } |x| < R \\ w = 0 \text{ on } |x| = R \end{gathered} \tag{56}$$

are also radially symmetric. Therefore, the authors were lead to study the ODE version of (48). Also, without lost of generality, they take the unit ball; $R = 1$, and consider in two dimensions

$$\begin{gathered} u''(r) + \frac{1}{r}u'(r) + \lambda f'(u) = 0, r \in (0,1) \\ u'(0) = u(1) = 0 \end{gathered} \tag{57}$$

and prove the following

Theorem 2.11 *[18] Assume that $f(u)$ satisfies assumption (54) and the conditions (49-53) and (55). Then there is a critical $\lambda_0 > 0$, such that for $\lambda < \lambda_0$ Problem (57) has no nontrivial solution, it has exactly one nontrivial solution for $\lambda = \lambda_0$, and exactly two nontrivial solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two branches denoted by $0 < u^-(r,\lambda) < u^+(r,\lambda)$, with $u^+(r,\lambda)$ strictly monotone increasing in λ and $\lim_{\lambda\to\infty} u^+(r,\lambda) = c$ for $r \in [0,1)$. For the lower branch, $\lim_{\lambda\to\infty} u^-(r,\lambda) = 0$ for $r \neq 0$, while $u^-(0,\lambda) > \gamma$ for $\lambda > \lambda_0$, where γ is the unique number $\in (b,c)$ such that $\int_0^\gamma f(u)du = 0$. (Recall that any nontrivial solution is positive by the maximum principle.)*

Notice that Theorem 2.11 deals with the exact multiplicity solutions of (57) for all $\lambda > 0$ but with the restriction to two dimensions. Thus it would be interesting to know what happens in higher dimensions. The main reason which makes Theorem 2.11 holds only in two dimensions was proving positivity of any nontrivial solution of the linearized Problem (56). This difficult task was recently overcome by Ouyang and Shi [21] by using Pohozhaev type identity. Ouyang and Shi [21] consider

$$\begin{gathered} u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u) = 0,\ r \in (0,1),\ n \geq 1, \\ u'(0) = u(1) = 0. \end{gathered} \tag{58}$$

They assume that $f \in C^2(\mathbb{R}_+)$ satisfies the following properties:

$$f(0) \leq 0,\ f(b) = f(c) = 0 \text{ for some constants } 0 < b < c, \tag{59}$$

$$\begin{aligned} f(u) &< 0 \quad \text{for } u \in (0,b) \cup (c,\infty) \\ f(u) &> 0 \quad \text{for } u \in (-\infty,0) \cup (b,c), \end{aligned} \tag{60}$$

$$\int_0^c f(u)du > 0, \tag{61}$$

$$\begin{gathered} \text{There exists } \alpha \in (0,c), \text{ such that} \\ f''(u) > 0 \text{ for } u \in (0,\alpha) \text{ and } f''(u) < 0 \text{ for } u \in (\alpha,c). \end{gathered} \tag{62}$$

Let θ be the smallest positive number such that $\int_0^\theta f(s)ds = 0$ and $\rho = \alpha - \frac{f(\alpha)}{f'(\alpha)}$. Clearly, $\theta \in (b,c)$. Define $K(u) = \dfrac{uf'(u)}{f(u)}$. If $\theta < \rho$, They assume that

$$\begin{gathered} K(u) > K(\theta) \quad \text{on } (b,\theta) \\ K(u) \text{ is non increasing on } (\theta,\rho) \\ K(u) < K(\rho) \text{ on } (\rho,\alpha). \end{gathered} \tag{63}$$

(If $\theta \geq \rho$ this condition is empty.) Next, they prove the following

Theorem 2.12 *[21] Assume that $f(u)$ satisfies the conditions listed above.*

(a) *If $f(0) = 0$, there exists a critical $\lambda_0 > 0$, such that for $0 < \lambda < \lambda_0$ Problem (58) has no nontrivial solution, it has exactly one nontrivial solution for $\lambda = \lambda_0$, and exactly two nontrivial solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve Σ, which for $\lambda > \lambda_0$ has two branches denoted by Σ^+ (the upper branch) and Σ^- (the lower branch); Σ^+ continues to the right up to (∞, c); Σ^- continues to the right down to (∞, g) for some $g \geq \beta$; there exists a unique turning point on the curve, the curve bend to the right at the turning point. (See Fig 12 in [21].)*

(b) *If $f(0) < 0$, there exist $\bar{\lambda} > \lambda_0 > 0$, such that for $0 < \lambda < \lambda_0$ Problem (58) has no solution, it has exactly one solution for $\lambda > \bar{\lambda}$ or $\lambda = \lambda_0$, and exactly two solutions for $\lambda_0 < \lambda \leq \bar{\lambda}$. Moreover, all solutions lie on a single smooth solution curve Σ, which for $\lambda > \lambda_0$ has two branches denoted by Σ^+ (the upper branch) and Σ^- (the lower branch); Σ^+ continues to the right up to (∞, c); Σ^- continues to the right down to $(\bar{\lambda}, g)$ for some $g \geq \beta$; there exists a unique turning point on the curve, the curve bend to the right at the turning point. (See Fig. 10 in [21].)*

More recently Korman [12] was able to avoid having to prove this positivity condition, and replaced it by an indirect argument. He shows that it is sufficient to prove that any nontrivial solution of (56) cannot vanish exactly once. This way he considerably simplifies the proof of Ouyang and Shi and make it more elegant.

Korman [12] considers Problem (48) for $n \geq 1$. He assumes that $f \in C^2(\overline{\mathbb{R}}_+)$, $f(0) = 0$ and satisfies (59)-(63). Next, he proves the following

Theorem 2.13 *[12] Assume that $f(u)$ satisfies the conditions listed above. For Problem (48) there is a critical $\lambda_0 > 0$ such that Problem (48) has exactly 0, 1 or 2 nontrivial solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two branches denoted by $0 < u^-(r, \lambda) < u^+(r, \lambda)$, with $u^+(r, \lambda)$ strictly monotone increasing in λ and $\lim_{\lambda\to\infty} u^+(\rho, \lambda) = c$ for $r \in [0, 1)$. For the lower branch $\lim_{\lambda\to\infty} u^-(r, \lambda) = 0$ for $r \neq 0$, while $u^-(0, \lambda) > b$ for all $\lambda > \lambda_0$.*

Other related results are available in the literature. (See for instance, Pimbley, [22], Wang and Kazarinoff [28], [29], Schaaf [23], Korman [11], Shi and Shivaji [24]). But as we have indicated at the beginning of this section, we shall not attempt to make a complete historical review. Thus, we apologize to all authors whose results are close to cubic-like nonlinearities and which have not been either described in this section or listed in the references.

3 The method used

We shall make use of the quadrature method. Denote by g a nonlinearity and by p a real parameter, and we assume,

$$g \in C(\mathbb{R}) \quad \text{and} \quad 1 < p < +\infty, \tag{64}$$

and consider the boundary value problem,

$$-(|u'|^{p-2}u')' = g(u) \quad \text{in } (0,1),\ u(0) = u(1) = 0. \tag{65}$$

Denote by $p' = p/(p-1)$ the conjugate exponent of p. Let $G(s) = \int_0^s g(t)dt$. For any $E \geq 0$ and $\kappa = +, -$, let,

$$X_\kappa(E) = \{s \in \mathbb{R} : \kappa s > 0 \text{ and } E^p - p'G(\xi) > 0, \forall\, \xi, 0 < \kappa\xi < \kappa s\} \text{ and,}$$

$$r_\kappa(E) = 0,\ \text{if } X_\kappa(E) = \emptyset \quad \text{and} \quad r_\kappa(E) = \kappa \sup(\kappa X_\kappa(E)) \ \text{otherwise.}$$

Let

$$\tilde{D}_\kappa = \left\{ E \geq 0 : 0 < |r_\kappa(E)| < +\infty \text{ and } \kappa \int_0^{r_\kappa(E)} (E^p - p'G(t))^{\frac{-1}{p}} dt < +\infty \right\}$$

and $\tilde{D} = \tilde{D}_+ \cap \tilde{D}_-$. Also, let $\tilde{D}_k^\kappa := \tilde{D}$ if $k \geq 2$, and $\tilde{D}_1^\kappa := \tilde{D}_\kappa$. Define the following time-maps,

$$T_\kappa(E) = \kappa \int_0^{r_\kappa(E)} (E^p - p'G(t))^{\frac{-1}{p}} dt, E \in \tilde{D}_\kappa.$$

$$\begin{aligned} T_{2n}^\kappa(E) &= n(T_+(E) + T_-(E)), \quad n \in \mathbb{N}, \quad E \in \tilde{D}, \\ T_{2n+1}^\kappa(E) &= T_{2n}^\kappa(E) + T_\kappa(E), \qquad n \in \mathbb{N}, \quad E \in \tilde{D}. \end{aligned}$$

Theorem 3.1 (Quadrature method) *Assume that (64) holds. Let $E > 0$, $\kappa = +, -$. Then, Problem (65) admits a solution $u_k^\kappa \in A_k^\kappa$ satisfying $(u_k^\kappa)'(0) = \kappa E$ if and only if $E \in \tilde{D}_k^\kappa - \{0\}$ and $T_k^\kappa(E) = (1/2)$, and in this case the solution is unique.*

Remark 3.2 *In practice, to compute $\tilde{D}_\kappa$ we first compute the set*

$$D_\kappa = \{E > 0 : 0 < |r_\kappa(E)| < +\infty \text{ and } \kappa g(r_\kappa(E)) > 0\}$$

and then we deduce $\tilde{D}_\kappa$ by observing that, $D_\kappa \subset \tilde{D}_\kappa \subset \overline{D}_\kappa$.

4 Some preliminary lemmas

To apply Theorem 3.1 we have, first, to determine the definition domains $\tilde{D}_+$ and $\tilde{D}_-$ of the time-maps T_+ and T_- respectively. Lemma 4.1 is used to. Next, we have to compute $\tilde{D} = \tilde{D}_+ \cap \tilde{D}_-$ which is the definition domain of the time-maps T_k^κ for all $k \geq 2$, $\kappa = +, -$. This is done in Lemma 4.2. Next, the aim of Lemma 4.3 is to compare the maximum and minimum of any solution of our problem. This comparison may be used subsequently. Next, we define all the time-maps and make some useful transformations. Lemmas 4.4, 4.5, and 4.6 are dedicated to the computation of the limits of all the time-maps, at each boundary point of their domains. In Lemma 4.7 we show that under some appropriate conditions, the time-maps may be monotonic increasing. The following step is crucial; when the time maps are not monotonic, usually one tries to compare there maximum and/or minimum with the real number $\frac{1}{2}$. This task is much simpler when f is odd. Indeed, the time-maps T_+ and T_- are always equal if f is odd. Thus; (*) $T_k^\pm = kT_+$ for all $k \geq 2$. This way, to study the maximum and/or minimum value of $T_k^\pm$ it suffices to handle those of T_+ only. Unfortunately, in our *p.h.o.* case it seems that the identity (*) is not satisfied. To overcome this difficulty we define two maps such that both T_+ and T_- are bounded from below by the first one, and from above by the second one. Thus, $T_k^\pm$ is bounded from below by k times the first map and from above by k times the second one. So, it suffices to study the two bounding maps. Moreover, these estimates seem to be optimal in the sense that in the particular odd case, they imply that T_+ and T_- are equal!

In Lemma 4.8 we compare the time-maps T_+ and T_- with the defined two maps. In Lemma 4.10 we deduces the estimates of $T_k^\pm$ for $k \geq 2$. In Lemma 4.11 we provide an identity which may be used in the sequel. This identity seems to be interesting for its own right and motivates us to ask a question in Section 6. Lemma 4.12 is dedicated to the limits of these two bounding maps.

Under appropriate conditions we provide, in Lemma 4.13, estimates which are used to prove uniqueness of the minimum of the time-maps T_+ and T_- respectively. This kind of estimates was introduced by Smoller and Wasserman [25] and was crucial in their study of uniqueness of critical points.

Lemma 4.1 *Consider the function defined in $\mathbb{R}^\pm$ by,*

$$s \longmapsto G_\pm(\lambda, E, s) := E^p - p'\lambda F(s), \tag{66}$$

where $E, \lambda > 0$ and $p, q > 1$ are real parameters, $F(s) = \int_0^s f(t)dt$. Assume that (2)-(5) hold. Then,

(i) *If $E > E_*^\pm(p,\lambda) := (p'\lambda F(\alpha_\pm))^{1/p}$, the function $G_\pm(\lambda, E, \cdot)$ is strictly positive in $\mathbb{R}^\pm$.*

(ii) *If $E = E_*^\pm(p,\lambda)$, the function $G_\pm(\lambda, E, \cdot)$ is strictly positive in $(0, \alpha_+)$ (resp. in $(\alpha_-, 0)$) and vanishes at $\alpha_\pm$.*

(iii) *If $0 < E < E_*^\pm(p,\lambda)$, the function $G_\pm(\lambda, E, \cdot)$ admits in the open interval $(0, \alpha_+)$ (resp. $(\alpha_-, 0)$) a unique zero $s_\pm(\lambda, E)$ and is strictly positive in the open interval $(0, s_+(\lambda, E))$ (resp. $(s_-(\lambda, E), 0)$). Moreover,*

(a) *The function $E \mapsto s_\pm(\lambda, E)$ is C^1 in $(0, E_*^\pm(p,\lambda))$ and,*

$$\pm\frac{\partial s_\pm}{\partial E}(\lambda, E) = \frac{\pm(p-1)E^{p-1}}{\lambda f(s_\pm(\lambda, E))} > 0, \tag{67}$$

for all $E \in (0, E_*^\pm(p,\lambda))$.

(b) $\lim_{E\to 0^+} s_\pm(\lambda,E)=0$ *and* $\lim_{E\to E_*^\pm} s_\pm(\lambda,E)=\alpha_\pm$,

(c) $$\lim_{E\to 0^+} |s_\pm(\lambda,E)|/E = \begin{cases} +\infty & \text{if } q-p>0 \\ (\dfrac{p-1}{\lambda a_0})^{1/p} & \text{if } q-p=0 \\ 0 & \text{if } q-p<0, \end{cases}$$

(d) $\lim_{E\to 0^+} F(\kappa s_\eta(\lambda,E)\xi)/E^p = \xi^q/(\lambda p')$,
for all $\xi>0$ *and all* $(\kappa,\eta)\in\{+,-\}^2$.

Proof. For any fixed $p>1$ and $E\geq 0$, consider the function,

$$s\longmapsto G_\pm(\lambda,E,s) := E^p - p'\lambda F(s), \tag{68}$$

defined in $\mathbb{R}^\pm$. One has, $\frac{dG_\pm}{ds}(\lambda,E,s) = -p'\lambda f(s)$. Hence, according to (3), $G_+(\lambda,E,\cdot)$ (resp. $G_-(\lambda,E,\cdot)$) is strictly decreasing in $(0,\alpha_+)$ (resp. in $(-\infty,\alpha_-)$) and according to (4), it is strictly increasing in $(\alpha_+,+\infty)$ (resp. in $(\alpha_-,0)$). Moreover, according to (2), $\frac{dG_\pm}{ds}(\lambda,E,\alpha_\pm)=0$. Therefore, it follows that $G_\pm(\lambda,E,\cdot)$ is strictly positive in $\mathbb{R}^\pm$ for all $E>E_*^\pm := (p'\lambda F(\alpha_\pm))^{1/p}$, admits a unique positive (resp. negative) zero, $\alpha_\pm$, and is strictly positive in $(0,\alpha+)$ (resp. in $(\alpha_-,0)$) at $E=E_*^\pm$, and finally admits a first positive (resp. negative) zero $s_\pm = s_\pm(\lambda,E)$ and is strictly positive in $(0,s_+)$ (resp. in $(s_-,0)$) for all $E: 0<E<E_*^\pm$, moreover $|s_\pm|<|\alpha_\pm|$.

Proof of (a) For any $p>1$ and $\lambda>0$, consider the real valued function,

$$(E,s)\longmapsto G_\pm(E,s) := E^p - p'\lambda F(s),$$

defined in $\Omega_+ = (0,E_*^+)\times(0,\alpha_+)$ (resp. $\Omega_- = (0,E_*^-)\times(\alpha_-,0)$). One has $G_\pm\in C^1(\Omega_\pm)$ and, $\frac{\partial G_\pm}{\partial s}(E,s) = -p'\lambda f(s)$ in $\Omega_\pm$, hence, according to (3) (resp. (4)), it follows that, $\pm\frac{\partial G_\pm}{\partial s}(E,s)<0$ in $\Omega_\pm$, and one may observe that $s_\pm(\lambda,E)$ belongs to the open interval $(0,\alpha_+)$ (resp. $(\alpha_-,0)$) and satisfies, from its definition,

$$G_\pm(E,s_\pm(\lambda,E))=0. \tag{69}$$

So, one can make use of the implicit function theorem to show that the function $E\mapsto s_\pm(\lambda,E)$ is $C^1((0,E_*^\pm),\mathbb{R})$ and to obtain the expression of $\dfrac{\partial s_\pm}{\partial E}(\lambda,E)$ given in **(a)**. Its sign is given by (3) (resp. (4) together with the fact that $s_\pm(\lambda,E)$ belongs to $(0,\alpha_+)$ (resp. $(\alpha_-,0)$). Therefore, Assertion **(a)** is proved.

Proof of (b) For any fixed $p>1$ and $\lambda>0$, Assertion **(a)** of the current lemma implies that the function defined in $(0,E_*^\pm)$ by $E\mapsto s_\pm(\lambda,E)$ is strictly increasing (resp. strictly decreasing). It is bounded from below by 0 (resp. by

α_-) and from above by α_+ (resp. by 0). Then, the limits $\lim_{E\to 0^+} s_\pm(\lambda,E)=\ell_0^\pm$ and $\lim_{E\to E_*^\pm} s_\pm(\lambda,E)=\ell_*^\pm$) exist as real numbers and moreover,

$$\alpha_- \le \ell_*^- < \ell_0^- \le 0 \le \ell_0^+ < \ell_*^+ \le \alpha_+.$$

Let us observe that, for any fixed $p>1$ and $\lambda>0$, the function, $(E,s)\mapsto G_\pm(E,s)$, is continuous in $[0,E_*^+]\times[0,\alpha_+]$ (resp. in $[0,E_*^-]\times[\alpha_-,0]$) and the function $E\mapsto s_\pm(\lambda,E)$ is continuous in $(0,E_*^\pm)$ and satisfies (69). So, by passing to the limit in (69) as E tends to 0^+, one gets, $0=\lim_{E\to 0^+} G_\pm(E,s_\pm(\lambda,E))=G_\pm(0,\ell_0^\pm)$. Hence, $\ell_0^\pm$ is a zero, belonging to $[0,\alpha_+]$, (resp. $[\alpha_-,0]$) to the equation in s, $G_\pm(0,s)=0$. By resolving this equation one gets: $\ell_0^\pm=0$. Also, by passing to the limit in (69) as E tends to $E_*^\pm$, one gets,

$$0=\lim_{E\to E_*^\pm} G_\pm(E,s_\pm(\lambda,E))=G_\pm(E_*^\pm,\ell_0^\pm).$$

Hence, $\ell_*^\pm$ is a zero, belonging to $(0,\alpha_+]$, (resp. $[\alpha_-,0)$) to the equation in s,

$$G_\pm(E_*^\pm,s)=0.$$

By resolving this equation one gets: $\ell_*^\pm=\alpha_\pm$. Therefore, Assertion **(b)** follows.

Proof of (c) Let $\Phi_q(s)=\int_0^s \varphi(t)dt=(1/q)\left|s\right|^q$. Observe that from the definition of $s_\pm(\lambda,E)$ one has

$$E^p=p'\lambda F(s_\pm(\lambda,E)), \tag{70}$$

hence, dividing by $\left|s_\pm(\lambda,E)\right|^p$, using l'Hopital's rule and (5) one gets,

$$\begin{aligned}\lim_{E\to 0^+} E^p/\left|s_\pm(\lambda,E)\right|^p &= \lim_{E\to 0^+}\frac{p'\lambda}{q}\left|s_\pm(\lambda,E)\right|^{q-p}\frac{F(s_\pm(\lambda,E))}{\Phi_q(s_\pm(\lambda,E))}\\ &= \frac{p'\lambda}{q}\lim_{E\to 0^+}\left|s_\pm(\lambda,E)\right|^{q-p}\lim_{s\to 0}\frac{f(s)}{\varphi_q(s)}\\ &= \frac{p'\lambda}{q}a_0\cdot\lim_{E\to 0^+}\left|s_\pm(\lambda,E)\right|^{q-p}.\end{aligned}$$

Therefore, Assertion **(c)** follows.

Proof of (d) Remark that for all $\xi>0$ and all $(\kappa,\eta)\in\{+,-\}^2$ one has,

$$\frac{F(\kappa s_\eta(\lambda,E)\xi)}{E^p}=\frac{F(\kappa s_\eta(\lambda,E)\xi)}{\Phi_q(\kappa s_\eta(\lambda,E)\xi)}\frac{\Phi_q(\kappa s_\eta(\lambda,E)\xi)}{E^p}.$$

Using l'Hopital's rule and (5) one gets,

$$\lim_{E\to 0^+}\frac{F(\kappa s_\eta(\lambda,E)\xi)}{\Phi_q(\kappa s_\eta(\lambda,E)\xi)}=\lim_{E\to 0^+}\frac{f(\kappa s_\eta(\lambda,E)\xi)}{\varphi_q(\kappa s_\eta(\lambda,E)\xi)}=a_0.$$

On the other hand, since $\Phi_q(\cdot)$ is an odd function then

$$\frac{\Phi_q(\kappa s_\eta(\lambda,E)\xi)}{E^p} = \frac{\Phi_q(s_\eta(\lambda,E)\xi)}{E^p}.$$

Thus, for all $\xi > 0$, using l'Hopital's rule, (67) and (5) one gets,

$$\begin{aligned}
\lim_{E\to 0^+} \frac{\Phi_q(s_\eta(\lambda,E)\xi)}{E^p} &= \lim_{E\to 0^+} \frac{\xi s'_\eta(\lambda,E)\varphi_q(s_\eta(\lambda,E)\xi)}{pE^{p-1}} \\
&= \lim_{E\to 0^+} \frac{\xi^q}{\lambda p'} \frac{\varphi_q(s_\eta(\lambda,E))}{f(s_\eta(\lambda,E))} = \frac{\xi^q}{a_0\lambda p'}.
\end{aligned}$$

Hence, Assertion **(d)** follows. Therefore, Lemma 4.1 is proved. ♢

Remark. We have used condition (5) in, and only in, the process of the proofs of assertions **(c)** and **(d)**.

Now we are ready to compute $X_\pm(\lambda,E)$ as defined in Section 3, for any $E > 0$ and $\lambda > 0$. In fact,

$$X_+(\lambda,E) = \begin{cases} (0,+\infty) & \text{if } \ E > E_*^+ \\ (0,\alpha_+) & \text{if } \ E = E_*^+ \\ (0,s_+(\lambda,E)) & \text{if } \ 0 < E < E_*^+, \end{cases}$$

$$X_-(\lambda,E) = \begin{cases} (-\infty,0) & \text{if } \ E > E_*^- \\ (\alpha_-,0) & \text{if } \ E = E_*^- \\ (0,s_-(\lambda,E)) & \text{if } \ 0 < E < E_*^-, \end{cases}$$

where $s_\pm(\lambda,E)$ is defined in Lemma 4.1. Then

$$r_+(\lambda,E) := \sup X_+(\lambda,E) = \begin{cases} +\infty & \text{if } \ E > E_*^+ \\ \alpha_+ & \text{if } \ E = E_*^+ \\ s_+(\lambda,E) & \text{if } \ 0 < E < E_*^+, \end{cases} \tag{71}$$

$$r_-(\lambda,E) := \inf X_-(\lambda,E) = \begin{cases} -\infty & \text{if } \ E > E_*^- \\ \alpha_- & \text{if } \ E = E_*^- \\ s_-(\lambda,E) & \text{if } \ 0 < E < E_*^-, \end{cases} \tag{72}$$

and one deduces from Lemma 4.1 the following limits

$$\pm\frac{\partial r_\pm}{\partial E}(\lambda,E) > 0, \forall\lambda > 0, E \in (0,E_*^\pm(p,\lambda)), \tag{73}$$

$$\lim_{E\to 0^+} r_\pm(\lambda,E) = 0 \qquad \text{and} \qquad \lim_{E\to E_*^\pm} r_\pm(\lambda,E) = \alpha_\pm. \tag{74}$$

$$\lim_{E\to 0^+} |r_\pm(\lambda,E)|/E = \begin{cases} +\infty & \text{if } \quad q-p>0 \\ (\dfrac{p-1}{\lambda a_0})^{1/p} & \text{if } \quad q-p=0 \\ 0 & \text{if } \quad q-p<0, \end{cases} \tag{75}$$

$$\lim_{E\to 0} F(\kappa r_\eta(\lambda,E)\xi)/E^p = \xi^q/(\lambda p'), \quad \text{for all } \xi>0 \text{ and } (\kappa,\eta)\in\{+,-\}^2. \tag{76}$$

On the other hand,

$$0<|r_\pm(\lambda,E)|<+\infty \quad \text{if and only if } \ 0<E\le E_*^\pm,$$

and,

$$\pm\lambda f(r_\pm(\lambda,E))>0 \quad \text{if and only if } \ 0<E<E_*^\pm.$$

Then,

$$\begin{aligned} D_\pm(\lambda) \quad &:= \quad \{E>0 : 0<|r_\pm(p,\lambda,E)|<+\infty \text{ and } \pm\lambda f(r_\pm(\lambda,E))>0\} \\ &= \quad (0,E_*^\pm), \end{aligned}$$

and

$$D(\lambda) := D_+(\lambda)\cap D_-(\lambda) = (0,\inf(E_*^+(\lambda),E_*^-(\lambda))).$$

Lemma 4.2 *If f satisfies (2)-(4) and (9), then for all $\lambda>0$: $E_*^-(\lambda)\le E_*^+(\lambda)$. Therefore, $D(\lambda)=(0,E_*^-(\lambda))$.*

Remark. For all $\lambda>0$, $k\ge 1$ and $\kappa=+,-$, define

$$E_k^\kappa(\lambda) = \begin{cases} E_*^\kappa(\lambda) \text{ if } k=1 \\ E_*^-(\lambda) \text{ if } k\ge 2 \text{ and } \kappa \text{ arbitrary.} \end{cases}$$

Therefore, for all $\lambda>0$, $k\ge 1$ and $\kappa=+,-$, one has $D_k^\kappa(\lambda)=(0,E_k^\kappa(\lambda))$, where D_k^κ is defined in Section 3.

Proof of Lemma 4.2. For all $\lambda>0$, one has $E_*^-(\lambda)\le E_*^+(\lambda)$ if and only if, $p'\lambda F(\alpha_-)\le p'\lambda F(\alpha_+)$, which is equivalent to,

$$-\int_0^{-\alpha_-} f(-t)dt \le \int_0^{\alpha_+} f(t)dt,$$

that is,

$$0\le \int_0^{-\alpha_-}(f(-t)+f(t))dt + \int_{-\alpha_-}^{\alpha_+} f(t)dt.$$

The first integral is positive from (9) and the second one is too from (3) and (11) (Recall that (11) is a consequence of (2), (4) and (9), see the Introduction). Therefore, Lemma 4.2 is proved. ♢

Lemma 4.3 *Assume that (2)-(4) and (9) hold. For all* $\lambda > 0$ *and* $E \in (0, E_*^-(\lambda))$*, one has*

$$y(r_-(\lambda, E)) = r_+(\lambda, E)$$

where $y(\cdot)$ *is defined in (12). Explicitly, for all* $\lambda > 0$ *and* $E \in (0, E_*^-(\lambda))$ *one has,*

$$0 < r_+(\lambda, E) \leq -r_-(\lambda, E),$$

and

$$F(t) = F(r_-(\lambda, E)) \Longleftrightarrow t = r_+(\lambda, E), \ \forall t \in (0, -r_-(\lambda, E)].$$

Proof. For all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$, one has from the definition of $r_\pm(\lambda, E) : E^p - p'\lambda F(r_\pm(\lambda, E)) = 0$. Thus,

$$F(r_+(\lambda, E)) - F(r_-(\lambda, E)) = 0. \tag{77}$$

On the other hand

$$F(r_-(\lambda, E)) = \int_0^{r_-(\lambda,E)} f(t)dt = -\int_0^{-r_-(\lambda,E)} f(-t)dt,$$

and

$$\begin{aligned} F(r_+(\lambda, E)) &= \int_0^{r_+(\lambda,E)} f(t)dt \\ &= \int_0^{-r_-(\lambda,E)} f(t)dt + \int_{-r_-(\lambda,E)}^{r_+(\lambda,E)} f(t)dt. \end{aligned}$$

Thus,

$$\begin{aligned} F(r_+(\lambda, E)) - F(r_-(\lambda, E)) &= \int_0^{-r_-(\lambda,E)} (f(t) + f(-t))dt \\ &\quad + \int_{-r_-(\lambda,E)}^{r_+(\lambda,E)} f(t)dt. \end{aligned} \tag{78}$$

Observe that by (73) and (74), $0 < -r_-(\lambda, E) < -\alpha_-$ and by (9) the function $t \mapsto (f(t) + f(-t))$ is positive in $(0, -\alpha_-)$, thus the first integral in (78) is positive. Now, if we assume that there exists at least a $\lambda_0 > 0$ and an $E_0 \in (0, E_*^-(\lambda_0))$ such that $r_+(\lambda_0, E_0) + r_-(\lambda_0, E_0) > 0$, it follows that $0 < -r_-(\lambda_0, E_0) < r_+(\lambda_0, E_0) < \alpha_+$. Therefore, since f is strictly positive in $(0, \alpha_+)$ it follows that the second integral in (78) is strictly positive and thus, $F(r_+(\lambda_0, E_0)) - F(r_-(\lambda_0, E_0)) > 0$ which is a contradiction with (77). Therefore, for all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$ one has,

$$0 < r_+(\lambda, E) \leq -r_-(\lambda, E).$$

Regarding (77), it remains to prove that for all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$ one has,

$$F(t) = F(r_-(\lambda, E)) \Longrightarrow t = r_+(\lambda, E),\ \forall t \in (0, -r_-(\lambda, E)].$$

But, this is immediate, since F is strictly increasing in $(0, -r_-(\lambda, E)]$. Therefore, Lemma 4.3 is proved. $\diamondsuit$

At present, if f satisfies (2)-(4) we define, for any $p, q > 1$, $\lambda > 0$ and $E \in D_\pm(\lambda)$, the time map $T_\pm$ by

$$T_\pm(\lambda, E) := \pm \int_0^{r_\pm(\lambda,E)} (E^p - p'\lambda F(\xi))^{-1/p} d\xi,\ E \in D_\pm(\lambda) = (0, E_*^\pm).$$

Actually, $T_\pm(\lambda, E)$ is defined for all $\lambda > 0$ and $E \in \tilde{D}_\pm(\lambda)$ (see Remark 3.2). A simple change of variables shows that,

$$T_\pm(\lambda, E) = |r_\pm(\lambda, E)| \int_0^1 (E^p - p'\lambda F(r_\pm(\lambda, E)\xi))^{-1/p} d\xi, \tag{79}$$

which can be written as,

$$T_\pm(\lambda, E) = (|r_\pm(\lambda, E)|\,/E) \int_0^1 (1 - p'\lambda F(r_\pm(\lambda, E)\xi)/E^p)^{-1/p} d\xi. \tag{80}$$

Also, observe that one has from (70), (71) and (72), $E^p = \lambda p' F(r_\pm(\lambda, E))$, so, (79) may be written as,

$$T_\pm(\lambda, E) = \pm(\lambda p')^{-1/p} \int_0^{r_\pm(\lambda,E)} (F(r_\pm(\lambda, E)) - F(r_\pm(\lambda, E)\xi))^{-1/p} d\xi. \tag{81}$$

For any $p > 1$ and $x \in [0, \alpha_+]$ (resp. $x \in [\alpha_-, 0]$ let us define $S_+(x)$ (resp. $S_-(x)$) by,

$$S_\pm(x) := \pm \int_0^x (F(x) - F(\xi))^{-1/p} d\xi \in [0, +\infty].$$

Then, (81) may be written as,

$$T_\pm(\lambda, E) = (\lambda p')^{-1/p} S_\pm(r_\pm(\lambda, E)). \tag{82}$$

On the other hand, we define for any $p > 1$, $\lambda > 0$ and $E \in D(\lambda) = (0, E_*^-)$, the time maps,

$$T_{2n}^\pm(\lambda, E) := n(T_+(\lambda, E) + T_-(\lambda, E)), \lambda > 0, E \in (0, E_*^-), n \geq 0, \tag{83}$$

$$T_{2n+1}^\pm(\lambda, E) := T_{2n}^\pm(\lambda, E) + T_\pm(\lambda, E), \lambda > 0, E \in (0, E_*^-), n \geq 0. \tag{84}$$

The limits of these time maps are the aim of the following Lemmas.

Lemma 4.4 *Assume that (2)-(4) hold.*

(i) *If* $(6)_\pm$ *holds,* $\lim_{E\to E_*^\pm} T_\pm(\lambda,E) = (\lambda p')^{-1/p} S_\pm(\alpha_\pm)$ *and* $S_\pm(\alpha_\pm) < +\infty$ *if and only if* $q-p<0$.

(ii) *If (5) holds,* $\lim_{E\to 0^+} T_\pm(\lambda,E) = \begin{cases} +\infty & \text{if} \quad q-p>0 \\ \frac{1}{2}(\frac{\lambda_1}{\lambda a_0})^{1/p} & \text{if} \quad q-p=0 \\ 0 & \text{if} \quad q-p<0. \end{cases}$

(iii) *If (9) holds, for all* $\lambda>0$, $r_+(\lambda,E_*^-(\lambda)) = y(\alpha_-)$, *where* $y(\cdot)$ *is defined in (12) and* $T_+(\lambda,E_*^-(\lambda)) = (\lambda p')^{-1/p} S_+(y(\alpha_-))$.

Proof of Lemma 4.4

Proof of (i) The value of the limit follows by passing to the limit in (82) as E tends to $E_*^\pm$. In order to show the second assertion of **(i)** one observes that

$$S_\pm(\alpha_\pm) = \pm\int_0^{\alpha_\pm\mp\delta} \{F(\alpha_\pm)-F(\xi)\}^{-1/p}\,d\xi \pm \int_{\alpha_\pm\mp\delta}^{\alpha_\pm} \{\cdots\}^{-1/p}\,d\xi,$$

where $\delta>0$ is given by (6). The first integral converges because the integrand function is continuous on the compact interval whose extremities are 0 and $\alpha_\pm\mp\delta$. For the second one, it follows from $(6)_\pm$ that, for all ξ, satisfying $(\pm\xi)\in(\pm\alpha_\pm-\delta,\pm\alpha_\pm)$, one has

$$\pm m_\pm\varphi_q(\alpha_\pm-\xi) \le \pm f(\xi) \le \pm M_\pm\varphi_q(\alpha_\pm-\xi),$$

and since for any E near $E_*^\pm$ one has $\pm r_\pm(\lambda,E)\in(\pm\alpha_\pm-\delta,\pm\alpha_\pm)$ then for all ξ, satisfying $(\pm\xi)\in(\pm\alpha_\pm-\delta,\pm r_\pm(\lambda,E))$, one has

$$m_\pm\int_\xi^{r_\pm(\lambda,E)} \varphi_q(\alpha_\pm-x)dx \le \int_\xi^{r_\pm(\lambda,E)} f(x)dx \le M_\pm\int_\xi^{r_\pm(\lambda,E)} \varphi_q(\alpha_\pm-x)dx$$

that is

$$\frac{m_\pm}{q}\{|\alpha_\pm-\xi|^q-|\alpha_\pm-r_\pm(\lambda,E)|^q\} \le F(r_\pm(\lambda,E))-F(\xi)$$

$$\le \frac{M_\pm}{q}\{|\alpha_\pm-\xi|^q-|\alpha_\pm-r_\pm(\lambda,E)|^q\}$$

then

$$\pm(\frac{M_\pm}{q})^{-1/p}\int_{\alpha_\pm\mp\delta_\pm}^{r_\pm(\lambda,E)} \{|\alpha_\pm-\xi|^q-|\alpha_\pm-r_\pm(\lambda,E)|^q\}^{-1/p}\,d\xi$$

$$\leq \pm \int_{\alpha_\pm \mp \delta}^{r_\pm(\lambda,E)} \{F(r_\pm(\lambda,E)) - F(\xi)\}^{-1/p}\, d\xi$$

$$\leq \pm (\frac{m_\pm}{q})^{-1/p} \int_{\alpha_\pm \mp \delta}^{r_\pm(\lambda,E)} \{|\alpha_\pm - \xi|^q - |\alpha_\pm - r_\pm(\lambda,E)|^q\}^{-1/p}\, d\xi.$$

By passing to the limit in these inequalities as E tends to $E_*^\pm$, one gets

$$\pm (\frac{M_\pm}{q})^{1/p} \int_{\alpha_\pm \mp \delta}^{\alpha_\pm} |\alpha_\pm - \xi|^{-q/p}\, d\xi \leq \pm \int_{\alpha_\pm \mp \delta}^{\alpha_\pm} \{F(\alpha_\pm) - F(\xi)\}^{-1/p}\, d\xi$$

$$\leq \pm (\frac{m_\pm}{q})^{-1/p} \int_{\alpha_\pm \mp \delta}^{\alpha_\pm} |\alpha_\pm - \xi|^{-q/p}\, d\xi,$$

and from the well-known fact

$$\pm \int_{\alpha_\pm \mp \delta}^{\alpha_\pm} |\alpha_\pm - \xi|^{-q/p}\, d\xi < +\infty \text{ if and only if } p > q,$$

the second assertion of **(i)** follows.

Proof of (ii) By passing to the limit in (80) as E tends to 0^+, the limit of $T_\pm(\lambda,E)$ follows immediately from (75), (76) and the fact that,

$$\int_0^1 (1-\xi^q)^{-1/p} = \frac{1}{q} B(\frac{1}{q}, 1 - \frac{1}{p}) \in \mathbb{R},$$

where $B(a,b)$ denotes the beta function. Remark that in the particular case $q = p$ one has

$$\int_0^1 (1-\xi^p)^{-1/p} = \frac{1}{p} B(\frac{1}{p}, 1 - \frac{1}{p}) = \pi/(p \sin(\frac{\pi}{p})) = \frac{1}{2}(\lambda_1/(p-1))^{1/p}.$$

Notice that condition (5) was used implicitly in this proof. In fact, to derive (75), (76) we have used (5). See the remark located before the proof of Lemma 4.1.

Proof of (iii) By Lemma 4.3 it follows that for all $\lambda > 0$, one has $r_+(\lambda, E_*^-(\lambda)) = y(r_-(\lambda, E_*^-(\lambda)))$ and by (74) it follows that $r_-(\lambda, E_*^-(\lambda)) = \alpha_-$. Thus, $r_+(\lambda, E_*^-(\lambda)) = y(\alpha_-)$.

The formula of $T_+(\lambda, E_*^-(\lambda))$ follows from a simple substitution in (82). Therefore, Lemma 4.4 is proved. $\diamondsuit$

Lemma 4.5 *Assume that (2)-(5) hold, $p, q > 1$, $\lambda > 0$ and $n \in \mathbb{N}^*$. Then,*

$$\lim_{E \to 0^+} T_n^\pm(\lambda, E) = \begin{cases} +\infty & \text{if} \quad q - p > 0 \\ \frac{1}{2}(\frac{\lambda_n}{\lambda a_0})^{1/p} & \text{if} \quad q - p = 0 \\ 0 & \text{if} \quad q - p < 0. \end{cases}$$

Lemma 4.6 *Assume that (2)-(4),* $(6)_-$*, and (9) hold,* p*,* $q > 1$*,* $\lambda > 0$ *and* $n \in \mathbb{N}^*$*. Then,*

- *In case* $q - p \geq 0$*, one has* $\lim_{E \to E_*^-} T_n^{\pm}(\lambda, E) = +\infty$.
- *In case* $q - p < 0$*, one has*

$$\begin{aligned}
\lim_{E \to E_*^-} T_{2n}^{\pm}(\lambda, E) &= (\lambda p')^{-1/p}(nS_-(\alpha_-) + nS_+(\alpha_*)) \\
\lim_{E \to E_*^-} T_{2n+1}^{+}(\lambda, E) &= (\lambda p')^{-1/p}(nS_-(\alpha_-) + (n+1)S_+(\alpha_*)) \\
\lim_{E \to E_*^-} T_{2n+1}^{-}(\lambda, E) &= (\lambda p')^{-1/p}((n+1)S_-(\alpha_-) + nS_+(\alpha_*)).
\end{aligned}$$

Proofs of Lemmas 4.5 and 4.6. These ones follow from Lemma 4.4 and the definitions (83) and (84) of the time maps $T_k^{\pm}$. ♢

Lemma 4.7 *For any* p*,* $q > 1$ *and* $\lambda > 0$*, assume that* $(15)_{\pm}$ *holds. Then,*

$$\frac{\partial T_{\pm}}{\partial E}(\lambda, E) > 0, \ \forall E \in D_{\pm}(\lambda) = (0, E_*^{\pm}(\lambda)).$$

Therefore, if both $(15)_+$ *and* $(15)_-$ *hold, then, for all* $k \geq 2$*,*

$$\frac{\partial T_k^{\pm}}{\partial E}(\lambda, E) > 0, \ \forall E \in D(\lambda) = (0, E_*^-(\lambda)).$$

Proof. A simple computation shows that

$$\begin{aligned}
\frac{\partial T_{\pm}}{\partial E}(\lambda, E) &= \frac{1}{p}(p')^{-1/p}(\pm\frac{\partial r_{\pm}}{\partial E}(\lambda, E))\lambda^{-1/p} \\
&\quad \times \int_0^1 \frac{H(r_{\pm}(\lambda, E)) - H(r_{\pm}(\lambda, E)\xi)}{(F(r_{\pm}(\lambda, E)) - F(r_{\pm}(\lambda, E)\xi))^{1+(1/p)}} d\xi.
\end{aligned}$$

where $H(x) = pF(x) - xf(x)$. Condition $(15)_{\pm}$ implies that

$$H(r_{\pm}(\lambda, E)) - H(r_{\pm}(\lambda, E)\xi) > 0, \forall \xi \in (0, 1).$$

Thus, the integral above is positive, and by (73) it follows that $\frac{\partial T_{\pm}}{\partial E}(\lambda, E) > 0$, for all $E \in D_{\pm}(\lambda)$. The last assertion is immediate from the definition of $T_k^{\pm}(\lambda, \cdot)$. The proof of Lemma 4.7 is complete. ♢

When f is odd, the time-maps $T_+(\lambda, \cdot)$ and $T_-(\lambda, \cdot)$ are always equal. In our *p.h.o.* case, we show that both $T_+(\lambda, \cdot)$ and $T_-(\lambda, \cdot)$ are bounded from below and from above by a same function respectively. These estimates imply in the particular odd case that the two time-maps $T_+(\lambda, \cdot)$ and $T_-(\lambda, \cdot)$ are equal. The following lemma is pioneer in our analysis.

Let us define

$$\Theta_+(x) = \int_0^x \{F(x) - F(-\xi)\}^{-1/p} d\xi, x \in (0, -\alpha_-),$$

and

$$\Theta_-(x) = \int_x^{-y(x)} \{F(x) - F(\xi)\}^{-1/p} d\xi + \int_{-y(x)}^0 \{F(x) - F(-\xi)\}^{-1/p} d\xi,$$

for all $x \in (\alpha_-, 0)$, where $y(x)$ is defined in (12).

Lemma 4.8 *Assume that (2)-(4) and (9) hold. Then, for all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$, one has*

(i) $\Theta_+(r_+(\lambda, E)) \leq S_-(r_-(\lambda, E))$,

(ii) $\Theta_+(r_+(\lambda, E)) \leq S_+(r_+(\lambda, E))$,

(iii) $S_-(r_-(\lambda, E)) \leq \Theta_-(r_-(\lambda, E))$,

(iv) $S_+(r_+(\lambda, E)) \leq \Theta_-(r_-(\lambda, E))$.

Proof of Lemma 4.8

Proof of (i) For all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$, one has

$$S_-(r_-(\lambda, E)) = -\int_0^{r_-(\lambda,E)} \{F(r_-(\lambda, E)) - F(\xi)\}^{-1/p} d\xi.$$

Using a simple change of variables one deduces,

$$S_-(r_-(\lambda, E)) = \int_0^{-r_-(\lambda,E)} \{F(r_-(\lambda, E)) - F(-\xi)\}^{-1/p} d\xi. \tag{85}$$

Using (77) one gets

$$\begin{aligned} S_-(r_-(\lambda, E)) &= \Theta_+(r_+(\lambda, E)) \\ &\quad + \int_{r_+(\lambda,E)}^{-r_-(\lambda,E)} \{F(r_-(\lambda, E)) - F(-\xi)\}^{-1/p} d\xi \end{aligned}$$

and by Lemma 4.3 it follows that the integral above is positive. Therefore, Assertion **(i)** is proved.

Proof of (ii) Recall that F is *p.h.e.* in $[\alpha_-, -\alpha_-]$, (see (10)), hence, for all $\lambda > 0$, $E \in (0, E_*^-(\lambda))$, and $0 < \xi < r_+(\lambda, E) \leq -r_-(\lambda, E) \leq -\alpha_-$, one has

$$\{F(r_-(\lambda, E)) - F(-\xi)\}^{-1/p} \leq \{F(r_-(\lambda, E)) - F(\xi)\}^{-1/p}, \tag{86}$$

Thus,

$$
\begin{aligned}
&\int_0^{r_+(\lambda,E)}\{F(r_-(\lambda,E))-F(-\xi)\}^{-1/p}d\xi\\
\leq\ &\int_0^{r_+(\lambda,E)}\{F(r_-(\lambda,E))-F(\xi)\}^{-1/p}d\xi.
\end{aligned}
$$

Using (77), it follows that

$$
\begin{aligned}
\Theta_+(r_+(\lambda,E)) &= \int_0^{r_+(\lambda,E)}\{F(r_+(\lambda,E))-F(-\xi)\}^{-1/p}d\xi\\
&\leq \int_0^{r_+(\lambda,E)}\{F(r_+(\lambda,E))-F(\xi)\}^{-1/p}d\xi\\
&= S_+(r_+(\lambda,E)).
\end{aligned}
$$

Therefore, Assertion **(ii)** is proved.

Proof of (iii) Recall that F is *p.h.e.* in $[\alpha_-,-\alpha_-]$, (see (10)), and for all $\lambda>0$ and $E\in(0,E_*^-(\lambda))$ one has $y(r_-(\lambda,E))=r_+(\lambda,E)$. Thus, for $\xi\in(-y(r_-(\lambda,E)),0)$,

$$
\{F(r_-(\lambda,E))-F(\xi)\}^{-1/p}\leq\{F(r_-(\lambda,E))-F(-\xi)\}^{-1/p}.
$$

Therefore,

$$
\begin{aligned}
&\int_{-y(r_-(\lambda,E))}^{0}\{F(r_-(\lambda,E))-F(\xi)\}^{-1/p}d\xi\\
\leq\ &\int_{-y(r_-(\lambda,E))}^{0}\{F(r_-(\lambda,E))-F(-\xi)\}^{-1/p}d\xi.
\end{aligned}
$$

So,

$$
\begin{aligned}
S_-(r_-(\lambda,E)) &= \int_{r_-(\lambda,E)}^{0}\{F(r_-(\lambda,E))-F(\xi)\}^{-1/p}d\xi\\
&\leq \int_{r_-(\lambda,E)}^{-y(r_-(\lambda,E))}\{F(r_-(\lambda,E))-F(\xi)\}^{-1/p}d\xi\\
&\quad+\int_{-y(r_-(\lambda,E))}^{0}\{F(r_-(\lambda,E))-F(-\xi)\}^{-1/p}d\xi\\
&= \Theta_-(r_-(\lambda,E)).
\end{aligned}
$$

Assertion **(iii)** is proved.

Proof of (iv) For all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$ one has

$$\begin{aligned}\Theta_-(r_-(\lambda,E)) &= \int_{r_-(\lambda,E)}^{-y(r_-(\lambda,E))} \{F(r_-(\lambda,E)) - F(\xi)\}^{-1/p} d\xi \\ &\quad + \int_{-y(r_-(\lambda,E))}^{0} \{F(r_-(\lambda,E)) - F(-\xi)\}^{-1/p} d\xi.\end{aligned} \tag{87}$$

Then by Lemma 4.3, one has $-y(r_-(\lambda,E)) = -r_+(\lambda,E)$ and $F(r_-(\lambda,E)) = F(r_+(\lambda,E))$. Thus, using a simple change of variable, it follows that

$$\int_{-y(r_-(\lambda,E))}^{0} \{F(r_-(\lambda,E)) - F(-\xi)\}^{-1/p} d\xi = S_+(r_+(\lambda,E)).$$

On the other hand, the first integral in (87) is positive, since $r_-(\lambda,E) \le -y(r_-(\lambda,E))$. Thus, $\Theta_-(r_-(\lambda,E)) \ge S_+(r_+(\lambda,E))$.

Therefore, Assertion **(iv)** is proved which completes the proof of Lemma 4.8. $\diamondsuit$

Remark 4.9 *It is well known that when f is odd then*

$$T_+(\lambda,\cdot) = T_-(\lambda,\cdot), \text{ for all } \lambda > 0. \tag{88}$$

The estimates in Lemma 4.8 imply (88) in the odd case. In fact if f is odd then $\alpha_+ = -\alpha_-$, F is even,

$$E_*^+(\lambda) = E_*^-(\lambda), \text{ for all } \lambda > 0$$

$$r_+(\lambda,E) = -r_-(\lambda,E), \text{ for all } \lambda > 0.$$

Also, since for all $\lambda > 0$ and $E \in (0, E_*^\pm(\lambda))$,

$$\Theta_-(r_-(\lambda,E)) = \int_{r_-}^{-y(r_-)} \{F(r_-) - F(\xi)\}^{-1/p} d\xi + \int_{-y(r_-)}^{0} \{F(r_-) - F(-\xi)\}^{-1/p} d\xi,$$

then by Lemma 4.3, it follows that

$$\begin{aligned}\Theta_-(r_-(\lambda,E)) &= \int_{r_-}^{-r_+} \{F(r_+) - F(\xi)\}^{-1/p} d\xi + \int_{-r_+}^{0} \{F(r_+) - F(-\xi)\}^{-1/p} d\xi, \\ &= \int_{-r_+}^{0} \{F(r_+) - F(-\xi)\}^{-1/p} d\xi.\end{aligned}$$

On the other hand, since the function defined on $(-r_+, r_+)$ by $\xi \mapsto \{F(r_+) - F(-\xi)\}^{-1/p}$ is even then

$$\int_{-r_+}^{0} \{F(r_+) - F(-\xi)\}^{-1/p} d\xi = \int_{0}^{r_+} \{F(r_+) - F(-\xi)\}^{-1/p} d\xi.$$

Therefore,

$$\Theta_-(r_-(\lambda,E)) = \Theta_+(r_+(\lambda,E)),\ \textit{for all } \lambda>0 \textit{ and all } E\in(0,E_*(\lambda)). \qquad (89)$$

Now, by the estimates of Lemma 4.8 and (89), equation (88) follows.

Lemma 4.10 *Assume that (2)-(4) and (9) hold. Then, for all $\lambda > 0$, $E \in (0, E_*^-(\lambda))$, and for all integer $k \geq 2$,*

$$k(p'\lambda)^{-1/p}\Theta_+(r_+(\lambda,E)) \leq T_k^\kappa(\lambda,E) \leq k(p'\lambda)^{-1/p}\Theta_-(r_-(\lambda,E)).$$

Proof. According to (82) and the definition for $T_k^\kappa(\lambda,E)$, the proof is an immediate consequence of Lemma 4.8. ♢

Lemma 4.11 *For all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$ the following identity holds;*

$$\Theta_+(r_+(\lambda,E)) + \Theta_-(r_-(\lambda,E)) = S_+(r_+(\lambda,E)) + S_-(r_-(\lambda,E)).$$

Proof. For all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$, we write $\Theta_-(r_-(\lambda,E))$ as follows

$$\begin{aligned}\Theta_-(r_-(\lambda,E)) &= \int_{r_-(\lambda,E)}^0 \{F(r_-(\lambda,E)) - F(\xi)\}^{-1/p}d\xi \\ &\quad - \int_{-r_+(\lambda,E)}^0 \{F(r_-(\lambda,E)) - F(\xi)\}^{-1/p}d\xi \\ &\quad + \int_{-r_+(\lambda,E)}^0 \{F(r_-(\lambda,E)) - F(-\xi)\}^{-1/p}d\xi.\end{aligned}$$

The first integral is equal to $S_-(r_-(\lambda,E))$. On the other hand, the change of variable $\xi = -r_+(\lambda,E)t$, and (77) imply that the second integral is equal to $\Theta_+(r_+(\lambda,E))$. The change of variable $\xi = -t$, and (77) imply that the third integral is equal to $S_+(r_+(\lambda,E))$. Therefore, Lemma 4.11 is proved. ♢

Lemma 4.12 *Assume that (2)-(5), $(6)_-$ and (9) hold. Then, for all $\lambda > 0$, one has*

(i) $$\lim_{E\to 0^+}\Theta_\pm(r_\pm(\lambda,E)) = \begin{cases} +\infty & \textit{if} \quad q-p>0 \\ \frac{1}{2}(\frac{p'\lambda_1}{a_0})^{1/p} & \textit{if} \quad q-p=0 \\ 0 & \textit{if} \quad q-p<0. \end{cases}$$

(ii) $\lim_{E\to E_*^-(\lambda)}\Theta_\pm(r_\pm(\lambda,E)) = \ell_\pm$ *with*

$$\begin{cases} \ell_\pm = +\infty & \textit{if} \quad q-p\geq 0 \textit{ and } -\alpha_* = \alpha_- \\ \ell_+ \in (0,+\infty) \textit{ and } \ell_- = +\infty & \textit{if} \quad q-p\geq 0 \textit{ and } -\alpha_* > \alpha_- \\ \ell_\pm \in (0,+\infty) & \textit{if} \quad q-p<0. \end{cases}$$

Proof of Lemma 4.12

Proof of Assertion (i). For all $\lambda > 0$ and $E \in (0, E_*^-(\lambda))$ one has $E^p/(p'\lambda) = F(r_+(\lambda,E))$, hence,

$$\Theta_+(r_+(\lambda,E)) = \int_0^{r_+(\lambda,E)} \{(E^p/p'\lambda) - F(-\xi)\}^{-1/p}\, d\xi.$$

The change of variable $\xi = r_+(\lambda,E)t$, yields

$$\Theta_+(r_+(\lambda,E)) = \frac{r_+(\lambda,E)}{E}(p'\lambda)^{1/p}\int_0^1 \{1 - p'\lambda F(-r_+(\lambda,E)t)/E^p\}^{-1/p}\, dt.$$

By (75) one has

$$\lim_{E\to 0^+}\frac{r_+(\lambda,E)}{E} = \begin{cases} +\infty & \text{if} \quad q-p>0 \\ (\frac{p-1}{\lambda a_0})^{1/p} & \text{if} \quad q-p=0 \\ 0 & \text{if} \quad q-p<0. \end{cases} \tag{90}$$

On the other hand, by (76) one has for all $(\kappa,\eta) \in \{+,-\}^2$,

$$\lim_{E\to 0^+}\int_0^1 \{1 - p'\lambda F(\kappa r_\eta(\lambda,E)t)/E^p\}^{-1/p}\, dt = \int_0^1 \{1-t^q\}^{-1/p} \tag{91}$$

with

$$\int_0^1 \{1-t^q\}^{-1/p} = \begin{cases} \frac{1}{q}B(\frac{1}{q}, 1-\frac{1}{p}) \in \mathbb{R} & \text{if} \quad q-p \neq 0 \\ \frac{1}{2}(\frac{\lambda_1}{p-1})^{1/p} & \text{if} \quad q-p=0. \end{cases} \tag{92}$$

Therefore, the limit $\lim_{E\to 0^+}\Theta_+(r_+(\lambda,E))$ follows.

Assume that $q-p>0$. By Assertion **(ii)** of Lemma 4.4, one has $\lim_{E\to 0^+} T_-(\lambda,E) = +\infty$, and by (82) and Assertion **(iii)** of Lemma 4.8, it follows that $\lim_{E\to 0^+}\Theta_-(r_-(\lambda,E)) = +\infty$.

Assume that $q-p=0$. In this case, for all $\lambda>0$ and $E \in (0,E_*^-(\lambda))$, we use the identity in Lemma 4.11. That is, we write $\Theta_-(r_-(\lambda,E))$ as follows

$$\Theta_-(r_-(\lambda,E)) = S_+(r_+(\lambda,E)) + S_-(r_-(\lambda,E)) - \Theta_+(r_+(\lambda,E))$$

and we prove that each term of the right hand side tends to the same limit; $\frac{1}{2}(p'\lambda_1/a_0)^{1/p}$.

The limits of $S_-(r_-(\lambda,E))$ and $S_+(r_+(\lambda,E))$ follows by (82) and Assertion **(ii)** of Lemma 4.4, and the limit of $\Theta_+(r_+(\lambda,E))$ was computed above. Therefore, $\lim_{E\to 0^+}\Theta_-(r_-(\lambda,E)) = \frac{1}{2}(p'\lambda_1/a_0)^{1/p}$ which completes the proof of Assertion **(i)**.

Proof of Assertion (ii). By (77) it follows that

$$\Theta_+(r_+(\lambda,E)) = \int_0^{r_+(\lambda,E)} \{F(r_-(\lambda,E)) - F(-\xi)\}^{-1/p}\,d\xi.$$

A simple change of variable yields

$$\Theta_+(r_+(\lambda,E)) = \int_{-r_+(\lambda,E)}^{0} \{F(r_-(\lambda,E)) - F(\xi)\}^{-1/p}\,d\xi$$

and thus,

$$\lim_{E\to E_*^-(\lambda)} \Theta_+(r_+(\lambda,E)) = \int_{-r_+(\lambda,E_*^-(\lambda))}^{0} \left\{F(r_-(\lambda,E_*^-(\lambda))) - F(\xi)\right\}^{-1/p}\,d\xi.$$

By (74), one has $r_-(\lambda,E_*^-(\lambda)) = \alpha_-$ and by Assertion **(iii)** of Lemma 4.4, it follows that $r_+(\lambda,E_*^-(\lambda)) = \alpha_*$.

Therefore,

$$\lim_{E\to E_*^-(\lambda)} \Theta_+(r_+(\lambda,E)) = \int_{-\alpha_*}^{0} \{F(\alpha_-) - F(\xi)\}^{-1/p}\,d\xi.$$

According to the definition of α_* (see (12)), one has, $\alpha_- \le -\alpha_*$. Thus, one has to distinguish two cases:

If $-\alpha_* = \alpha_-$,

$$\lim_{E\to E_*^-(\lambda)} \Theta_+(r_+(\lambda,E)) = \int_{\alpha_-}^{0} \{F(\alpha_-) - F(\xi)\}^{-1/p}\,d\xi = S_-(\alpha_-).$$

Thus, by Assertion **(i)** of Lemma 4.4, it follows that

$$S_-(\alpha_-) = \begin{cases} +\infty & \text{if} \quad q-p \ge 0 \\ \ell_+ \in (0,+\infty) & \text{if} \quad q-p < 0. \end{cases}$$

If $-\alpha_* > \alpha_-$, it follows that the integral $\int_{-\alpha_*}^{0} \{F(\alpha_-) - F(\xi)\}^{-1/p}\,d\xi$ is a positive real number, since the integrand function is continuous on the compact interval $[-\alpha_*, 0]$.

Therefore, the claims related to the limit $\lim_{E\to E_*^-} \Theta_+(r_+(\lambda,E))$ follows.

Assume that $q-p \ge 0$. By Assertion **(i)** of Lemma 4.4, one has $\lim_{E\to E_*^-} T_-(\lambda,E) = +\infty$, and by (82) and Assertion **(iii)** of Lemma 4.8, it follows that $\lim_{E\to E_*^-} \Theta_-(r_-(\lambda,E)) = +\infty$.

Assume that $q-p < 0$. In this case one has, by Assertion **(i)** of Lemma 4.4, $\lim_{E\to E_*^-} S_-(r_-(\lambda,E)) = S_-(\alpha_-) \in (0,+\infty)$, and by Assertion **(iii)** of Lemma 4.4,

$$\lim_{E\to E_*^-} S_+(r_+(\lambda,E)) = S_+(\alpha_*) = (\lambda p')^{1/p} T_+(\lambda, E_*^-(\lambda)),$$

and $T_+(\lambda, E_*^-(\lambda)) \in (0,+\infty)$, since $E_*^-(\lambda) \in int(D_+(\lambda)) = int(dom(T_+(\lambda,\cdot)))$. Also, $\lim_{E\to E_*^-} \Theta_+(r_+(\lambda,E)) = \ell_+ \in (0,+\infty)$ (proved above). Thus, by the identity of Lemma 4.11, it follows that $\lim_{E\to E_*^-} \Theta_-(r_-(\lambda,E)) = \ell_- \in (0,+\infty)$. Therefore, Assertion **(ii)** follows, which completes the proof of Lemma 4.12. $\diamondsuit$

Lemma 4.13 *Assume that (2)-(4) and* $(13)_\pm$ *hold. Then, for any* $p > 1$ *one has,*

$$r^2\frac{d^2S_\pm}{dr^2}(r) + 2(p-1)r\frac{dS_\pm}{dr}(r) > 0, \text{ for all } r \in I_\pm(\alpha_\pm),$$

where $I_+(\alpha_+) = (0,\alpha_+)$ *and* $I_-(\alpha_-) = (\alpha_-,0)$.

Proof. Some easy computations show that for all $\lambda > 0$ and $r \in I_\pm(\alpha_\pm)$

$$r^2\frac{d^2S_\pm}{dr^2}(r) + 2(p-1)r\frac{dS_\pm}{dr}(r)$$

$$= \frac{\pm1}{p}\int_0^r \frac{\Psi(r)-\Psi(s)}{(F(r)-F(s))^{(p+1)/p}}ds \pm \left(\frac{p+1}{p}\right)\int_0^r \frac{(H(r)-H(s))^2}{(F(r)-F(s))^{(2p+1)/p}}ds$$

where,

$$\begin{aligned} H(x) &= pF(x) - xf(x), \text{for all } x \in I_\pm(\alpha_\pm) \\ \Psi(x) &= p(p-3)F(x) + 2xf(x) - x^2f'(x), \text{for all } x \in I_\pm(\alpha_\pm). \end{aligned}$$

A differentiation yields

$$\Psi'(x) = (p-2)(p-1)f(x) - x^2f''(x), \text{ for all } x \in I_\pm(\alpha_\pm).$$

Thus, $(13)_\pm$ implies that $\pm\Psi$ is strictly increasing in $I_\pm(\alpha_\pm)$. Then, $\Psi(r) - \Psi(s) > 0$, for all $s \in I_\pm(r) \subset I_\pm(\alpha_\pm)$. Therefore, Lemma 4.13 is proved.

5 Proof of the main results

To prove our main results we make use of the quadrature method; Theorem 3.1. Hence, we have to resolve equations of the type $T(E) = (1/2)$, where T designates, in each case, the appropriate time map.

Proof of Theorem 1.1 Let us assume that $1 < q < p$.

Proof of Assertion (i) For all $\kappa = +,-$, if (2)-(5) and $(6)_\kappa$ hold, then for each $\lambda > 0$, the function $E \mapsto T_\kappa(\lambda,E)$ is defined in $D_\kappa(\lambda) = (0, E_*^\kappa(\lambda))$ and

$$\lim_{E\to0^+} T_\kappa(\lambda,E) = 0, \quad \lim_{E\mapsto E_*^\kappa} T_\kappa(\lambda,E) = (\lambda p')^{-1/p}S_\kappa(\alpha_\kappa) < +\infty.$$

(Lemma 4.4). Therefore, $\tilde{D}_\kappa(\lambda) = [0, E_*^\kappa(\lambda)]$ (see Remark 3.2), and the equation $T_\kappa(\lambda, E) = (1/2)$ in the variable $E \in \tilde{D}_\kappa(\lambda) - \{0\}$ admits a solution in $\tilde{D}_\kappa(\lambda) - \{0\}$ provided that

$$(\lambda p')^{-1/p} S_\kappa(\alpha_\kappa) \geq (1/2),$$

that is, provided that $\lambda \leq (2S_\kappa(\alpha_\kappa))^p/p'$. Furthermore, by Lemma 4.7, if $(15)_\kappa$ holds, the function $E \mapsto T_\kappa(\lambda, E)$ is strictly increasing in $\tilde{D}_\kappa(\lambda)$. Thus, the equation $T_\kappa(\lambda, E) = (1/2)$ in the variable $E \in \tilde{D}_\kappa(\lambda) - \{0\}$ admits a solution in $\tilde{D}_\kappa(\lambda) - \{0\}$ if and only if $\lambda \leq (2S_\kappa(\alpha_\kappa))^p/p'$, and in this case the solution is unique since $T_\kappa(\lambda, \cdot)$ is strictly increasing.

Proof of Assertion (ii) For all $k \geq 2$, if (2)-(5), $(6)_-$, and (9) hold, then for each $\lambda > 0$, the function $E \mapsto T_k^\kappa(\lambda, E)$ is defined in $D(\lambda) = (0, E_*^-(\lambda))$ and by Lemma 4.5 its limit at 0 is 0 and by Lemma 4.6 its limit at $E_*^-(\lambda)$ is

- $n(\lambda p')^{-1/p}(S_-(\alpha_-) + S_+(\alpha_*))$ if $k = 2n$,
- $(\lambda p')^{-1/p}(nS_-(\alpha_-) + (n+1)S_+(\alpha_*))$ if $k = 2n+1$, and $\kappa = +$,
- $(\lambda p')^{-1/p}((n+1)S_-(\alpha_-) + nS_+(\alpha_*))$ if $k = 2n+1$, and $\kappa = -$.

Thus, $\tilde{D}(\lambda) = [0, E_*^-(\lambda)]$ and the equation $T_k^\kappa(\lambda, E) = \frac{1}{2}$ in the variable $E \in \tilde{D}(\lambda) - \{0\}$ admits a solution in $\tilde{D}(\lambda) - \{0\}$ provided that,

- $\frac{1}{2} \leq (\lambda p')^{-1/p}(nS_-(\alpha_-) + nS_+(\alpha_*))$, if $k = 2n$,
- $\frac{1}{2} \leq (\lambda p')^{-1/p}(nS_-(\alpha_-) + (n+1)S_+(\alpha_*))$, if $k = 2n+1$ and $\kappa = +$,
- $\frac{1}{2} \leq (\lambda p')^{-1/p}((n+1)S_-(\alpha_-) + nS_+(\alpha_*))$, if $k = 2n+1$ and $\kappa = -$,

that is, provided that,

- $0 < \lambda \leq (n\frac{2S_-(\alpha_-)}{(p')^{1/p}} + n\frac{2S_+(\alpha_*)}{(p')^{1/p}})^p$, if $k = 2n$.
- $0 < \lambda \leq (n\frac{2S_-(\alpha_-)}{(p')^{1/p}} + (n+1)\frac{2S_+(\alpha_*)}{(p')^{1/p}})^p$, if $k = 2n+1$ and $\kappa = +$.
- $0 < \lambda \leq ((n+1)\frac{2S_-(\alpha_-)}{(p')^{1/p}} + n\frac{2S_+(\alpha_*)}{(p')^{1/p}})^p$, if $k = 2n+1$ and $\kappa = -$.

Furthermore, by Lemma 4.7, if $(15)_+$ and $(15)_-$ hold, the function $E \mapsto T_k^\kappa(\lambda, E)$ is strictly increasing in $\tilde{D}(\lambda)$ so that, the equation $T_k^\kappa(\lambda, E) = \frac{1}{2}$ in the variable $E \in \tilde{D}(\lambda)$ admits a solution in $\tilde{D}(\lambda)$ if and only if

- $0 < \lambda \leq (n\frac{2S_-(\alpha_-)}{(p')^{1/p}} + n\frac{2S_+(\alpha_*)}{(p')^{1/p}})^p$, if $k = 2n$
- $0 < \lambda \leq (n\frac{2S_-(\alpha_-)}{(p')^{1/p}} + (n+1)\frac{2S_+(\alpha_*)}{(p')^{1/p}})^p$, if $k = 2n+1$ and $\kappa = +$
- $0 < \lambda \leq ((n+1)\frac{2S_-(\alpha_-)}{(p')^{1/p}} + n\frac{2S_+(\alpha_*)}{(p')^{1/p}})^p$, if $k = 2n+1$ and $\kappa = -$,

and in this case the solution is unique since the function $E \mapsto T_k^\kappa(\lambda, E)$ is strictly increasing in $\tilde{D}(\lambda)$.

Therefore, if we put $J_\pm := \frac{2S_\pm(\alpha_\pm)}{(p')^{1/p}}$ and $J_* := \frac{2S_+(\alpha_*)}{(p')^{1/p}}$, Theorem 1.1 is proved. $\diamondsuit$

Proof of Theorem 1.2 Let us assume that $1 < q = p$.

For all $k \geq 1$ and $\kappa = +, -$, assume that conditions (2)-(5), and $(\mathbf{H})_k^\kappa$ hold. It follows for each $\lambda > 0$, the function $E \mapsto T_k^\kappa(\lambda, E)$ is defined in $D_k^\kappa(\lambda) = (0, E_k^\kappa(\lambda))$ and by Lemma 4.5 and 4.6, one has

$$\lim_{E \to 0^+} T_k^\kappa(\lambda, E) = \frac{1}{2}(\frac{\lambda_k}{\lambda a_0})^{1/p}, \qquad \lim_{E \to E_k^\kappa} T_k^\kappa(\lambda, E) = +\infty.$$

Therefore, $\tilde{D}_k^\kappa(\lambda) = [0, E_k^\kappa(\lambda))$ (see Remark 3.2). So, the equation $T_k^\kappa(\lambda, E) = \frac{1}{2}$ in the variable $E \in \tilde{D}_k^\kappa(\lambda) - \{0\}$, admits a solution in $\tilde{D}_k^\kappa(\lambda) - \{0\}$ provided that

$$\frac{1}{2}(\frac{\lambda_k}{\lambda a_0})^{1/p} < \frac{1}{2},$$

that is, provided that, $\lambda > \lambda_k / a_0$. Furthermore, by Lemma 4.7, if $(\mathbf{K})_k^\kappa$ holds, the function $E \mapsto T_k^\kappa(\lambda, E)$ is strictly increasing in $\tilde{D}_k^\kappa(\lambda) - \{0\}$ and then, the equation $T_k^\kappa(\lambda, E) = \frac{1}{2}$ in the variable $E \in \tilde{D}_k^\kappa(\lambda) - \{0\}$, admits a solution in $\tilde{D}_k^\kappa(\lambda) - \{0\}$ if and only if $\lambda > \lambda_k / a_0$, and in this case the solution is unique since $T_k^\kappa(\lambda, \cdot)$ is strictly increasing.

Proof of Theorem 1.3 Let us assume that $1 < p < q$.

Proof of Assertion (i) If (2)-(5), and $(6)_\pm$ hold, then for each $\lambda > 0$, the function $E \mapsto T_\pm(\lambda, E)$ is defined in $D_\pm(\lambda) = (0, E_*^\pm(\lambda))$ and by Lemma 4.4,

$$\lim_{E \to 0^+} T_\pm(\lambda, E) = \lim_{E \to E_*^\pm} T_\pm(\lambda, E) = +\infty. \tag{93}$$

Therefore, $\tilde{D}_\pm(\lambda) = D_\pm(\lambda) = (0, E_*^\pm(\lambda))$, and the function $E \mapsto T_\pm(\lambda, E)$ admits at least a minimum value. Recall that for all $\lambda > 0$, the function $E \mapsto \pm r_\pm(\lambda, E)$ is an increasing C^1-diffeomorphism from $(0, E_*^\pm(\lambda))$ onto $(0, \pm\alpha_\pm)$. So, regarding (82) it follows that the local maximum and minimum values of $T_\pm(\lambda, \cdot)$ are in a one to one correspondence with those of $S_\pm(\cdot)$ respectively. That is, $S_\pm(\cdot)$ attains a local maximum (resp. minimum) value at $r_\pm^* \in I_\pm(\alpha_\pm)$ if and only if $T_\pm(\lambda, \cdot)$ does so at $r_\pm^{-1}(\lambda, r_\pm^*)$, where $r_\pm^{-1}(\lambda, \cdot)$ is the function inverse to $r_\pm(\lambda, \cdot)$. Let $r_\pm^* \in I_\pm(\alpha_\pm)$ be a point where $S_\pm$ attains its global minimum value in $I_\pm(\alpha_\pm)$. (The existence of $r_\pm^*$ is guaranteed by the limits $\lim_{r \to 0^+} S_\pm(r) = \lim_{r \to \pm\alpha_\pm} S_\pm(r) = +\infty$ which follow from (82) and (93)). Thus, for each fixed $\lambda > 0$, there exists a unique $\tilde{E}^\pm = \tilde{E}^\pm(\lambda) \in (0, E_*^\pm(\lambda))$

such that $r_\pm^* = r_\pm(\lambda, \tilde{E}^\pm)$ and then for all $\lambda > 0$, and all $E \in (0, E_*^\pm(\lambda))$,

$$\begin{aligned} T_\pm(\lambda, E) &= (\lambda p')^{-1/p} S_\pm(r_\pm(\lambda, E)) \\ &\geq (\lambda p')^{-1/p} S_\pm(r_\pm^*) \\ &= T_\pm(\lambda, \tilde{E}^\pm), \end{aligned}$$

that is, $T_\pm(\lambda, \cdot)$ attains its global minimum value at $\tilde{E}^\pm(\lambda) \in (0, E_*^\pm(\lambda))$. It follows that

- If $(\lambda p')^{-1/p} S_\pm(r_\pm^*) > (1/2)$, the equation $T_\pm(\lambda, E) = (1/2)$ in the variable $E \in (0, E_*^\pm(\lambda))$ admits no solution.
- If $(\lambda p')^{-1/p} S_\pm(r_\pm^*) = (1/2)$, the equation $T_\pm(\lambda, E) = (1/2)$ in the variable $E \in (0, E_*^\pm(\lambda))$ admits at least a solution (Notice that $S_\pm$ may attains its global minimum at two-or more-distinct points; $r_\pm^*$ and other point(s)!).
- If $(\lambda p')^{-1/p} S_\pm(r_\pm^*) < (1/2)$, the equation $T_\pm(\lambda, E) = (1/2)$ in the variable $E \in (0, E_*^\pm(\lambda))$ admits at least two solutions.

Hence, the first part of Assertion **(i)** of Theorem 1.3 follows if we put $\mu_1^\pm = (2S_\pm(r_\pm^*))^p/p'$. Notice that $S_\pm(r_\pm^*)$ is the (unique) global minimum value of the function $S_\pm(\cdot)$, and do not depends on $r_\pm^*$ which may be not unique.

Now, if $(13)_\pm$ holds, let us show that the function $S_\pm(\cdot)$ admits at most a minimum value (and no maximum one) in $I_\pm(\alpha_\pm)$. To this end, since the set $I_\pm(\alpha_\pm)$ is an interval, that is, a connected set, it suffices to show that $S_\pm(\cdot)$ admits a minimum value at each of its critical points. This follows if $S_\pm(\cdot)$ is convex in a neighborhood of each of its critical points, that is

$$\frac{dS_\pm}{dr}(r) = 0 \Longrightarrow \frac{d^2 S_\pm}{dr^2}(r) > 0, \quad \text{for all } r \in I_\pm(\alpha_\pm).$$

But this holds as an immediate consequence of Lemma 4.13. Therefore, $S_\pm(\cdot)$ admits at most a minimum value (and no maximum one) in $I_\pm(\alpha_\pm)$, and for all $\lambda > 0$, $T_\pm(\lambda, \cdot)$ admits a unique minimum value (and no maximum one) in $\tilde{D}_\pm(\lambda)$. So, if we denote (as above) $r_\pm^*$ the point of $I_\pm(\alpha_\pm)$ at which the function $S_\pm(\cdot)$ attains its global minimum, it follows that,

- If $(\lambda p')^{-1/p} S_\pm(r_\pm^*) > (1/2)$, the equation $T_\pm(\lambda, E) = (1/2)$ in the variable $E \in \tilde{D}_\pm(\lambda)$ admits no solution.
- If $(\lambda p')^{-1/p} S_\pm(r_\pm^*) = (1/2)$, the equation $T_\pm(\lambda, E) = (1/2)$ in the variable $E \in \tilde{D}_\pm(\lambda)$ admits a unique solution.
- If $(\lambda p')^{-1/p} S_\pm(r_\pm^*) < (1/2)$, the equation $T_\pm(\lambda, E) = (1/2)$ in the variable $E \in \tilde{D}_\pm(\lambda)$ admits exactly two solutions.

Hence, the second part of **(i)** of Theorem 1.3 follows with $\mu_1^\pm = (2S_\pm(r_\pm^*))^p/p'$. Notice that this $\mu_1^\pm$ is the same as that of the first part of Assertion **(i)** above. Therefore, Assertion (**i**) of Theorem 1.3 is proved.

Proof of Assertion (ii) Assume that (2)-(5), $(6)_-$ and (9) hold. For each $\lambda > 0$, the function $E \mapsto \Theta_-(r_-(\lambda, E))$ is defined in $D(\lambda) = (0, E_*^-(\lambda))$ and by Lemma 4.12

$$\lim_{E \mapsto 0^+} \Theta_-(r_-(\lambda, E)) = \lim_{E \mapsto (E_*^-)^-} \Theta_-(r_-(\lambda, E)) = +\infty. \tag{94}$$

Recall that for all $\lambda > 0$, the function $E \mapsto r_-(\lambda, E)$ is a decreasing C^1-diffeomorphism from $(0, E_*^-(\lambda))$ onto $(\alpha_-, 0)$. Thus, (94) implies that

$$\lim_{x \mapsto \alpha_-} \Theta_-(x) = \lim_{x \mapsto 0^-} \Theta_-(x) = +\infty.$$

Therefore, there exists at least a $x_* \in (\alpha_-, 0)$ at which the function $x \mapsto \Theta_-(x)$ attains its global minimum on $(\alpha_-, 0)$, say $\Theta_-(x_*)$. Note that x_* is independent of λ. On the other hand, for each $\lambda > 0$, the function $E \mapsto T_k^\kappa(\lambda, E)$ is defined on $D(\lambda)$ and by Lemmas 4.5 and 4.6

$$\lim_{E \mapsto 0^+} T_k^\kappa(\lambda, E) = \lim_{E \mapsto (E_*^-)^-} T_k^\kappa(\lambda, E) = +\infty.$$

Therefore, $\tilde{D}(\lambda) = D(\lambda)$ (see Remark 3.2). Moreover, it attains its global minimum value at a certain point in $(0, E_*^-(\lambda))$ which may depends on λ, k and κ. Using Lemma 4.10 it follows that for $\lambda > 0$,

$$\min_{E \in (0, E_*^-(\lambda))} T_k^\kappa(\lambda, E) \le k(p'\lambda)^{-1/p} \Theta_-(r_-(\lambda, E)), \text{for all } E \in (0, E_*^-(\lambda)).$$

In particular,

$$\min_{E \in (0, E_*^-(\lambda))} T_k^\kappa(\lambda, E) \le k(p'\lambda)^{-1/p} \Theta_-(r_-(\lambda, \tilde{E}_-(\lambda))),$$

where $\tilde{E}_-(\lambda) \in (0, E_*^-(\lambda))$ is such that $r_-(\lambda, \tilde{E}_-(\lambda)) = x_*$. The existence of $\tilde{E}_-(\lambda)$ is guaranteed from the fact that $E \mapsto r_-(\lambda, E)$ is a C^1-diffeomorphism from $(0, E_*^-(\lambda))$ onto $(\alpha_-, 0)$. Thus

$$\min_{E \in (0, E_*^-(\lambda))} T_k^\kappa(\lambda, E) \le k(p'\lambda)^{-1/p} \Theta_-(x_*), \text{for all } \lambda > 0.$$

Therefore, if $k(p'\lambda)^{-1/p}\Theta_-(x_*) < (1/2)$, it follows that $\min_{E \in (0, E_*^-(\lambda))} T_k^\kappa(\lambda, E) < (1/2)$. Thus, the equation $T_k^\kappa(\lambda, E) = \frac{1}{2}$ in the variable $E \in \tilde{D}(\lambda)$ admits at least two solutions in $\tilde{D}(\lambda)$, for all λ satisfying: $k(p'\lambda)^{-1/p}\Theta_-(x_*) < (1/2)$, that is, for all $\lambda > (2k\Theta_-(x_*))^p/p'$. Therefore, the existence part of Assertion **(ii)** of Theorem 1.3 follows by taking $\mu_k = (2k\Theta_-(x_*))^p/p'$. (Notice that $\mu_k > 0$ since $\Theta_-(x) > 0$ for all $x \in (\alpha_-, 0)$, in particular, $\Theta_-(x_*) > 0$).

We have find a range of λ where there is existence of at least two solutions for the equation $T_k^\kappa(\lambda, E) = (1/2)$. Let us, now, look for an other range where there is no solution.

Using Lemma 4.10 it follows that for $\lambda > 0$,

$$k(p'\lambda)^{-1/p}(\inf_{E\in(0,E_*^-(\lambda))}\Theta_+(r_+(\lambda,E))) \le T_k^\kappa(\lambda,E),$$

for all $E \in (0, E_*^-(\lambda))$. In particular, for all $\lambda > 0$, one has

$$k(p'\lambda)^{-1/p}(\inf_{E\in(0,E_*^-(\lambda))}\Theta_+(r_+(\lambda,E))) \le \min_{E\in(0,E_*^-(\lambda))} T_k^\kappa(\lambda,E). \tag{95}$$

Recall that for all $\lambda > 0$, the function $E \mapsto r_+(\lambda, E)$ is strictly increasing on the interval $D(\lambda) = (0, E_*^-(\lambda)) \subset (0, E_*^+(\lambda))$. Hence, for all $\lambda > 0$, and $E \in D(\lambda)$, $r_+(\lambda, E) \in (0, r_+(\lambda, E_*^-(\lambda)))$. On the other hand, by Assertion **(iii)** of Lemma 4.4, the quantity $r_+(\lambda, E_*^-(\lambda))$ is independent of $\lambda > 0$ and $r_+(\lambda, E_*^-(\lambda)) = y(\alpha_-)$. Thus, for all $\lambda > 0$ the function $E \mapsto r_+(\lambda, E)$ is an increasing diffeomorphism from $(0, E_*^-(\lambda))$ onto $(0, y(\alpha_-))$. Therefore, the quantity $\inf_{E\in(0,E_*^-(\lambda))}\Theta_+(r_+(\lambda,E))$ is independent of $\lambda > 0$ and

$$\inf_{E\in(0,E_*^-(\lambda))}\Theta_+(r_+(\lambda,E)) = \inf_{0<x<y(\alpha_-)}\Theta_+(x).$$

Thus, (95) becomes,

$$k(p'\lambda)^{-1/p}\inf_{0<x<y(\alpha_-)}\Theta_+(x) \le \min_{E\in(0,E_*^-(\lambda))} T_k^\kappa(\lambda,E), \quad \text{for all } \lambda > 0.$$

Let us assume momently that $\inf_{0<x<y(\alpha_-)}\Theta_+(x) > 0$. It follows that for all λ satisfying $k(p'\lambda)^{-1/p}\inf_{0<x<y(\alpha_-)}\Theta_+(x) > (1/2)$, one has

$$\min_{E\in(0,E_*^-(\lambda))} T_k^\kappa(\lambda,E) > (1/2).$$

Hence, the equation $T_k^\kappa(\lambda,E) = \frac{1}{2}$ in the variable $E \in \tilde{D}(\lambda)$ admits no solution in $\tilde{D}(\lambda)$ for all λ satisfying $k(p'\lambda)^{-1/p}\inf_{0<x<y(\alpha_-)}\Theta_+(x) > (1/2)$, that is, for all $0 < \lambda < (2k\inf_{0<x<y(\alpha_-)}\Theta_+(x))^p/p'$, provided that $\inf_{0<x<y(\alpha_-)}\Theta_+(x)$ is strictly positive. In this case, we put, for all $k \ge 2$,

$$\nu_k = (2k\inf_{0<x<y(\alpha_-)}\Theta_+(x))^p/p',$$

and thus, Problem (1) admits no solution in A_k^κ for $\lambda \in (0, \nu_k)$, $k \ge 2$.

Now, let us prove that $\inf_{0<x<y(\alpha_-)}\Theta_+(x) > 0$. By Lemma 4.12, one has

$$\lim_{E\mapsto 0^+}\Theta_+(r_+(\lambda,E)) = +\infty, \text{ and } \lim_{E\mapsto(E_*^-)^-}\Theta_+(r_+(\lambda,E)) = \ell_+ \tag{96}$$

$$\text{with } \begin{cases} \ell_+ \in (0,+\infty) & \text{if} \quad y(\alpha_-) < -\alpha_- \\ \ell_+ = +\infty & \text{if} \quad y(\alpha_-) = -\alpha_-. \end{cases}$$

However, since for all $\lambda > 0$, the function $E \mapsto r_+(\lambda, E)$ is an increasing diffeomorphism from $(0, E_*^-(\lambda))$ onto $(0, y(\alpha_-))$, it follows from (96) that

$$\begin{aligned} \lim_{x \mapsto 0^+} \Theta_+(x) &= +\infty, \text{ and} \\ \lim_{x \to y(\alpha_-)} \Theta_+(x) &= \ell_+ \text{ with } \begin{cases} \ell_+ \in (0, +\infty) & \text{if} \quad y(\alpha_-) < -\alpha_- \\ \ell_+ = +\infty & \text{if} \quad y(\alpha_-) = -\alpha_-. \end{cases} \end{aligned}$$

Therefore, if $y(\alpha_-) < -\alpha_-$ (resp. $y(\alpha_-) = -\alpha_-$) there exists at least a $x^* \in (0, y(\alpha_-)]$ (resp. $x^* \in (0, y(\alpha_-))$) at which the function $x \mapsto \Theta_+(x)$ attains its global minimum on $(0, y(\alpha_-)]$ (resp. on $(0, y(\alpha_-))$).

Thus $\inf_{0<x<y(\alpha_-)} \Theta_+(x) = \Theta_+(x^*)$. But, it is clear from its definition that $\Theta_+(x) > 0$ for all $x \in (0, -\alpha_-)$. In particular $\Theta_+(x^*) > 0$. Therefore, the non existence part of Assertion **(ii)** of Theorem 1.3 is proved, which completes the proof of Theorem 1.3. $\diamondsuit$

6 Open questions

1. Regarding the identity of Lemma 4.11 one may ask if there exists a nonlinearity $\tilde{f}$ such that the corresponding time-maps would be

$$\tilde{T}_\pm(\lambda, E) = (\lambda p')^{-1/p} \Theta_\pm(r_\pm(\lambda, E)).$$

   In the affirmative, the identity of Lemma 4.11 implies that

$$\tilde{T}_{2n}(\lambda, E) = T_{2n}(\lambda, E).$$

   So, does $\tilde{T}^\pm_{2n+1}(\lambda, E) = T^\pm_{2n+1}(\lambda, E)$, and if not can one compare them? On the other hand what kind of symmetry does $\tilde{f}$ have: odd, *p.h.o.*, *n.h.o.*, or something else ? A comparison of the two solution sets corresponding to f and $\tilde{f}$ would be interesting.

   Notice that this is an inverse problem. Related results are available in the literature, see for instance Urabe [26], Schaaf [23, Chap. 4].

2. A description of the entire solution set should be interesting. Indeed the main results of this paper describe only the solutions which are inside the set $\cup_k A_k$. So, how does the solution set of Problem (1) look like outside $\cup_k A_k$? (Such kinds of descriptions can be found in Guedda and Veron [10] or Addou [6]).

3. Open questions are numerous. In fact, in view of the known results for $p = 2$ described in Section 2, one can ask to extend each one of them to the general case where $p > 1$, either in one or higher dimensions. But a question very close to our main results is to consider Problem (1) with f satisfying our conditions but with q_0 instead of q in (5) and $q_\pm$ instead of q in $(6)_\pm$ and to give a description of the solution set with respect to all these parameters: p, q_0, q_+, $q_- > 1$ when f is either *p.h.o.* or *n.h.o.* We

have considered only the cases $p > 1$, and $q_0 = q_+ = q_- > 1$, when f is *p.h.o.* We believe that the same method (Theorem 3.1) works, but much more patience is required!

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