*Electronic Journal of Differential Equations*, Vol. 2000(2000), No. 54, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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# SMOOTHNESS OF SOLUTIONS OF CONJUGATE BOUNDARY-VALUE PROBLEMS ON A MEASURE CHAIN

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ABSTRACT. In this paper we consider the $n^{th}$ order $\Delta$-differential equation (often refered to as a differential equation on a measure chain)

$$u^{\Delta_n}(t) = f(t, u(\sigma(t)), \ldots, u^{\Delta_{n-1}}(\sigma(t)))$$

satisfying $n$-point conjugate boundary conditions. We show that solutions depend continuously and smoothly on the boundary values.

## 1. Introduction

Differential equations on a measure chain (also called differential equations on time scales) have received much attention since Hilger's [17] work unifying continuous and discrete calculus. Subsequent works by Agarwal and Bohner [1], Aulback and Hilger [3], Erbe and Hilger [8], and Kaymakcalan, *et al.* [18] have furthered the development of calculus on measure chains. There are many recent papers that consider a variety of different problem for differential equations on a measure chain. See [2, 4, 9, 10] for example.

In this paper we are concerned with the continuous dependence and smoothness of solutions of differential equations on a measure chain with respect to boundary values. The results of this paper are patterned after those found in Henderson and Lee [16] and Henderson [13]. In [16], the authors considered the continuous dependence and smoothness of solutions of conjugate boundary-value problems for difference equations with respect to boundary conditions. In [13], the author considered the continuous dependence and smoothness of solutions of conjugate boundary-value problems for differential equations with respect to boundary conditions. Other works devoted to continuous dependence and smoothness of solutions with repsect to boundary values include [5, 6, 7, 11, 14, 15] and references therein.

Let $T$ be a nonempty closed subset of $\mathbb{R}$, and let $T$ have the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. Then $T$ is called a *measure chain*, (in some of the literature $T$ is called a *time scale*).

*Mathematics Subject Classification.* 34B15, 34B99, 39A10, 34A99 .
*Key words.* Measure chain, initial-value problem, boundary-value problem.

Submitted June 27, 2000. Published July 14, 2000.

**Definition 1.1.** *For $t < \sup T$ and $r > \inf T$, we define the* forward jump operator*,* $\sigma$*, and the* backward jump operator*,* $\rho$*, respectively, by*

$$\sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T,$$
$$\rho(r) = \sup\{\tau \in T \mid \tau < r\} \in T,$$

*for all $t, r \in T$. If $\sigma(t) > t$, $t$ is said to be* right scattered*, and if $\sigma(t) = t$, $t$ is said to be* right dense*. If $\rho(r) < r$, $r$ is said to be* left scattered*, and if $\rho(r) = r$, $r$ is said to be* left dense*.*

**Definition 1.2.** *For $x : T \to \mathbb{R}$ and $t \in T$ (assume $t$ is not left scattered if $t = \sup T$), we define the* delta derivative *of $x(t)$, $x^{\Delta}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, $U$, of $t$ such that*

$$\Big|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\Big| \le \varepsilon\Big|\sigma(t) - s\Big|,$$

*for all $s \in U$. Higher order delta derivatives are defined recursively,*

$$x^{\Delta_n}(t) = (x^{\Delta_{n-1}})^{\Delta}(t).$$

For convenience, we will use the notation $x^{\Delta_0}(t)$ to represent the function $x(t)$. That is, $x^{\Delta_0}(t) = x(t)$.

**Remarks:** If $x : T \to \mathbb{R}$ is continuous at $t \in T$, $t < \sup T$, and $t$ is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

In particular, if $T = \mathbb{Z}$, the integers, then

$$x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t),$$

whereas, if $t$ is right dense, then

$$x^{\Delta}(t) = x'(t).$$

Let $a, b \in T$. We define the closed interval $[a, b]$ by $[a, b] = \{t \in T | a \le t \le b\}$. Other closed, open, and half-open intervals in $T$ are similarly defined.

We consider solutions of the $\Delta$-differential equation

$$u^{\Delta_n}(t) = f(t, u(\sigma(t)), u^{\Delta}(\sigma(t)), \dots, u^{\Delta_{n-1}}(\sigma(t)),)\, t \in T, \tag{1}$$

We will assume throughout that

(A) $f(t, x_1, x_2, \dots, x_n) \colon T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

At times we will need to assume that

(B) $\frac{\partial f}{\partial x_i}(t, x_1, x_2, \dots, x_n) \colon T \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $1 \le i \le n$.

Given a solution, $u(t)$, of (1), we will also have need of the *variational equation along* $u(t)$,

$$z^{\Delta_n}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, u(\sigma(t)), u^{\Delta}(\sigma(t)), \dots, u^{\Delta_{n-1}}(\sigma(t)))z^{\Delta_i}(\sigma(t)), \tag{2}$$

In Section 2 we state two results for solutions of initial-value problems of (1). The first result is that solutions of initial-value problems depend continuously on initial data provided condition (A) holds. The second results states that if conditions (A) and (B) hold then solutions of initial-value problems can be differentiated with respect to initial values.

In Section 3 we state our main results which are analogues of the Theorems in section 2 for $n$-point conjugate boundary-value problems. The proofs of these Theoresm depend on the uniqueness of solutions of conjugate boundary value problems.

## 2. Smoothness with Respect to Initial Values

In this section we present theorems on continuous dependence and smoothness of solutions of initial-value problems with respect to initial values. The $\Delta$-differential equation along with the conditions

$$u^{\Delta_i}(t_0) = c_{i+1},\ 0 \le i \le n-1, \tag{3}$$

where $t_0 \in T$ is called an initial-value problem. The authors in [18] have shown that under a weaker condition than (A) initial value problems of the form (1), (3) have unique solutions. Furthermore they have shown that that the initial-value problem (1), (3) depends continuously on the initial values under this weaker condition. Theorem 2.1 is similar to the theorem on continuous dependence presented in [18].

**Theorem 2.1** (Continuous Dependence on Initial Values)**.** *Suppose that condition (A) is satisfied. Let $u(t; t_0, c_1, c_2, \dots, c_n)$ be the solution of (1), (3) where $t_0 \in T$ and $c_1, c_2, \dots c_n \in \mathbb{R}$. Then for each $\varepsilon > 0$ and $\tau$ such that $t_0 + \tau \in T$ there exists a $\delta(\varepsilon, t_0, \tau, c_1, \dots, c_n)$ such that if $|c_i - d_i| < \delta$, $1 \le i \le n$ then*

$$|u(t; t_0, c_1, c_2, \dots, c_n) - u(t; t_0, d_1, d_2, \dots, d_n)| < \varepsilon$$

*for all $t \in [t_0, t_0 + \tau]$.*

**Theorem 2.2.** *Assume that conditions (A) and (B) are satisfied. Let $u(t) = u(t; t_0, c_1, c_2, \dots, c_n)$ denote the solution of the initial-value problem (1), (3) where $t_0 \in T$ and $c_1, c_2, \dots c_n \in \mathbb{R}$. Then, given $\gamma_1, \dots, \gamma_n \in \mathbb{R}$, for each $1 \le j \le n$*

$$\beta_j(t) = \frac{\partial u}{\partial c_j}(t; t_0, \gamma_1, \dots, \gamma_n)$$

*exists and is the solution of the variational equation*

$$\begin{aligned} \beta_j^{\Delta_n}(t) \quad = \quad & \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, u(\sigma(t); t_0, \gamma_1, \dots, \gamma_n), u^{\Delta}(\sigma(t); t_0, \gamma_1, \dots, \gamma_n), \\ & \qquad \dots, u^{\Delta_{n-1}}(\sigma(t); t_0, \gamma_1, \dots, \gamma_n))\beta_j^{\Delta_i}(\sigma(t)) \end{aligned}$$

*and satisfies*

$$\beta_j^{\Delta_i}(t_0) = \delta_{i,j},\ 1 \le i \le n,$$

*where*

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

## 3. Smoothness with Respect to Boundary Values

In this section we state and prove analogues to Theorems 2.1 and 2.2 for $n$-point conjugate boundary-value problems.

**Definition 3.1.** *Let $t_1 < t_2 < \cdots < t_n \in T$ and let $u_1, u_2, \dots, u_n \in \mathbb{R}$. A boundary-value problem satisfying*

$$u(t_i) = u_i,\ 1 \le i \le n, \tag{4}$$

*is called an* $n$-point conjugate boundary-value problem.

We give some conditions characterizing disconjugacy for linear $\Delta$-differential equations in terms of *generalized zeros*. These conditions parallel those given by Hartman [12] for the disconjugacy for difference equations.

**Definition 3.2.** *Let $u : T \to \mathbb{R}$. We say that $u$ has a* generalized zero *at $t_0$ if either $u(t_0) = 0$ or if there is a $k \in \mathbb{N}$ such that $(-1)^k u(\rho^k(t_0))u(t_0) > 0$ and $u(\rho(t_0)) = u(\rho^2(t_0)) = \cdots = u(\rho^{k-1}(t_0)) = 0$.*

**Definition 3.3.** *The nonlinear $\Delta$-differential (1) is said to be $n$-point* disconjugate *on $T$ provided that whenever $u(t)$ and $v(t)$ are solutions of (1) such that if $u(t)-v(t)$ has $n$ generalized zeros at $t_1 < t_2 < \cdots < t_n \in T$ then $u(t) - v(t) \equiv 0$ on $[t_1, +\infty)$.*

In the case when (1) is linear, say

$$v^{\Delta_n}(t) = \sum_{i=1}^{n} \alpha_i(t) v^{\Delta_{i-1}}(\sigma(t)), \tag{5}$$

where $\alpha_i : T \to \mathbb{R}$, $1 \le i \le n$, we may reformulate Definition 3.3 as follows.

**Definition 3.4.** *The linear equation (5) is said to be $n$-*point disconjugate *on $T$ provided no nontrivial solution $u$ of (5) has $n$-generalized zeros on $T$.*

We adopt the following notation to distinguish initial-value problems from boundary value problems. Given $t_0 \in T$ and $c_1, \dots, c_n \in \mathbb{R}$, let $v(t) = v(t; t_0, c_1, \dots, c_n)$ denote the solution of the initial-value problem (1), (3). Given $t_1, \dots, t_n \in T$ and $v_1, \dots, v_n \in \mathbb{R}$, let $v(t) = v(t; t_1, \dots, t_n, v_1, \dots, v_n)$ denote the solution of the boundary value problem (1), (4).

We will use the Brouwer Theorem on Invariance of Domain, Theorem 3.5 below, to prove that solutions of (1) depend continuously on the boundary values when (1) is $n$-point disconjugate. To show that (1) depends smoothly on the boundary values we must further assume that the variational equation, (2), is $n$-point disconjugate.

**Theorem 3.5.** *If $U$ is an open subset of $\mathbb{R}^n$, $n$ dimensional Euclidean space, and $\varphi : U \to \mathbb{R}^n$ is one-to-one and continuous on $U$ then $\varphi$ is a homeomorphism and $\varphi(U)$ is an open subset of $\mathbb{R}^n$.*

**Theorem 3.6** (Continuous Dependence on Boundary Values)**.** *Suppose that condition (A) is satisfied and that (1) is $n$-point disconjugate on $T$. Let $y(t)$ be a solution of (1) on $[t_1, +\infty)$ and let $t_1 < t_2 < \cdots < t_n \in T$ be given. Then there exists an $\varepsilon > 0$ such that if $\gamma_i \in \mathbb{R}$, $1 \le i \le n$ where $|\gamma_i| < \varepsilon$, $1 \le i \le n$, then the boundary-value problem (1) satisfying*

$$u(t_i) = y(t_i) + \gamma_i, \quad 1 \le i \le n,$$

*has a unique solution $u(t; t_1, \dots, t_n, y(t_1) + \gamma_1, \dots, y(t_n) + \gamma_n)$. Furthermore we have $u(t; t_1, \dots, t_n,\, y(t_1) + \gamma_1, \dots, y(t_n) + \gamma_n)$ converging to $y(t)$ as $\varepsilon \to 0$.*

**Proof:** Let $t_1 < t_2 < \cdots < t_n \in T$ be given and define a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by $\varphi(c_1, c_2, \dots, c_n) = (v(t_1), v(t_2), \dots, v(t_n))$ where $v(t) = v(t; t_1, c_1, \dots c_n)$ is the solution of the (1) satisfying the initial conditions

$$v^{\Delta_{i-1}}(t_1) = c_i, \quad 1 \le i \le n.$$

We will show that $\varphi$ is one-to-one and continuous. It will then follow from Theorem 3.5, that $\varphi$ is a homeomorphism.

Suppose that $\varphi(c_1, c_2, \dots, c_n) = \varphi(c_1', c_2', \dots, c_n')$. Then,

$$\begin{aligned}&(v(t_1; t_1, c_1, \dots, c_n), v(t_2; t_1, c_1, \dots, c_n), \dots, v(t_n; t_1, c_1, \dots, c_n))\\&\quad = (v(t_1; t_1, c_1', \dots, c_n'), v(t_2; t_1, c_1', \dots, c_n'), \dots, v(t_n; t_1, c_1', \dots, c_n')).\end{aligned}$$

Now, equation (1) is $n$-point disconjugate on $T$ and hence solutions to (1), (4) are unique. And so, for all $t \in [t_1, +\infty)$ we have

$$v(t; t_1, c_1, \dots, c_n) = v(t; t_1, c_1', \dots, c_n').$$

In particular,

$$v^{\Delta_{i-1}}(t_1; t_1, c_1, \dots, c_n) = v^{\Delta_{i-1}}(t_1; t_1, c_1', \dots, c_n'),\ 1 \le i \le n.$$

Recalling our notation, we see that $(c_1, c_2, \dots, c_n) = (c_1', c_2', \dots, c_n')$. Hence $\varphi$ is one-to-one.

To show that $\varphi$ is continuous we consider a sequence $\{(c_1^\ell, c_2^\ell, \dots, c_n^\ell)\}_{\ell=1}^\infty$ which converges to $(c_1, c_2, \dots, c_n)$ as $\ell \to \infty$. By the continuous dependence on initial values, Theorem 2.1, $v(t; t_1, c_1^\ell, \dots, c_n^\ell) \to v(t; t_1, c_1, \dots, c_n)$ for all $t \in [t_1, +\infty)$ as $\ell \to \infty$. That is,

$$\lim_{\ell\to\infty} v(t; t_1, c_1^\ell, \dots, c_n^\ell) = v(t; t_1, c_1, \dots, c_n).$$

Thus, $\{\varphi(c_1^\ell, c_2^\ell, \dots, c_n^\ell)\}$ converges to $\varphi(c_1, c_2, \dots, c_n)$ as $\ell \to \infty$ and so $\varphi$ is continuous. By the Brouwer Theorem on Invariance of Domain, $\varphi$ is a homeomorphism onto its range, $\varphi(\mathbb{R}^n)$, and $\varphi(\mathbb{R}^n)$ is open in $\mathbb{R}^n$.

Let $y(t)$ be a solution of (1). Then $(y(t_1), \dots, y(t_n)) \in \varphi(\mathbb{R}^n)$. Since $\varphi(\mathbb{R}^n)$ is open, there exists an $\varepsilon > 0$ such that if $|\gamma_i| < \varepsilon$, $1 \le i \le n$, then $(y(t_1) + \gamma_1, \dots, y(t_n) + \gamma_n) \in \varphi(\mathbb{R}^n)$. Since $\varphi$ is one-to-one there exists a unique $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ such that $\varphi(r_1, \dots, r_n) = (y(t_1) + \gamma_1, \dots, y(t_n) + \gamma_n)$. By our definition,

$$\varphi(r_1, \dots, r_n) = (v(t_1; t_1, r_1, \dots, r_n), \dots, v(t_n; t_1, r_1, \dots, r_n))$$

where $v(t; t_1, r_1, \dots, r_n)$ is the solution of (1) satisfying the initial conditions

$$v^{\Delta_{i-1}}(t_1) = r_i, \quad 1 \le i \le n.$$

Thus,

$$(y(t_1) + \gamma_1, \dots, y(t_n) + \gamma_n) = (v(t_1; t_1, r_1, \dots, r_n), \dots, v(t_n; t_1, r_1, \dots, r_n)).$$

That is, $v(t; t_1, r_1, \dots, r_n)$ is the solution of (1) satisfying the boundary conditions,

$$v(t_i; t_1, r_1, \dots, r_n) = y(t_i) + \gamma_i, \quad 1 \le i \le n.$$

Now consider a sequence $\{(y(t_1)+\gamma_1^\ell, \dots, y(t_n)+\gamma_n^\ell)\}_{\ell=1}^\infty \subset \varphi(\mathbb{R}^n)$ where $|\gamma_i^\ell| < \varepsilon$, $1 \le i \le n$ and

$$\lim_{\ell\to\infty} (y(t_1) + \gamma_1^\ell, \dots, y(t_n) + \gamma_n^\ell) = (y(t_1), \dots, y(t_n)).$$

Let

$$u_\ell(t) = u(t; t_1, \dots, t_n, y(t_1) + \gamma_1^\ell, \dots, y(t_n) + \gamma_n^\ell).$$

Since $\varphi$ is a homeomorphism then $\varphi^{-1}$ is continuous and so,

$$\begin{aligned}\lim_{\ell\to\infty} \varphi^{-1}(u_\ell(t_1), \dots, u_\ell(t_n)) &= \lim_{\ell\to\infty} \varphi^{-1}(y(t_1) + \gamma_1^\ell, \dots, y(t_n) + \gamma_n^\ell)\\&= \varphi^{-1}(\lim_{\ell\to\infty} y(t_1) + \gamma_1^\ell, \dots, \lim_{\ell\to\infty} y(t_n) + \gamma_n^\ell)\\&= \varphi^{-1}(y(t_1), \dots, y(t_n)).\end{aligned}$$

That is, the initial values of $u_\ell(t)$ converge to the initial values of $y(t)$. By Theorem 2.1 $u_\ell(t)$ converges uniformly to $y(t)$ on each compact subset of $[t_1,+\infty)$. Thus, $u(t;t_1,\dots,t_n,y(t_1)+\gamma_1^\ell,\dots,y(t_n)+\gamma_n^\ell)$ converges to $y(t)$ as $\varepsilon\to 0$ and the proof is complete.

**Theorem 3.7.** *Assume that $f$ satisfies (A) and (B), that (1) is $n$-point disconjugate on $T$, and that the variational equation (2) is $n$-point disconjugate along all solutions of (1). Let $u(t)=u(t;t_1,\dots t_n,u_1,\dots,u_n)$ be the solution of (1), (4) on $[t_1,+\infty)$. Then for $1\le j\le n$, $\frac{\partial u}{\partial u_j}$ exists on $[t_1,+\infty)$ and $z_j(t)=\frac{\partial u}{\partial u_j}$ is the solution of the variational equation (2) along $u(t)$ and satisfies*

$$z_j(t_i)=\delta_{ij},\quad 1\le i\le n.$$

**Proof:** Fix $j$, $1\le j\le n$. Let $\varepsilon>0$ be as Theorem 3.6 and let $h$ be such that $0<|h|<\varepsilon$. Define the difference quotient

$$z_{jh}(t)=\frac{1}{h}[u(t;t_1,\dots,t_n,u_1,\dots,u_j+h,\dots,u_n)-u(t;t_1,\dots,t_n,u_1,\dots,u_n)]$$

It suffices to show that $\lim_{h\to\infty} z_{jh}(t)$ exists on $[t_1,+\infty)$. Note that for all $h\ne 0$,

$$z_{jh}(t_i)=\delta_{ij},\quad 1\le i\le n.$$

For each $2\le i\le n$, define $\alpha_i=u^{\Delta_{i-1}}(t_j;t_1,\dots,t_n,u_1,\dots u_n)$ and $\varepsilon_i=\varepsilon_i(h)=u(t_i;t_1,\dots,t_n,u_1,\dots,u_i+h,\dots,u_n)-\alpha_i$. Recalling our notation we see that $u(t_i;t_1,\dots,t_n,u_1,\dots,u_i+h,\dots,u_n)=u_i+h$ and $u(t_i;t_1,\dots,t_n,u_1,\dots,u_n)=u_i$. As a consequence of Theorem 3.6 $\varepsilon_i\to 0$ as $h\to 0$ for $2\le i\le n$.

Recall that $v(t;t_j,v_1,v_2,\dots v_n)$ is the solution of (1) satisfying the initial conditions

$$v^{\Delta_{i-1}}(t_j)=v_i,\ 1\le i\le n.$$

In particular, $v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)$ is the solution of (1) satisfying $v(t_j)=u_j$ and for $2\le i\le n$, $v^{\Delta_{i-1}}(t_j)=\alpha_i$. Likewise $v(t;t_j,u_j+h,\alpha_2+\varepsilon_2,\dots,\alpha_n+\varepsilon_n)$ is the solution of (1) satisfying $v(t_j)=u_j+h$ and for $2\le i\le n$, $v^{\Delta_{i-1}}(t_j)=\alpha_i+\varepsilon_i$. Since solutions to initial-value problems are unique then $v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)=u(t;t_1,\dots,t_n,u_1,\dots,u_n)$. Similarly, we have $v(t;t_j,u_j+h,\alpha_2+\varepsilon_2,\dots,\alpha_n+\varepsilon_n)=u(t,t_1,\dots,t_n,u_1,\dots,u_j+h,\dots,u_n)$.

Using a telescoping sum, we have

$$\begin{aligned}
&z_{jh}(t)\\
&=\frac{1}{h}[u(t;t_1,\dots,t_n,u_1,\dots,u_j+h,\dots,u_n)-u(t;t_1,\dots,t_n,u_1,\dots,u_n)]\\
&=\frac{1}{h}[v(t;t_j,u_j+h,\alpha_2+\varepsilon_2,\dots,\alpha_n+\varepsilon_n)-v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)]\\
&=\frac{1}{h}[[v(t;t_j,u_j+h,\alpha_2+\varepsilon_2,\dots,\alpha_n+\varepsilon_n)-v(t;t_j,u_j,\alpha_2+\varepsilon_2,\dots,\alpha_n+\varepsilon_n)]\\
&\quad+[v(t;t_j,u_j,\alpha_2+\varepsilon_2,\dots,\alpha_n+\varepsilon_n)-v(t;t_j,u_j,\alpha_2,\dots,\alpha_n+\varepsilon_n)]\\
&\quad+\dots+[v(t;t_j,u_j,\alpha_2,\dots,\alpha_{n-1},\alpha_n+\varepsilon_n)-v(t;t_j,u_j,\alpha_2,\dots,\alpha_{n-1},\alpha_n)]]
\end{aligned}$$

By Theorem 2.2, solutions of (1) can be differentiated with respect to initial values. That is $\beta_1=\frac{\partial v}{\partial v_1},\beta_2=\frac{\partial v}{\partial v_2},\dots,\beta_n=\frac{\partial v}{\partial v_n}$ exist. By Theorem 2.2 and the Mean

Value Theorem, we see that

$$
\begin{aligned}
z_{jh}(t) &= \frac{1}{h}[\beta_1(t, v(t; t_1, u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n))h \\
&\quad + \beta_2(t, v(t; t_1, u_j, \alpha_2 + \bar{\varepsilon}_2, \ldots, \alpha_n + \varepsilon_n))\varepsilon_2 \\
&\quad + \ldots + \ \beta_n(t; v(t, t_1, u_j, \alpha_2, \ldots, \alpha_n + +\bar{\varepsilon}_n))\varepsilon_n]
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
&\beta_1(t; v(t; t_1, u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n)) \\
&\qquad\qquad = \frac{\partial v}{\partial v_1}(t; t_1, u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n) \\
&\qquad \vdots \\
&\beta_n(t; v(t; t_1, u_j, \alpha_2, \ldots, \alpha_n + \bar{\varepsilon}_n)) = \frac{\partial v}{\partial v_n}(t; t_1, u_j, \alpha_2, \ldots, \alpha_n + +\bar{\varepsilon}_n)
\end{aligned}
$$

and $\bar{h}$ is between 0 and $h$ and $\bar{\varepsilon}_\ell$ is between 0 and $\varepsilon_\ell$, $2 \leq \ell \leq n$. That is, $\beta_1(t; v(t, t_1, u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n))$ is the solution of the variational equation (2) along $v(t; t_1, u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n)$ satisfying $\beta_1^{\Delta_{i-1}}(t_j) = \delta_{i1}, 1 \leq i \leq n$. For $2 \leq \ell \leq n$, $\beta_\ell(t; v(t; t_1, u_j, \alpha_2 + \varepsilon_2, \ldots, \alpha_\ell + \bar{\varepsilon}_\ell, \ldots, \alpha_n + \varepsilon_n))$ is the solution of the variational equation (2) along $v(t; t_1, u_j, \alpha_2 + \varepsilon_2, \ldots, \alpha_\ell + \bar{\varepsilon}_\ell, \ldots, \alpha_n + \varepsilon_n)$ satisfying $\beta_\ell^{\Delta_{i-1}}(t_j) = \delta_{i\ell}, 1 \leq i \leq n$. In particular note that

$$
\beta_2(t_j) = \cdots = \beta_n(t_j) = 0.
$$

Distribute the factor $\frac{1}{h}$ in equation (6).

$$
\begin{aligned}
z_{jh}(t) &= \beta_1(t, v(t; t_1, u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n)) \\
&\quad + \beta_2(t, v(t; t_1, u_j, \alpha_2 + \bar{\varepsilon}_2, \ldots, \alpha_n + \varepsilon_n))\frac{\varepsilon_2}{h} \\
&\quad + \cdots + \beta_n(t; v(t, t_1, u_j, \alpha_2, \ldots, \alpha_n + +\bar{\varepsilon}_n))\frac{\varepsilon_n}{h}.
\end{aligned}
\tag{7}
$$

To show that $\lim_{h\to 0} z_{jh}(t)$ exists, it suffices to show that $\lim_{h\to 0} \frac{\varepsilon_\ell}{h}$ exists for $2 \leq \ell \leq n$.

Recall that $z_{jh}(t_1) = \cdots = z_{jh}(t_{j-1}) = z_{jh}(t_{j+1}) = \cdots = z_{jh}(t_n) = 0$. Evaluate (7) at $t_\ell, 1 \leq \ell \leq n, \ell \neq j$ to obtain the system of equations

$$
\begin{aligned}
&-\beta(t_\ell; v(t_j; u_j + \bar{h}, \alpha_2 + \varepsilon_2, \ldots, \alpha_n + \varepsilon_n) \\
&\qquad = \beta_2(t_\ell; v(t_j; u_j, \alpha_2 + \bar{\varepsilon}_2, \ldots, \alpha_n + \varepsilon_n)\frac{\varepsilon_2}{h} \\
&\qquad + \cdots + \beta_n(t_\ell; v(t_j; u_j, \alpha_2, \ldots, \alpha_n + \bar{\varepsilon}_n)\frac{\varepsilon_n}{h}, \ 1 \leq \ell \leq n, \ \ell \neq j.
\end{aligned}
$$

This is a system of $n-1$ equations in the $n-1$ unknowns $\frac{\varepsilon_2}{h}, \frac{\varepsilon_3}{h}, \ldots, \frac{\varepsilon_n}{h}$.

By Cramer's rule we have, (after surpressing the variable dependency in $v(\cdot)$),

$$\frac{\varepsilon_2}{h} = \frac{\begin{vmatrix} -\beta_1(t_1;v(\cdot)) & \beta_3(t_1;v(\cdot)) & \cdots & \beta_n(t_1;v(\cdot)) \\ \vdots & \vdots & & \vdots \\ -\beta_1(t_{j-1};v(\cdot)) & \beta_3(t_{j-1};v(\cdot)) & \cdots & \beta_n(t_{j-1};v(\cdot)) \\ -\beta_1(t_{j+1};v(\cdot)) & \beta_3(t_{j+1};v(\cdot)) & \cdots & \beta_n(t_{j+1};v(\cdot)) \\ \vdots & \vdots & & \vdots \\ -\beta_1(t_n;v(\cdot)) & \beta_3(t_n;v(\cdot)) & \cdots & \beta_n(t_n;v(\cdot)) \end{vmatrix}}{D(h)},$$

$$\vdots$$

$$\frac{\varepsilon_n}{h} = \frac{\begin{vmatrix} \beta_2(t_1;v(\cdot)) & \beta_3(t_1;v(\cdot)) & \cdots & -\beta_1(t_1;v(\cdot)) \\ \vdots & \vdots & & \vdots \\ \beta_2(t_{j-1};v(\cdot)) & \beta_3(t_{j-1};v(\cdot)) & \cdots & -\beta_1(t_{j-1};v(\cdot)) \\ \beta_2(t_{j+1};v(\cdot)) & \beta_3(t_{j+1};v(\cdot)) & \cdots & -\beta_1(t_{j+1};v(\cdot)) \\ \vdots & \vdots & & \vdots \\ \beta_1(t_n;v(\cdot)) & \beta_3(t_n;v(\cdot)) & \cdots & -\beta_1(t_n;v(\cdot)) \end{vmatrix}}{D(h)},$$

provided that

$$D(h) \equiv \begin{vmatrix} \beta_2(t_1;v(\cdot)) & \beta_3(t_1;v(\cdot)) & \cdots & \beta_n(t_1;v(\cdot)) \\ \vdots & \vdots & & \vdots \\ \beta_2(t_{j-1};v(\cdot)) & \beta_3(t_{j-1};v(\cdot)) & \cdots & \beta_n(t_{j-1};v(\cdot)) \\ \beta_2(t_{j+1};v(\cdot)) & \beta_3(t_{j+1};v(\cdot)) & \cdots & \beta_n(t_{j+1};v(\cdot)) \\ \vdots & \vdots & & \vdots \\ \beta_1(t_n;v(\cdot)) & \beta_3(t_n;v(\cdot)) & \cdots & \beta_n(t_n;v(\cdot)) \end{vmatrix} \neq 0.$$

To see that $D(h) \neq 0$ for small values of $h$, consider the determinant

$$D = \begin{vmatrix} \beta_2(t_1;v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) & \cdots & \beta_n(t_1;v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) \\ \vdots & & \vdots \\ \beta_2(t_{j-1};v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) & \cdots & \beta_n(t_{j-1};v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) \\ \beta_2(t_{j+1};v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) & \cdots & \beta_n(t_{j+1};v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) \\ \vdots & & \vdots \\ \beta_1(t_n;v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) & \cdots & \beta_n(t_n;v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)) \end{vmatrix}$$

If $D = 0$ then there exists a set of numbers $r_2, \dots, r_n$, at least one of which is nonzero, such that

$$\gamma(t) = \sum_{\ell=2}^{n} r_\ell \beta_\ell(t; v(t;t_j,u_j,\alpha_2,\dots,\alpha_n))$$

is a nontrivial solution of (2) along $v(t;t_j,u_j,\alpha_2,\dots,\alpha_n)$ that vanishes at $t = t_1, \dots t_{j-1},\ t_{j+1}, \dots, t_n$. Since $\beta_\ell(t_j) = 0$ for $2 \le \ell \le n$ then $\gamma(t_j) = 0$. That is $\gamma(t)$ is a nontrivial solution of (2) that has $n$ zeros in $T$ contradicting the $n$-point disconjugacy of the variational equation.

Consequently $D \neq 0$. By continuity, $D(h) \neq 0$ for $h$ sufficiently small. Thus $\lim_{h\to 0} \frac{\varepsilon_\ell}{h}$ exists for each $2 \leq \ell \leq n$. Let

$$\lim_{h\to 0} \frac{\varepsilon_\ell}{h} = k_\ell,\ 2 \leq \ell \leq n.$$

Then,

$$\begin{aligned} z_j(t) &= \lim_{h\to 0} z_{jh}(t) \\ &= \beta_1(t; v(t_j; u_j, \alpha_2, \dots, \alpha_n)) + \sum_{\ell=2}^{n} k_\ell \beta_\ell(t; v(t_j; u_j, \alpha_2, \dots, \alpha_n)) \end{aligned}$$

exists. That is, $\frac{\partial u}{\partial u_j}(t; t_1, \dots, t_n, u_1, \dots, u_n)$ exists and $z_j(t) = \frac{\partial u}{\partial u_j}$. Furthermore, since each $\beta_\ell(t; v(t; t_j, u_j, \alpha_2, \dots, \alpha_n))$, $1 \leq \ell \leq n$ is a solution of the variational equation (2) along $v(t; t_j, u_j, \alpha_2, \dots, \alpha_n) = u(t; t_1, \dots, t_n, u_1, \dots, u_n)$ then $z_j(t) = \frac{\partial u}{\partial u_j}$ is also a solution of (2) along $u(t; t_1, \dots, t_n, u_1, \dots, u_n)$. Finally we note that

$$z_j(t_i) = \lim_{h\to 0} z_{jh}(t_i) = \delta_{ij},\ 1 \leq i \leq n$$

and the proof is complete.

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