Electronic Journal of Differential Equations, Vol. 2000(2000), No. 68, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ejde.math.unt.edu (login: ftp)

REGULARITY OF THE INTERFACE FOR THE POROUS MEDIUM EQUATION

YOUNGSANG KO

ABSTRACT. We establish the interface equation and prove the C^∞ regularity of the interface for the porous medium equation whose solution is radial symmetry.

1. INTRODUCTION

We consider the Cauchy problem of the form

$$u_t = \Delta(u^m) \quad \text{in } S = \mathbb{R}^N \times (0, \infty), \tag{1.1}$$

$$u(x, 0) = u_0 \quad \text{on } \mathbb{R}^N. \tag{1.2}$$

Here we suppose that $m > 1$, and u_0 is a nonzero bounded nonnegative function with compact support.

It is well known that (1.1) describes the evolution in time of various diffusion processes, in particular the flow of a gas through a porous medium. Here u stands for the density, while $v = \frac{m}{m-1} u^{m-1}$ represents the pressure of the gas. Then v satisfies

$$v_t = (m - 1)v\Delta v + |\nabla v|^2. \tag{1.3}$$

If the solution is radial symmetry, then v satisfies

$$v_t = (m - 1)vv_{rr} + \frac{\bar{m}}{r} vv_r + v_r^2, \tag{1.4}$$

where $r = \sqrt{\sum_{i=1}^N x_i^2}$ and $\bar{m} = (m - 1)(N - 1)$. Since we are concerning about the regularity of the interface we may assume $r \geq \epsilon_0$ for some positive number ϵ_0. In this paper we will show that, if the solution is radial symmetry then the interface of (1.4) can be represented by a C^∞ function after a large time.

In the one-dimensional case Aronson and Vázuez [5] and independently Höllig and Kreiss [10] showed that the interfaces are smooth after the waiting time. Angenent [1] showed that the interfaces are real analytic after the waiting time. For the dimensions ≥ 2, Daskalopoulos and Hamilton [8] showed that for $t \in (0, T)$, for some $T > 0$, the interface can be described as a C^∞ surface if the initial data u_0 satisfies some assumptions. On the other hand, Caffarelli, Vazquez and Wolanski

Mathematics Subject Classification. 35K65.
Key words. porous medium, radial symmetry, interface, C-infinity regularity.

Submitted October 7, 2000. Published November 13, 2000.
Supported by a research grant form Kyonggi University.

[6] showed that after a large time the interface can be described as a Lipschitz surface if the u_0 satisfies some non-degeneracy conditions. Caffarelli and Wolanski [7] improved this result by showing that the interface can be described as a $C^{1,\alpha}$ surface under the same non-degeneracy conditions on the initial data. But many people believe that even after a large time, the interface can be described as a smooth surface.

In this paper, assuming the solution to (1.1) is radial symmetry and the initial data u_0 satisfies the same non-degeneracy conditions as in [7], we obtain the following result :

Theorem 1.1. *If v is a solution to (1.4), then v is a C^∞ function near the interface where $v > 0$ and the interface is a C^∞ function for $t > T$, for some $T > 0$.*

This paper is divided into three parts : In Part I, we obtain the interface equation. In Part II we obtain the upper and lower bound of v_{rr} by constructing a barrier function. In Part III, we obtain the upper and lower bound of $\left(\frac{\partial}{\partial r}\right)^j v \equiv v^{(j)}$ by constructing another barrier function. In showing our results, we adapt the methods used in [5].

2. Preliminaries

In this section, we introduce some basic results which are necessary in showing the uniform boundedness of the derivatives of the pressure v. First, since we are interested in the radial symmetry solution, let

$$P[u] = \{(r,t) \mid u(r,t) > 0, r \geq \epsilon_0 > 0\}$$

for some $\epsilon_0 > 0$, be the positivity set. Then by [6], we can express the interface as a nondecreasing and Lipschitz continuous $r = \zeta(t) = \sup\{r \geq \epsilon_0 \mid u(r,t) > 0\}$ and $r = \zeta(t)$ on $[T, \infty)$, for some $T > 0$. In showing the interface is a C^∞ function, we need the following :

Theorem 2.1. *Assume that $u_0 > 0$ on $I = (2\epsilon_0, a)$ and $u_0 = 0$ on $[a, \infty)$. Let $v_0 = \frac{m}{m-1} u_0^{m-1} \in C^1(\bar{I})$. Then*

$$\lim_{r \to \zeta(t)} v_r(r,t) \equiv v_r(\zeta(t), t)$$

exists for all $t \geq 0$ and

$$v_r(\zeta(t), t) \left\{ \zeta^{'}(t) + v_r(\zeta(t), t) \right\} = 0$$

for almost all $t \geq 0$.

In proving Theorem 2.1, we need

Lemma 2.2. *Assume that v_0 has bounded derivatives of all orders, and that there exist positive constants μ, M such that*

$$\mu \leq v_0(r) \leq M \quad \text{on } [\epsilon_0, \infty).$$

Then the Cauchy problem (1.4) has a unique classical solution v in $S = [\epsilon_0, \infty) \times [0, \infty)$ such that

$$\mu \leq v(r,t) \leq M \quad \text{on } S$$

and $v \in C^\infty(S)$.

Proof. To show the existence of v, we need to have an a priori lower bound for v. To this end, let $\varphi = \varphi(s)$ denote a $C^\infty(\mathbb{R}^1)$ function such that $\varphi(s) = s$ for $s \geq \mu$, $\varphi(s) = \mu/2$ for $s \leq 0$, and φ increases from $\mu/2$ to μ as s increases from 0 to μ. Now consider

$$\begin{gathered} v_t = (m-1)\varphi(v)v_{rr} + \frac{\bar{m}}{r} v v_r + v_r^2 \quad \text{in } [\epsilon_0, \infty) \times (0, \infty), \\ v(x,0) = v_0(x) \quad \text{on } [\epsilon_0, \infty). \end{gathered} \tag{2.1}$$

Then it is easily verified that equation (2.1) satisfies all the hypotheses of Theorem 5.2 in [11](pp. 564-565). Let $\tau \in (0, \infty)$ and $\beta \in (0, 1)$ be arbitrary, and let $R_\tau = [\epsilon_0, \infty) \times [0, \tau]$. Then there exists a unique solution v of equation (2.1) such that $|v| \leq M$ in R_τ and $v \in H^{2+\beta, 1+\beta/2}(R_\tau)$.In [11], the solution of (2.1) is obtained as the limit as $n \to \infty$ of the solutions v^n of the sequence of the first boundary value problems

$$\begin{gathered} v_t = (m-1)\varphi(v)v_{rr} + \frac{\bar{m}}{r} v v_r + v_r^2 \quad \text{in } [\epsilon_0 + 1/n, n] \times (0, \tau), \\ v(r,0) = v_0(r) \quad \text{in } [\epsilon_0 + \frac{1}{n}, n], \\ v(n,t) = v_0(n) \quad \text{in } [0, \tau]. \end{gathered}$$

Then by the method used in proving Theorem 2 in [2], we have $v = \lim_{n\to\infty} v^n$ belongs to $C^\infty(S)$ and $\mu \leq v \leq M$. □

Now let us prove Theorem 2.1. Let $t > 0$ and assume $r < r^{'} \in I_t = [2\epsilon_0, \zeta(t))$. By mean value theorem,

$$v_r(r^{'}, t) = v_r(r,t) + (r - r^{'})v_{rr}(\tilde{r}, t)$$

for some $\tilde{r} \in (r, r^{'})$. Since $|\nabla v| = |v_r| \leq L$ [6], by the following famous result

$$\Delta v = v_{rr} + \frac{N-1}{r} v_r \geq -\frac{1}{(m-1+2/N)t}$$

established in [4], together with the assumption that $r \geq \epsilon_0$, the lower bound for v_{rr} is obtained. Hence

$$v_r(r^{'}, t) \geq v_r(r,t) - \alpha(r^{'} - r),$$

for some positive constant α. Also, since v_r is bounded above,

$$\sup_{I_t} |v_r(r,t)| \leq C.$$

Therefore the inferior and superior limits of $v_r(r,t)$ exist and are finite when $r \to \zeta(t)$. It follows that

$$\liminf_{r^{'} \to \zeta(t)} v_r(r^{'}, t) \geq v_r(r,t) - \alpha\{\zeta(t) - r\}$$

for all $r \in I_t$, and

$$\liminf_{r^{'} \to \zeta(t)} v_r(r^{'}, t) \geq \limsup_{r \to \zeta(t)} v_r(r^{'}, t).$$

Therefore

$$\lim_{r \to \zeta(t)} v_r(r,t) \equiv v_r(\zeta(t), t)$$

exists.

Next, let ψ be a $C(S)$ function which has compact support in $\mathbb{R}^1$ for each fixed $t \geq 0$. Suppose further that $\psi_r \in C(S)$ and that ψ possesses a weak derivative ψ_t with respect to t in S. Define

$$\tilde{\psi}(r,t) = \begin{cases} \psi(r,t) & \text{for } t \geq 0 \\ \psi(r,0) & \text{for } t \leq 0\,. \end{cases}$$

Then it is shown [3] that $\tilde{\psi} \in C(\mathbb{R}^2)$, $\tilde{\psi}$ has compact support as a function of r for each fixed t, $\tilde{\psi}_r \in C(\mathbb{R}^2)$, and $\tilde{\psi}$ is weakly differentiable with respect to t in $\mathbb{R}^2$. Moreover, $\tilde{\psi}_t$ coincides with ψ_t for $t > 0$. Let

$$\psi_n(r,t) = \iint_{\mathbb{R}^2} k_n(r-\xi, t-\tau)\tilde{\psi}(\xi,\tau)d\xi d\tau,$$

where $k_n(r,t)$ denotes an averaging kernel with support in $(-\frac{1}{n},\frac{1}{n}) \times (-\frac{1}{n},\frac{1}{n})$ for each integer $n \geq 1$. Then ψ_n satisfies

$$\begin{aligned} &\int_{\mathbb{R}^1} \psi_n(r,t_2)v(r,t_2)dr + \int_{t_1}^{t_2}\int_{\mathbb{R}^1} \{(m-1+\frac{c}{r})vv_r\psi_{nr} + (m-2)v_r^2\psi_n - v\psi_{nt}\}\,dr\,dt \\ &= \int_{\mathbb{R}^1} \psi_n(r,t_1)v(r,t_1)dr. \end{aligned} \tag{2.2}$$

Recall that v and its weak derivative v_r are bounded in S. On the other hand, it is shown in [3] that $\psi_n \to \tilde{\psi}$ and $\psi_{nr} \to \tilde{\psi}_r$ uniformly on any compact subsets of $\mathbb{R}^2$, while $\psi_{nt} \to \tilde{\psi}_t$ strongly in $L^1_{loc}(\mathbb{R}^2)$. Therefore (2.2) also holds for the limit of the sequence ψ_n, that is, for any set function which satisfies the conditions listed at the beginning of this paragraph. Now define a function

$$K(r) = \begin{cases} C \cdot \exp\{-1/(1-r^2)\} & \text{for } |r| \leq 1 \\ 0 & \text{for } |r| \geq 1\,, \end{cases}$$

where C is chosen so that

$$\int_{\mathbb{R}^1} K(r)dr = 1\,,$$

and set $k_n(r) = nK(nr)$ for each integer $n \geq 1$. Then $k_n(r)$ is an even averaging kernel and $k_n(\zeta(t)-r)$ belongs to $C(S)$ and has compact support in $\mathbb{R}^1$ for each $t \geq 0$. Also $\frac{d}{dr}k_n(\zeta(t)-r) \in C(S)$. Since ζ is Lipschitz continuous, $\zeta^{'}$ exists almost everywhere and is bounded above. Moreover ζ is weakly differentiable and its weak derivative can be represented by $\zeta^{'}$. It follows that $k_n(\zeta(t)-r)$ is also weakly differentiable with respect to t in S with weak derivative given by $\zeta^{'}(t)k_n'(\zeta(t)-r)$ which belongs to $L^1(\mathbb{R}^1 \times (t_1,t_2)$ for any $0 \leq t_1 < t_2 < \infty$. Thus $k_n(\zeta(t)-r)$ is an admissible test function in (2.2). In particular, if we set $\psi_n(r,t) = k_n(\zeta(t)-r)$ in (2.2), we have

$$\begin{aligned} \int_{\mathbb{R}^1} k_n(\zeta(t_2)-r)v(r,t_2)dr + \int_{t_1}^{t_2}\int_{\mathbb{R}^1} k_n'(\zeta-r)v(r,t)\{-(m-1)v_r - \zeta'\}\,dr\,dt \\ +(m-2)\int_{t_1}^{t_2}\int_{\mathbb{R}^1} k_n v_r^2\,dr\,dt - \int_{t_1}^{t_2}\int_{\mathbb{R}^1} \frac{\bar{m}}{r}vv_r k_n(\zeta-r)\,dr\,dt \\ = \int_{\mathbb{R}^1} k_n(\zeta(t_1)-r)v(r,t_1)dr. \end{aligned} \tag{2.3}$$

For fixed t, $v(r,t)$ is a continuous function of r in $\mathbb{R}^1$ and $v(r,t)=0$ for $r\in\mathbb{R}^1\setminus I_t$. Hence for any $t\geq 0$

$$\lim_{n\to\infty}\int_{\mathbb{R}^1} k_n(\zeta(t)-r)v(r,t)dx = v(\zeta(t),t)=0.$$

Similarly, since $|v_r|$ is bounded above, we have

$$\lim_{n\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^1}\frac{\bar{m}}{r}vv_rk_n(\zeta-r)dr=0.$$

Since $v_r(r,t)\to v_r(\zeta(t),t)$ as $r\to\zeta-$ and $v_r(r,t)=0$ on $\mathbb{R}^1\setminus I_t$, we have

$$\int_{\mathbb{R}^1}k_n(\zeta-r)v_r^2(r,t)dr=\int_{\zeta-\frac{1}{n}}^{\zeta}k_n(\zeta-r)v_r^2(r,t)dr.$$

Since k_n is even, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^1}k_n(\zeta(t)-r)v_r^2(r,t)dx=\frac{1}{2}v_r^2(\zeta,t)$$

for each $t>0$. Moreover, since $|v_r|\leq L$,

$$\left|\int_{\mathbb{R}^1}k_n(\zeta-r)v_r^2(r,t)dr\right|\leq L^2.$$

Thus by the Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^1}k_n(\zeta(t)-r)v_r^2(r,t)dxdt=\frac{1}{2}\int_{t_1}^{t_2}v_r^2(\zeta,t)dt.$$

Next define

$$w(r,t)=\begin{cases}\dfrac{-v(r,t)}{\zeta-r} & \text{for } r<\zeta(t)\\ v_r(\zeta(t),t) & \text{for } r=\zeta(t)\\ 0 & \text{for } r>\zeta(t)\,.\end{cases}$$

Note that for fixed t, w is continuous on $[\epsilon_0,\zeta(t)]$ and $|w(x,t)|\leq C$. Then the second integral on the left in (2.3) can be written in the form

$$I_n=\int_{t_1}^{t_2}\int_{\mathbb{R}^1}(\zeta-r)k_n'(\zeta-r)w(r,t)\{(m-1)v_r+\zeta'\}\,dr\,dt.$$

It is easily verified that the function $-rk_n'(r)$ is also an even averaging kernel. Thus for each t,

$$\lim_{n\to\infty}\int_{\mathbb{R}^1}(\zeta-r)k_n'(\zeta-r)w\{(m-1)v_r+\zeta'\}dr=-\frac{1}{2}v_r(\zeta,t)\{(m-1)v_r+\zeta'\}$$

and

$$\left|\int_{\mathbb{R}^1}(\zeta-r)k_n'(\zeta-r)w\{(m-1)v_r+\zeta'\}dr\right|\leq(m-1)L^2+L\zeta'\in L^1(t_1,t_2).$$

It follows that

$$\lim_{n\to\infty}I_n=-\frac{1}{2}\int_{t_1}^{t_2}v_r(\zeta,t)\{(m-1)v_r+\zeta'\}dt.$$

Hence if we let $n\to\infty$ in (2.3), we obtain

$$-\frac{1}{2}\int_{t_1}^{t_2}v_r(\zeta,t)(\zeta'(t)+v_r(\zeta,t))dt=0$$

for any $0 \le t_1 < t_2 < \infty$. Therefore
$$v_r(\zeta,t)(\zeta^{'}(t) + v_r(\zeta,t)) = 0$$
for almost all $t \ge 0$. □

3. Upper and Lower Bounds for v_{rr}

Let $v = v(x,t)$ be the pressure corresponding to a solution $u = u(x,t)$ of equation (1.1). If u is radial symmetry then $v = v(r,t)$ satisfies (1.4) in the positivity set $P[u] = \{(r,t) \mid u(r,t) > 0, r \ge \epsilon_0\}$. Assume that $v_0(r)$ has compact support containing $[2\epsilon_0, a]$ for some $a > 0$ and satisfies the non-degeneracy condition (1.4) of the Theorem 1 in [7]. Let $q = (r_0, t_0)$ be a point on the interface $r = \zeta(t)$ so that $r_0 = \zeta(t_0)$, $v(r,t_0) = 0$ for all $r \ge \zeta(t_0)$, and $v(r,t_0) > 0$ for all $0 < r < \zeta(t_0)$. Since the lower bound for v_{rr} is obtained already, we need to show v_{rr} is bounded above. In showing this we adopt the methods used in [5].

Now let $T > 0$ be the positive constant established in [6]. Assume $t_0 > T$ so that the interface is moving at q. Then from Theorem 2.1, we have
$$\zeta^{'}(t_0) = -v_r(\zeta,t) = a > 0, \tag{3.1}$$
and on the moving interface we have
$$v_t = v_r^2. \tag{3.2}$$
As in [5], we use the notation
$$R_{\delta,\eta} = R_{\delta,\eta}(t_0) \equiv \{(r,t) \in \mathbb{R}^2 \mid \zeta(t) - \delta < r \le \zeta(t),\ t_0 - \eta \le t \le t_0 + \eta\}.$$

Lemma 3.1. *Let q be a point on the interface and assume (3.1) holds. Then there exist positive constants C, δ and η depending only on N, ϵ_0, m, q and u such that*
$$|v_{rr}| \le C \qquad \text{in } R_{\delta,\eta/2}.$$

Proof. It is well known that v_t, v_r and vv_{rr} are continuous in a closed neighborhood in $\overline{P}[u]$ of any point on the interface if $t > T$, and that
$$(vv_{rr})(r,t) = \frac{1}{m-1}\left(v_t - \frac{\bar{m}}{r}vv_r - v_r^2\right) \to 0 \text{ as } P[u] \ni (r,t) \to (\zeta(t),t)$$
for any $t > T$. Choose now an $\epsilon > 0$ such that
$$(a - (8m-3)\epsilon)(a - \epsilon) \ge 4(m+1)\epsilon > 0. \tag{3.3}$$
Then there exist $\frac{\epsilon_0}{2\bar{m}} \le \delta = \delta(\epsilon) > 0$ and $\eta = \eta_1(\epsilon) \in (0, t_0 - T)$ such that $R_{\delta,\eta} \subset P[u]$,
$$-a - \epsilon < v_r < -a + \epsilon, \tag{3.4}$$
and
$$vv_{rr} \le \epsilon \tag{3.5}$$
in $R_{\delta,\eta}$. In view of (3.4) we have
$$(a-\epsilon)(\zeta(t) - r) < v(r,t) < (a+\epsilon)(\zeta(t) - r) \quad \text{in } R_{\delta,\eta} \tag{3.6}$$
and
$$a - \epsilon < \zeta^{'}(t) < a + \epsilon \quad \text{in } [t_1, t_2] \tag{3.7}$$
where $t_1 = t_0 - \eta$ and $t_2 = t_0 + \eta$. We set
$$\zeta^*(t) \equiv \zeta_1 + b(t - t_1), \tag{3.8}$$

where $\zeta_1 = \zeta(t_1)$ and $b = a + 2\epsilon$. Clearly $\zeta(t) < \zeta^*(t)$ in $(t_1, t_2]$.

On $P[u]$, $p \equiv v_{rr}$ satisfies

$$\begin{aligned} L(p) &= p_t - (m-1)vp_{rr} - \left(2mv_r + \frac{\bar{m}}{r}v\right)p_r \\ &\quad -(m+1)p^2 + \left(\frac{2\bar{m}}{r^2}v - \frac{3\bar{m}}{r}v_r\right)p - \frac{2\bar{m}}{r^3}(vv_r - rv_r^2) = 0 \end{aligned}$$

where $\bar{m} = (m-1)(N-1)$. As in [5] we construct a barrier function for p of the form

$$\phi(r,t) \equiv \frac{\alpha}{\zeta(t) - r} + \frac{\beta}{\zeta^*(t) - r}, \tag{3.9}$$

where α and β are positive constants and will be decided later.

$$\begin{aligned} L(\phi) &= \frac{\alpha}{(\zeta - r)^2}\left\{-\zeta' - 2(m-1)\frac{v}{\zeta - r} - 2mv_r - \frac{\bar{m}}{r}v\right\} \\ &\quad + \frac{\beta}{(\zeta^* - r)^2}\left\{-\zeta^{*'} - 2(m-1)\frac{v}{\zeta^* - r} - 2mv_r - \frac{\bar{m}}{r}v\right\} \\ &\quad -(m+1)\phi^2 + \left(\frac{2\bar{m}}{r^2}v - \frac{3\bar{m}}{r}v_r\right)\phi - \frac{2\bar{m}}{r^3}(vv_r - rv_r^2) \\ &\geq \frac{\alpha}{(\zeta - r)^2}\left\{-\zeta' - 2(m-1)\frac{v}{\zeta - r} - 2mv_r - \frac{\bar{m}}{r}v - 2(m+1)\alpha\right\} \\ &\quad + \frac{\beta}{(\zeta^* - r)^2}\left\{-\zeta^{*'} - 2(m-1)\frac{v}{\zeta^* - r} - 2mv_r - \frac{\bar{m}}{r} - 2(m+1)\beta\right\}, \end{aligned}$$

since $v_r < 0$. ¿From the choice of δ and the estimates (3.4), (3.6), (3.7) and the definition (3.8) of ζ^* we conclude that

$$\begin{aligned} L(\phi) &\geq \frac{\alpha}{2(\zeta - r)^2}\{a - (8m-5)\epsilon - 4(m+1)\alpha\} \\ &\quad + \frac{\beta}{2(\zeta^* - r)^2}\{a - (8m-3)\epsilon - 4(m+1)\beta\}. \end{aligned}$$

Now set

$$\beta = \frac{a - (8m-3)\epsilon}{8(m+1)} \tag{3.10}$$

and note that (3.3) implies that $\beta > 0$. Then $L(\phi) \geq 0$ in $R_{\delta,\eta}$ for all $\alpha \in (0, \alpha_0]$, where $\alpha_0 = \{a - (8m-5)\epsilon\}/4(m+1)$.

Let us compare p and ϕ on the parabolic boundary of $R_{\delta,\eta}$. In view of (3.5) and (3.6) we have

$$v_{rr} < \frac{\epsilon}{(a - \epsilon)(\zeta - r)} \quad \text{in } R_{\delta,\eta},$$

so that, in particular,

$$v_{rr}(\zeta(t) - \delta, t) \leq \frac{\epsilon}{(a - \epsilon)\delta} \quad \text{in } [t_1, t_2].$$

By the mean value theorem and (3.7) it follows that for some $\tau \in (t_1, t_2)$

$$\begin{aligned} \zeta^*(t) + \delta - \zeta(t) &= \delta + (a + 2\epsilon)(t - t_1) - \zeta'(\tau)(t - t_1) \\ &\leq \delta + 3\epsilon(t - t_1) \leq \delta + 6\epsilon\eta. \end{aligned}$$

Now set

$$\eta \equiv \min\{\eta_1(\epsilon), \delta(\epsilon)/6\epsilon\}.$$

Since ϵ satisfies (3.3) and β is given by (3.10) it follows that

$$\phi(\zeta+\delta,t) \geq \frac{\beta}{2\delta} \geq \frac{\epsilon}{(a-\epsilon)\delta} \geq v_{rr}(\zeta+\delta,t) \quad \text{on } [t_1,t_2].$$

Moreover,

$$\phi(r,t_1) \geq \frac{\beta}{\zeta_1 - r} > \frac{\epsilon}{(a-\epsilon)(\zeta_1-r)} > v_{rr}(r,t_1) \quad \text{on } [\zeta_1-\delta,\zeta_1).$$

Let $\Gamma = \{(r,t)\in\mathbb{R}^2 \ : \ r=\zeta(t), t_1\leq t\leq t_2\}$. Then Γ is a compact subset of $\mathbb{R}^2$. Now fix $\alpha\in(0,\alpha_0)$. For each point $s\in\Gamma$ there is an open ball B_s centered at s such that

$$(vv_{rr})(r,t) \leq \alpha(a-\epsilon) \quad \text{in } B_s\cap P[u],$$

In view of (3.6) we have

$$\phi(r,t) \geq \frac{\alpha}{\zeta-r} \geq v_{rr}(r,t) \quad \text{in } B_s\cap P[u].$$

Since Γ is compact, finite number of these balls can cover Γ and hence there is a $\gamma=\gamma(\alpha)\in(0,\delta)$ such that

$$\phi(r,t)\geq p(r,t) \quad \text{in } R_{\gamma,\eta}.$$

Thus for every $\alpha\in(0,\alpha_0)$, ϕ is a barrier for p in $R_{\delta,\eta}$. Hence by the comparison principle we conclude

$$v_{rr}(r,t) \leq \frac{\alpha}{\zeta-r} + \frac{\beta}{\zeta^*(t)-r} \quad \text{in } R_{\delta,\eta},$$

where β is given by (3.10) and $\alpha\in(0,\alpha_0)$ is arbitrary. Now let $\alpha\downarrow 0$ to obtain

$$v_{rr}(r,t) \leq \frac{\beta}{\zeta^*(t)-r} \leq \frac{2\beta}{\epsilon\eta} \quad \text{in } R_{\delta,\eta/2},$$

□

4. Bound for $\left(\frac{\partial}{\partial r}\right)^j v$

As in [5], if we can show

$$|v^{(j)} \equiv \left(\frac{\partial}{\partial x}\right)^j v| \leq C_j,$$

for each $j\geq 2$, then Theorem 1.1 follows. First by a direct computation for $j\geq 3$, $v^{(j)}$ satisfy the following equation

$$\begin{aligned} L_j v^{(j)} \equiv\ & v_t^{(j)} - (m-1)vv_{rr}^{(j)} - (2+j(m-1))v_r v_r^{(j)} - \frac{\bar{m}}{r} v v_r^{(j)} - F_j(r,t)v^{(j)} \\ & - G_j(r,t), \end{aligned} \tag{4.1}$$

where $F_j(r,t)$ and $G_j(r,t)$ are functions of r, v and derivatives of v of order $< j$ only. Then our result is

Proposition 4.1. *Let $q=(r_0,t_0)$ be a point on the interface for which (3.1) holds. For each integer $j\geq 2$ there exist constants C_j, δ and η depending only on N, ϵ_0, m, j, q and u such that*

$$\left|\left(\frac{\partial}{\partial r}\right)^j v\right| \leq C_j \quad \textit{in } R_{\delta,\eta/2}. \tag{4.2}$$

The proof proceeds as in [5] by induction on j. Suppose that q is a point on the interface for which (3.1) holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta_0(\epsilon) \in (0, t_0 - T)$ such that $R_0 \equiv R_{\delta_0 \eta_0}(t_0) \subset P[u]$ and (3.4) holds. Thus we also have (3.6) and (3.7) in R_0. Assume that there are constants $C_k \in \mathbb{R}^+$ for $k = 2, 3, \dots, j-1$ such that

$$|v^{(k)}| \le C_k \quad \text{on } R_0 \quad \text{for } k = 2, \dots, j-1. \tag{4.3}$$

Observe that, by Lemma 3.1, the estimate (4.3) holds for $k = 2$. As in [5] by rescaling and using interior estimates we obtain the following estimate near ζ.

Lemma 4.2. *There are constants $K \in \mathbb{R}^+$, $\delta \in (0, \delta_0)$ and $\eta \in (0, \eta_0)$, depending only on q, m and the C_k for $k \in [2, j-1]$ with $j \ge 3$, such that*

$$|v^{(j)}(r,t)| \le \frac{K}{\zeta(t) - r} \quad \text{in } R_{\delta,\eta}.$$

Proof. Set

$$\delta = \min\{\frac{2\delta_0}{3}, 2s\eta_0\},$$
$$\eta = \eta_0 - \frac{\delta}{4s},$$

and define

$$R(\bar r, \bar t) \equiv \left\{(r,t) \in \mathbb{R}^2 \ : \ |r - \bar r| < \frac{\lambda}{2},\ \bar t - \frac{\lambda}{4s} < t \le \bar t\right\}$$

for $(\bar r, \bar t) \in R_{\delta,\eta}$, where $s = a + \epsilon$ and $\lambda = \zeta(\bar t) - \bar r$. Then $(\bar r, \bar t) \in R_{\delta,\eta}$ implies that $R_{\bar\delta,\bar\eta} \subset R_0$. Also observe that for each $(\bar r, \bar t) \in R_{\delta,\eta}$, $R(\bar r, \bar t)$ lies to the left of the line $r = \zeta(\bar t) + s(t - \bar t)$. Now set $r = \lambda\xi + \bar r$ and $t = \lambda\tau + \bar t$. Then the function

$$V^{(j-1)}(\xi, \tau) \equiv v^{(j-1)}(\lambda\xi + \bar r, \lambda\tau + \bar t) = v^{(j-1)}(r,t)$$

satisfies the equation

$$\begin{aligned} V_\tau^{(j-1)} = {} & \left\{(m-1)\frac{v}{\lambda}V_\xi^{(j-1)} + (2 + (j-1)(m-1))v_r V^{(j-1)}\right\}_\xi \\ & - (m-1)v_r V_\xi^{(j-1)} + \frac{\bar m}{r} v V_\xi^{(j-1)} \\ & + \lambda(F_{j-1}(r,t) - (2 + (j-1)(m-1))v_{rr})V^{(j-1)} + \lambda G_{j-1}(r,t) \end{aligned} \tag{4.4}$$

in the region

$$B \equiv \left\{(\xi, \tau) \in \mathbb{R}^2 \ : \ |\xi| \le \frac{1}{2},\ -\frac{1}{4s} < \tau \le 0\right\},$$

and $|V^{(j-1)}| \le C_{j-1}$ in B. In view of (3.6) and (3.7) we have

$$(a - \epsilon)\frac{\zeta(t) - r}{\lambda} \le \frac{v(r,t)}{\lambda} \le (a + \epsilon)\frac{\zeta(t) - r}{\lambda}$$

and

$$\zeta(t) \le \zeta(\bar t) \le \zeta(t) + s(\bar t - t) \le \zeta(t) + \frac{\lambda}{4}.$$

Therefore

$$\frac{\lambda}{4} = \zeta(\bar t) - \frac{\lambda}{4} - \bar r - \frac{\lambda}{2} \le \zeta(t) - r \le \zeta(\bar t) - \bar r + \frac{\lambda}{2} = \frac{3\lambda}{2}$$

which implies

$$\frac{a - \epsilon}{4} \le \frac{v}{\lambda} \le \frac{3(a + \epsilon)}{2}.$$

that is, equation (4.4) is uniformly parabolic in B. Moreover, it follows from (3.4) and (4.3) that $V^{(j-1)}$ satisfies all of the hypotheses of the Theorem 5.3.1 of [11]. Thus we conclude that there is a constant $K = K(a, m, C_1, \dots, C_{j-1}) > 0$ such that

$$\left|\frac{\partial}{\partial \xi} V^{(j-1)}(0,0)\right| \le K,$$

that is

$$|v^{(j-1)}(\bar{r},\bar{t})| \le K/\lambda.$$

Since $(\bar{r},\bar{t}) \in R_{\delta,\eta}$ is arbitrary, this proves the lemma. □

We now turn to the barrier construction. If $\gamma \in (0,\delta)$ we will use the notation

$$R^{\gamma}_{\delta,\eta} = R^{\gamma}_{\delta,\eta}(t_0) \equiv \{(r,t) \in \mathbb{R}^2 \ : \ \zeta(t) - \delta \le r \le \zeta - \gamma, t_0 - \eta \le t \le t_0 + \eta\}.$$

Then we have

Lemma 4.3. *Let R_{δ_1,η_1} be the region constructed in the proof of Lemma 3.1. For $j \ge 3$ and $(r,t) \in R^{\gamma}_{\delta_1,\eta_1}$, let*

$$\phi_j(r,t) \equiv \frac{\alpha}{\zeta(t) - r - \gamma/3} + \frac{\beta}{\zeta^*(t) - r}$$

where ζ^* is given by (3.8), and α and β are positive constants. There exist $\delta \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ depending only on a, m, $C_1, \dots, C_{j-1}$ such that

$$L_j(\phi_j) \ge 0 \quad \text{in } R^{\gamma}_{\delta,\eta}$$

for all $\gamma \in (0,\delta)$.

Proof. Choose ϵ such that

$$0 < \epsilon < \frac{a}{2(4 + 2j(m-1))}. \tag{4.5}$$

There exist $\delta_2 \in (0,\delta_1)$ and $\eta \in (0,\eta_1)$ such that (3.4), (3.6) and (3.7) hold in $R_{\delta_2,\eta}$. Fix $\gamma \in (0,\delta_2)$. For $(r,t) \in R^{\gamma}_{\delta_2,\eta}$, we have

$$\begin{aligned}
L_j(\phi_j) \;&=\; \frac{\alpha}{(\zeta - r - \gamma/3)^2}\Big[-\zeta' - \frac{2(m-1)v}{\zeta - r - \gamma/3} - (2 + j(m-1))v_r - \frac{\bar{m}}{r}v \\
&\quad -F_j(r,t)(\zeta - r - \gamma/3) - \frac{(\zeta - r - \gamma/3)^2}{\alpha}G_j(r,t)\Big] \\
&\quad +\frac{\beta}{(\zeta^* - r)^2}\Big[-\zeta^{*'} - \frac{2(m-1)v}{\zeta^* - r} - (2 + j(m-1))v_r - \frac{\bar{m}}{r}v \\
&\quad -F_j(r,t)(\zeta^* - r) - \frac{(\zeta^* - r)^2}{\beta}G_j(r,t)\Big].
\end{aligned}$$

¿From (3.6) and by the fact that $\zeta^* - r \ge \zeta - r - \gamma/3$ we have

$$\frac{v}{\zeta^* - r} \le \frac{v}{\zeta - r - \gamma/3} \le (a+\epsilon)\frac{\gamma}{\gamma - \gamma/3} = \frac{3}{2}(a+\epsilon).$$

Thus it follows from (3.4), (3.7) and (4.3) that

$$\begin{aligned}
L_j(\phi_j) \;&\ge\; \frac{\alpha}{(r - \zeta - \gamma/3)^2}\Big\{a/2 - (3 + 2j(m-1))\epsilon - \delta_2\left(F_j + \frac{\delta_2}{\alpha}G_j\right)\Big\} \\
&\quad +\frac{\beta}{(r - \zeta^*)^2}\Big\{a/2 - (4 + 2j(m-1))\epsilon - \delta_2\left(F_j + \frac{\delta_2}{\beta}G_j\right)\Big\}.
\end{aligned}$$

Since ϵ satisfies (4.5) we can choose $\delta = \delta(\epsilon, m, C_2, \dots, C_{j-1}) > 0$ so small that $L_j(\phi_j) \geq 0$ in $R^{\gamma}_{\delta,\eta}$. $\square$

Lemma 4.4. *(Barrier Transformation). Let δ and η be as in Lemma 4.3 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}, \tag{4.6}$$

where ϵ satisfies (4.5). Suppose that for some nonnegative constants α and β

$$v^{(j)}(r,t) \leq \frac{\alpha}{\zeta(t) - r} + \frac{\beta}{\zeta^*(t) - r} \quad \text{in } R_{\delta,\eta}, \tag{4.7}$$

Then $v^{(j)}$ also satisfies

$$v^{(j)}(r,t) \leq \frac{2\alpha/3}{\zeta(t) - r} + \frac{\beta + 2\alpha/3}{\zeta^*(t) - r} \quad \text{in } R_{\delta,\eta}. \tag{4.8}$$

Proof. By Lemma 4.3, for any $\gamma \in (0,\delta)$ the function

$$\phi_j(r,t) = \frac{2\alpha/3}{\zeta(t) - r - \gamma/3} + \frac{\beta + 2\alpha/3}{\zeta^*(t) - r}$$

satisfies $L_j(\phi_j) \geq 0$ in $R^{\gamma}_{\delta,\eta}$. On the other hand, on the parabolic boundary of $R^{\gamma}_{\delta,\eta}$ we have $\phi_j \geq v^{(j)}$. In fact, for $t = t_1$ and $\zeta_1 - \delta \leq r \leq \zeta_1 - \gamma$, with $\zeta_1 = \zeta(t_1)$, we have

$$\phi_j(r,t_1) = \frac{2\alpha/3}{\zeta_1 - r - \gamma/3} + \frac{\beta + 2\alpha/3}{\zeta_1 - r} > \frac{4\alpha/3}{\zeta_1 - r} + \frac{\beta}{\zeta_1 - r} > v^{(j)}(r,t_1),$$

while for $r = \zeta - \delta$ and $t_1 \leq t \leq t_2$ we have, in view of (4.6),

$$\begin{aligned} \phi_j(\zeta - \delta, t) &\geq \frac{2\alpha/3}{\delta - \gamma/3} + \frac{\beta}{\zeta^* + \delta - \zeta} + \frac{2\alpha/3}{\delta + 6\epsilon\eta} \\ &\geq \frac{2\alpha/3}{\delta} + \frac{\beta}{\zeta^* + \delta - \zeta} + \frac{\alpha/3}{\delta} \geq v^{(j)}(\zeta - \delta, t). \end{aligned}$$

Finally, for $r = \zeta - \gamma$, $t_1 \leq t \leq t_2$ we have

$$\phi_j(\zeta - \gamma, t) = \frac{2\alpha/3}{\gamma - \gamma/3} + \frac{\beta + 2\alpha/3}{\zeta^* + \gamma - \zeta} \geq \frac{\alpha}{\gamma} + \frac{\beta}{\zeta^* + \gamma - \zeta} \geq v^{(j)}(\zeta - \gamma, t).$$

By the comparison principle we have

$$\phi_j \geq v^{(j)} \quad \text{in } R^{\gamma}_{\delta,\eta}$$

for any $\gamma \in (0,\delta)$, and (4.8) follows by letting $\gamma \downarrow 0$. $\square$

Now we are ready to prove our main proposition.

Completion of proof of Proposition 4.1. By Lemma 4.2, we have an estimate for $v^{(j)}$ of the form (4.7) with $\alpha = K$ and $\beta = 0$. Iterating this estimate by the Barrier Transformation Lemma we obtain the sequence of estimates

$$v^{(j)}(r,t) \leq \frac{\alpha_n}{\zeta(t) - r} + \frac{\beta_n}{\zeta^* - r}$$

with $\alpha_n = (2/3)^n K$ and $\beta_n = \{(\frac{2}{3} + \dots + (\frac{2}{3})^n\} K$. Thus if we let $n \to \infty$ we obtain an upper bound for $v^{(j)}$ of the form

$$v^{(j)}(r,t) \leq \frac{2K}{\zeta^* - r} \quad \text{in } R_{\delta,\eta}. \tag{4.9}$$

As in the proof of Lemma 3.1, this implies that $v^{(j)}$ is bounded above in$R_{\delta,\eta/2}$.

Since the equation (4.1) for $v^{(j)}$ is linear, a similar lower bound can be obtained in the same way and the induction step is complete. Therefore as we mentioned in the beginning of this section, Theorem 1.1 is proved.

Acknowledgement. The author would like to thank Professor Zhengfang Zhou for his encouragement and helpful discussions during preparation of this paper.

References

[1] S. Angenent, Analyticity of the interface of the porous media equation after the waiting time, *Proc. Amer. Math. Soc.* **102** (1988), 329-336.

[2] D. G. Aronson, Regularity Properties of Flow through Porous Media : A counterexample, *Siam J. Appl. Math.* **19**, no. 2, (1970), 299-307.

[3] D. G. Aronson, Regularity Properties of Flow through Porous Media : The Interface, *Arch, Rational Mech. Anal.* **37**(1970), 1-10.

[4] D. G. Aronson and P. Benilan, Regularité des solutions de l'equation des milieux poreux dans $\mathbb{R}^n$, *C. R. Acad. Sci. Paris, Ser. A-B* **288** (1979), 103-105.

[5] D. G. Aronson and J. L. Vazquez, Eventual C^∞-regularity and concavity for flows in one-dimensional porous media, *Arch. Rational Mech. Anal.* **99** (1987),no.4, 329-348.

[6] L. A. Caffarelli, J. L. Vázquez and N.I. Wolanski, Lipschtz continuity of solutions and interfaces of the N-dimensional porous medium equation,*Ind. Univ. Math. J.* **36** (1987), 373-401.

[7] L. A. Caffarelli and N. J. Wolanski, $C^{1,\alpha}$ regularity of the free boundary for the N-dimensional porous medium equation, *Comm. Pure Appl. Math.* **43** (1990), 885-902.

[8] P. Daskalopoulos and R. Hamilton, Regularity of the free boundary for the porous medium equation, *J. A.M.S.* **11**(**4**) (1998), 899-965.

[9] J. R. Esteban and J. L. Vazquez, Homogeneous diffusion in R with power-like nonlinear diffusivity, *Arch. Rational Mech. Anal.* **103**(1988) 39-80.

[10] K. Höllig and H. O. Kreiss, C^∞-Regularity for the porous medium equation, *Math. Z.* **192** (1986), 217-224.

[11] O. A. Ladyzhenskaya, N.A. Solonnikov and N.N. Uraltzeva, Linear and quasilinear equations of parabolic type, *Trans. Math. Monographs,* **23**, Amer. Math. Soc., Providence, R. I., 1968.

Youngsang Ko
Department of Mathematics, Kyonggi University, Suwon, Kyonggi-do, 442-760, Korea
E-mail address: ysgo@kuic.kyonggi.ac.kr