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Four-parameter bifurcation for a p-Laplacian system *

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Abstract

We study a four-parameter bifurcation phenomenum arising in a system involving p-Laplacians:

$$
\begin{aligned}
-\Delta_p u &= a\phi_p(u) + b\phi_p(v) + f(a, \phi_p(u), \phi_p(v)),\\
-\Delta_p v &= c\phi_p(u) + d\phi p(v)) + g(d, \phi_p(u), \phi_p(v)),
\end{aligned}
$$

with $u = v = 0$ on the boundary of a bounded and sufficiently smooth domain in $\mathbb{R}^N$; here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $p > 1$ and $p \neq 2$, is the p-Laplacian operator, and $\phi_p(s) = |s|^{p-2}s$ with $p > 1$. We assume that a, b, c, d are real parameters. Thwn we use a bifurcation method to exhibit some nontrivial solutions. The associated eigenvalue problem, with $f = g \equiv 0$, is also studied here.

1 Introduction and Hypotheses

We study some four-parameter bifurcation phenomena arising in the system

$$
\begin{gathered}
-\Delta_p u = a\phi_p(u) + b\phi_p(v) + f(a, \phi_p(u), \phi_p(v)),\\
-\Delta_p v = c\phi_p(u) + d\phi p(v) + g(d, \phi_p(u), \phi_p(v)), \quad \text{in } \Omega\\
u = v = 0\,, \quad \text{on } \partial\Omega\,.
\end{gathered} \tag{1.1}
$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $p > 1,\ p \neq 2$, is the p-Laplacian operator, $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s,\ p > 1$, $\Omega \subset \mathbb{R}^N$ is a sufficiently smooth bounded domain, and a, b, c, d are real parameters.

The operator $-\Delta_p$ occurs in problems arising in pure mathematics, such as the theory of quasiregular and quasiconformal mappings (see [24] and the references therein), and in a variety of applications, such as non-Newtonian fluids, reaction-diffusion problems, flow through porous media, nonlinear elasticity, glaciology, petroleum extraction, astronomy, etc (see [6, 7, 12, 5]). We also emphasize that systems such as (1.1) are not easy generalizations of equations

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because the solutions cannot be obtained by variational methods. Here we use a bifurcation method to exhibit some nontrivial solutions. Another approach for non variational systems can be found in [8]. Moreover the problem considered here where $p \neq 2$ is not a straightforward extension of the case $p = 2$ due to the fact that the translations of the p-Laplacian are not always invertible neither commutative. In this paper we obtain bifurcation results for (1.1). The linear case ($p = 2$) is studied in [15]. The case where $g \equiv 0$ is considered in [14].

We assume through this article that the functions f and g satisfy the following Hypothesis:

A continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies Hypothesis **(H)** if there exists ρ such that $1 \leq \rho < \frac{N+p'}{N-min(p,p')}$ for $\min(p, p') < N$ and $1 \leq \rho$ for $\min(p, p') \geq N$, and such that

$$\text{(H1)} \qquad \lim_{|(r,s)|\to 0} \frac{f(\lambda, r, s)}{|(r, s)|} = 0 \quad \text{uniformly with respect to } \lambda \text{ on bounded sets,}$$

$$\text{(H2)} \qquad \lim_{|(r,s)|\to \infty} \frac{f(\lambda, r, s)}{|(r, s)|^\rho} = 0 \quad \text{uniformly with respect to } \lambda \text{ on bounded sets.}$$

where, as usual, for a given $q > 1$, q' is defined by:

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Definitions: By a solution of the system (1.1) we mean a pair $(A, (u, v)) \in \mathbb{R}^4 \times (W_0^{1,p}(\Omega))^2$, with $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, satisfying (1.1) in the weak sense, i.e., for all $w, z \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla w &= \int_\Omega a|u|^{p-2}uw + b|v|^{p-2}vw + f(a, \phi_p(u), \phi_p(v))w \\ \int_\Omega |\nabla v|^{p-2}\nabla v.\nabla z &= \int_\Omega c|u|^{p-2}uz + d|v|^{p-2}vz + g(d, \phi_p(u), \phi_p(v))z \end{aligned} \tag{1.2}$$

The set of solutions will be denoted by $\mathcal{S}$. Obviously $(A, (0, 0))$ is a solution of (1.1) for every $(a, b, c, d) \in \mathbb{R}^4$. The set of these pairs will be called the trivial solution set, and will be denoted by $\mathcal{S}_0$.

We say that $(A_0, (0, 0)) \in \mathcal{S}_0$ is a *bifurcation point* of (1.1) with respect to the trivial solution set iff every neighborhood of $(A_0, (0, 0))$ contains solutions of (1.1) belonging to $\mathcal{S} \setminus \mathcal{S}_0$.

We will show that whenever (H) is satisfied, any matrix A_0 with a negative eigenvalue, the other being the principal eigenvalue of the p-Laplacian, is such that $(A_0, (0, 0)) \in \mathcal{S}_0$ is a bifurcation point to positive solutions for (1.1).

To establish our results, we combine and adapt methods of [15] and [14]. Our paper is organized as follows: In Section 2, we recalls some results concerning the p-Laplacian. We recall in particular several lemmas established in [14] concerning spaces that we will use. In Section 3, we show that if $(A_0, (0, 0)) \in \mathcal{S}_0$

is a bifurcation point, then the homogeneous system: $-\Delta_p U = A_0 U$ has a non trivial solution. In Section 4 we obtain conditions on A_0 for this to happen. In Section 5 we compute the Leray-Schauder degree for the eigenvalue problem and in Section 6 we state and establish our result.

2 Notation and preliminaries

In this section, we recall briefly some notation and results concerning the p-Laplacian.

The p-Laplacian, $-\Delta_p$, defined on $W_0^{1,p}(\Omega)$ has a first eigenvalue $\lambda_1(p) := \lambda_1$ which is simple and isolated [3]; it is associated to a simple eigenfunction φ (normalized as $\|\varphi\|_\infty = 1$) which is positive. Moreover, λ_1 is characterized by

$$\lambda_1 = \inf_{u \in W_0^{1,p};\int_\Omega |u|^p=1} \int_\Omega |\nabla u|^p \ . \tag{2.1}$$

The following results are known for the equation

$$-\Delta_p u = k|u|^{p-2}u + f \quad \text{in } \Omega \tag{2.2}$$

$$u = 0 \quad \text{on } \partial\Omega\,. \tag{2.3}$$

Lemma 2.1 ([25]) *If $f \in L^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$, Equation (2.2-2.3) has at least one solution and satisfies the maximum principle (i.e. any solution u is non-negative) if and only if $k < \lambda_1$.*

Lemma 2.2 ([13]) *For $f \in L^\infty$, $f \geq 0$, $f \not\equiv 0$, and for $k = \lambda_1$, Equation (2.2-2.3) has no solution in $W_0^{1,p}(\Omega)$.*

The operator T_q. We introduce now some notation and results used in [14]. Let

$$\mathcal{A}(q) = \begin{cases} \frac{Nq'}{N-min(q,q')} & \text{if } \min(q,q') < N \\ +\infty & \text{if } \min(q,q') \geq N \end{cases} \tag{2.4}$$

$$\mathcal{B}(q) = \begin{cases} \frac{Nq}{Nq-N+q} & \text{if } \min(q,q') < N \\ +1 & \text{if } \min(q,q') \geq N. \end{cases} \tag{2.5}$$

Then we introduce the operator $T_q := -\Delta_q \circ \phi_{q'}$ with domain

$$D(T_q) := \left\{ z \in L^{\alpha(q)}(\Omega) : \phi_{q'}(z) \in W_0^{1,p}(\Omega) \text{ and } -\Delta_q(\phi_{q'}(z)) \in L^{\beta(q)}(\Omega) \right\}$$

Then

$$\phi_q(W_0^{1,q}) \hookrightarrow L^\alpha \hookrightarrow L^\beta \hookrightarrow W^{-1,q'},$$

where $\alpha(q), \beta(q)$ are real numbers satisfying $\mathcal{B}(q) < \beta$, $\alpha < \mathcal{A}(q)$.

We notice that the operator T_q is homogeneous of degree 1. We also notice that the equation $T_q u = \lambda u$ has a solution $u \not\equiv 0$, $u \in W_0^{1,q}(\Omega)$ if and only if $-\Delta_q u = \lambda \phi_q(u)$ has a nontrivial solution. Such a λ is an eigenvalue and we denote by $\sigma(-\Delta_p) = \sigma(T_p)$ these eigenvalues. If this solution is positive, then $\lambda = \lambda_1$ and $u = k\varphi$, $k > 0$.

We have the following embeddings.

Lemma 2.3 (Lemma 2.2 in [14]) *If $\alpha < \mathcal{A}(q)$ the embedding $\phi_q(W^{1,q})$ into L^α is compact. If $\beta > \mathcal{B}(q)$, the embedding L^β into $W^{-1,q'}$ is compact.*

Lemma 2.4 ([14]) *For $\alpha < \mathcal{A}(q)$, $\beta > \mathcal{B}(q)$, and $k < 0$, the operators*

$$T_q : D(T_q) \subset L^\alpha \longrightarrow L^\beta \quad \text{and} \quad (T_q - k)^{-1} : L^\beta \longrightarrow L^\alpha$$

are well defined and $(T_q - k)^{-1} : L^\beta \longrightarrow L^\alpha$ is completely continuous.

Lemma 2.5 *For $\alpha < \mathcal{A}(q)$, $\beta > \mathcal{B}(q)$, $k < \lambda_1$ and $f \in L^\beta, f > 0$, $(T_q - k)^{-1} f$ is well defined or equivalently (2.2) has a unique solution.*

Remark 2.6 Obviously, $\Delta_q(-u) = -\Delta_q u$ and $\phi_q(-s) = -\phi_q(s)$, then it follows that $T_q(-u) = -T_q(u)$ and by the previous Lemma with $k < \lambda_1$ and $f \in L^\beta, f < 0$, $(T_q - k)^{-1} f$ is also well defined. When f changes sign several solutions may appear [21, 16].

We also introduce

$$a(q) = \begin{cases} \frac{Nq}{N-q} & \text{if } q < N \\ +\infty & \text{if } q \geq N. \end{cases}$$

From their definitions, it is easy to prove that, for any $q > 1$, we have

$$(a(q))' < q' < \mathcal{A}(q) \leq a(q'),$$

and that the functions $(a(q))', a(q')$ are decreasing in q.

Lemma 2.7 *Assume that $F : \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies* (H). *Choose $\alpha \in \mathbb{R}$ such that*

$$(a(p))' < \frac{\alpha}{\rho} < p' < \alpha < \mathcal{A}(p).$$

Then for any $\lambda \in \mathbb{R}$ and for any $(w, z) \in L^\alpha \times L^\alpha$, we have $F(\lambda, w, z) \in L^\beta$, where $\beta = \frac{\alpha}{\rho}$. Moreover, for any sequence $\{w_n, z_n\} \in L^\alpha \times L^\alpha$, satisfying $(w_n, z_n) \neq (0, 0)$ and $\lim_{n\to\infty} \|(w_n, z_n)\|_{L^\alpha \times L^\alpha} = 0$, we have that

$$\limsup_{n\to\infty} \left\| \frac{F(\lambda, w_n, z_n)}{\|(w_n, z_n)\|_{L^\alpha \times L^\alpha}} \right\|_{L^\beta} = 0$$

Proof. By (H) ρ satisfies $1 \leq \rho < \frac{N+q'}{N-\min(q,q')}$, if $\min(q, q') < N$ and $1 \leq \rho$ if $\min(q, q') \geq N$. It follows from Lemma 2.1 in [14] that we can choose α satisfying

$$(a(p))' < \frac{\alpha}{\rho} < p' < \alpha < \mathcal{A}(p).$$

Moreover, for any $\delta > 0$, there exists a constant C such that

$$|F(\lambda, r, s)| \leq \delta + C|(r,s)|^\rho, \quad \forall (r,s) \in \mathbb{R} \times \mathbb{R}\,;$$

hence the first assertion holds. Now, by Hölder s inequality,

$$\begin{aligned}
&\int_\Omega \Big|\frac{F(\lambda, w_n, z_n)}{\|(w_n, z_n)\|_{L^\alpha \times L^\alpha}}\Big|^\beta \\
\leq\ & \Big(\int_\Omega \Big|\frac{F(\lambda, w_n, z_n)}{|(w_n, z_n)|}\Big|^{\frac{\alpha}{\rho-1}}\Big)^{1/\rho'} \Big(\int_\Omega \Big|\frac{|(w_n, z_n)|}{\|(w_n, z_n)\|_{L^\alpha \times L^\alpha}}\Big|^\alpha\Big)^{1/\rho} \\
\leq\ & C_1 \Big(\int_\Omega \Big|\frac{F(\lambda, w_n, z_n)}{|(w_n, z_n)|}\Big|^{\alpha/(\rho-1)}\Big)^{1/\rho'}. \qquad (2.6)
\end{aligned}$$

From (H) we deduce

$$\left|\frac{F(\lambda, w_n, z_n)}{|(w_n, z_n)|}\right|^{\frac{\alpha}{\rho-1}} \leq \delta^{\frac{\alpha}{\rho-1}} + C_2|(w_n, z_n)|^\alpha.$$

Since $\lim_{n\to\infty} \|(w_n, z_n)\|_{L^\alpha \times L^\alpha} = 0$, for every $\delta > 0$,

$$\limsup_{n\to\infty} \int_\Omega \left|\frac{F(\lambda, w_n, z_n)}{|(w_n, z_n)|}\right|^{\frac{\alpha}{\rho-1}} \leq \delta^{\frac{\alpha}{\rho-1}} |\Omega|\,.$$

Taking into account (2.6) the results follows. ♢

3 Preliminary results

In this section we show that if $(A_0, (0,0))$ is a bifurcation point, then the eigenvalue problem

$$\begin{aligned}
-\Delta_p u &= a_0\phi_p(u) + b_0\phi_p(v), \\
-\Delta_p v &= c_0\phi_p(u) + d_0\phi_p(v), \quad \text{in } \Omega \\
u &= v = 0, \quad \text{on } \partial\Omega\,.
\end{aligned} \qquad (3.1)$$

has a non-trivial solution. This is well-known in the case $p = 2$, (cf. [10]), but due to the nonlinearity of T_p, the proof is much more delicate.

Theorem 3.1 *Let f, g satisfy $(H1)$, and $(A_0, (0,0))$ be a bifurcation point of (1.1) in $\mathbb{R}^4 \times (W_0^{1,p}(\Omega))^2$; then the eigenvalue problem (3.1) has a non-trivial solution .*

Proof. If $(A_0, (0,0))$ is a bifurcation point, then there exists a sequence $\{(A_n, (u_n, v_n))\}$ of nontrivial solutions of (1.1), with $A_n = (a_n, b_n, c_n, d_n) \in \mathbb{R}^4$ and $(u_n, v_n) \in (W_0^{1,p}(\Omega))^2$, such that

$$A_n \to A_0 \quad \text{in } \mathbb{R}^4 \quad \text{and} \quad (u_n, v_n) \to (0,0) \quad \text{in } (W_0^{1,p}(\Omega))^2.$$

Define $w_n = \phi_p(u_n),\ z_n = \phi_p(v_n)$. Due to Lemma 2.2 (cf. [14] ; Lemma 2.2), $w_n,\ z_n \in L^\alpha$ whenever $\alpha < \mathcal{A}(p)$. Moreover, $(A_n, (w_n, z_n))$ is a nontrivial solution of the system

$$\begin{aligned} T_p w_n &= a_n w_n + b_n z_n + f(a_n, w_n, z_n), \\ T_p z_n &= c_n w_n + d_n z_n + g(d_n, w_n, z_n) \quad \text{in } \Omega. \end{aligned} \tag{3.2}$$

Let $s_n = \max\{\|w_n\|_{L^\alpha}, \|z_n\|_{L^\alpha}\} > 0$. By Lemma 2.2 above it is obvious that $s_n \to 0$ as $n \to \infty$. We define

$$W_n = \frac{w_n}{s_n}, \quad Z_n = \frac{z_n}{s_n}, \quad n \in \mathbb{N}$$

Dividing each equation of System (3.2) by s_n we can write

$$\begin{aligned} W_n &= T_p^{-1}\left(a_n W_n + b_n Z_n + \tfrac{1}{s_n} f(a_n, w_n, z_n)\right), \\ Z_n &= T_p^{-1}\left(c_n W_n + d_n Z_n + \tfrac{1}{s_n} g(d_n, w_n, z_n)\right), \quad \text{in } \Omega\,. \end{aligned}$$

From Lemma 2.7, $f(\mathbb{R} \times L^\alpha \times L^\alpha) \subset L^\beta$ for $\beta = \frac{\alpha}{\rho}$ and

$$\limsup_{n\to\infty} \left\| \frac{f(a_n, w_n, z_n)}{s_n} \right\|_{L^\beta} = 0.$$

Of course an analogous result holds for g. Therefore,

$$a_n W_n + b_n Z_n + \frac{1}{s_n} f(a_n, w_n, z_n) \quad \text{and} \quad c_n W_n + d_n Z_n + \frac{1}{s_n} g(d_n, w_n, z_n)$$

are bounded sequences in L^β with $\beta < \alpha$. It follows from the compactness $T_p^{-1} : L^\beta \to L^\alpha$ that there exists two convergent subsequences

$$\begin{aligned} T_p^{-1}\Big(a_n W_n + b_n Z_n + \frac{1}{s_n} f(a_n, w_n, z_n)\Big) &\to W\,, \\ T_p^{-1}\Big(c_n W_n + d_n Z_n + \frac{1}{s_n} g(d_n, w_n, z_n)\Big) &\to Z \end{aligned}$$

in L^α and $(W, Z) \neq (0, 0)$. Moreover $(W_n, Z_n) \to (W, Z)$ in L^α and

$$\begin{aligned} T_p W &= a_0 W + b_0 Z, \\ T_p Z &= c_0 W + d_0 Z, \quad \text{in } \Omega, \end{aligned}$$

or equivalently, (W, Z) is a nontrivial solution of the eigenvalue problem (3.1).

4 An Eigenvalue problem

In this section we consider the eigenvalue problem (3.1) with $(a_0, b_0, c_0, d_0) = (a, b, c, d)$. We establish necessary and sufficient conditions so that System (3.1) has a nontrivial positive solution.

Definition. We say that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the *solvability condition*, and we write $A \in \mathcal{S}(T_p)$, if there exists a nontrivial solution of

$$T_p \begin{pmatrix} w \\ z \end{pmatrix} = A \begin{pmatrix} w \\ z \end{pmatrix}, \tag{4.1}$$

where $w := \phi_p(u)$, $z = \phi_p(v)$, $w, z \in D(T_p)$ with

$$D(T_p) := \{z \in L^{\alpha(p)}(\Omega) : \phi_{p'}(z) \in W_0^{1,p}(\Omega),\ -\Delta_p(\phi_{p'}(z)) \in L^{\beta(p)}(\Omega)\},$$

and $\alpha(p), \beta(p)$ satisfy

$$(\mathcal{B}(p))' < \beta(p) \le \alpha(p) < \mathcal{A}(p). \tag{4.2}$$

We remark that Problem (3.1) is equivalent to the operator equation (4.1).

Definition. Let $\sigma(A)$ denote the spectrum of the Matrix A. Let $\mathcal{M}^-$ be the set of matrices that have a negative eigenvalue.

Remark 4.1 Since A has real coefficients the eigenvalues are complex conjugate; and if one is real, both eigenvalues are real. The eigenvalues, denoted by γ and δ, are the roots of the equation

$$X^2 - (a + d)X + ad - bc = 0. \tag{4.3}$$

If the eigenvalues are not real, $\gamma = \xi + i\eta$ and $\delta = \xi - i\eta$; therefore, $\gamma\delta = \xi^2 + \eta^2 > 0$. since $\gamma\delta = ad - bc$, complex values occur only when $ad - bc > 0$.

When A in $\mathcal{M}^-$, we denote by γ the negative eigenvalue.

Proposition 4.2 **(a)** *If $\sigma(T_p) \cap \sigma(A)$ is not empty, then A is in $\mathcal{S}(T_p)$. More precisely, let λ be in $\sigma(T_p) \cap \sigma(A)$, let $D \in \mathbb{R}^2$ be its corresponding A-eigenvector, let $\phi \in D(T_p)$ be its corresponding T_p-eigenfunction, then $D\phi$ solves (4.1). Consequently, if $\lambda_1 \in \sigma(A)$, and either $b(\lambda_1 - a) > 0$, (≥ 0) or $c(\lambda_1 - d) > 0$, (≥ 0) the eigenvalue problem* (4.1) *has a positive (nonnegative) solution.*

(b) *Conversely, if $A \in \mathcal{M}^- \cap \mathcal{S}(T_p)$, then $\sigma(T_p) \cap \sigma(A)$ is not empty. Moreover if $A \in \mathcal{M}^-$ and if the eigenvalue problem* (4.1) *has a positive solution, then $\sigma(T_p) \cap \sigma(A) = \{\lambda_1\}$.*

This proposition can also be stated as follows:
(a) If one of the eigenvalues of A is in $\sigma(T_p)$ then there exists a nontrivial solution of (4.1).
(b) Conversely, if A has a negative eigenvalue, and if there exists a nontrivial solution of (4.1), then the other eigenvalue of A is in $\sigma(T_p)$.

Remark 4.3 *In part (b) above, if $\sigma(A) := \{\gamma, \delta\}$ and if $\gamma < 0$, necessarily $\delta > 0$, and we have $\gamma\delta = ad - bc < 0$.*

Proof of Proposition (4.2) **(a)** Assume that $\lambda \in \sigma(A) \cap \sigma(T_p)$. By definition of λ, there exists an eigenfunction $\varphi \in D(T_p)$, φ such that $T_p\varphi = \lambda\varphi$. Since $\lambda \in \sigma(A) \subset \mathbb{R}$, there exists an eigenvector $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{R}^2$ such that $AD = \lambda D$. Define $(\eta, \zeta) := (d_1\varphi, d_2\varphi)$. Since T_p is homogeneous of order 1,

$$T_p \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = T_p D\varphi = DT_p\varphi = \lambda D\varphi = AD\varphi = A \begin{pmatrix} \eta \\ \zeta \end{pmatrix}$$

i.e. (η, ζ) is a nontrivial solution of (4.1), and $(d_1\phi_{p'}(\varphi), d_2\phi_{p'}(\varphi)) \neq (0, 0)$ is a nontrivial solution of (3.1). Moreover, if $\lambda = \lambda_1$, we can take $\varphi > 0$, and either $(w, z) = (|b|, |\lambda_1 - a|)\phi$ or $(w, z) = (|c|, |\lambda_1 - d|)\phi$ is a positive (nonnegative) solution (or $b = c = 0, a = d = \lambda_1$ and $(1, 0)\phi$, $(0, 1)\phi$ are nonnegative solutions).

(b) Let $(w, z) \neq (0, 0)$ be a nontrivial solution of (4.1), i.e. (w, z) is a nontrivial solution of

$$\begin{aligned} T_p w &= aw + bz \\ T_p z &= cw + dz \\ w = z &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{4.4}$$

We first consider some obvious cases.

If $w = 0$, then $z \neq 0$ satisfies $T_p z = dz$, therefore $d \in \sigma(T_p)$ and $(T_p - aI)w = bz$ implies $b = 0$; consequently $d \in \sigma(A)$ is an eigenvalue of A and of T_p. Likewise, $z = 0$ implies that $a \in \sigma(A)$ is an eigenvalue of A. Hence we assume now that $w \neq 0$, $z \neq 0$.

If $b = 0$, then $T_p w = aw$ with $w \neq 0$, implies that $a \in \sigma(A) \cap \sigma(T_p)$. On the same way, $c = 0$ implies that $d \in \sigma(A) \cap \sigma(T_p)$. Hence we assume now that $bc \neq 0$.

Now $bc \neq 0$, and assume that γ is a negative eigenvalue of A. Moreover let us assume that $w \neq 0$, $z \neq 0$ are solutions of (4.1).

System (4.1) can also be written as

$$(T_p - \gamma I) \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} a - \gamma & b \\ c & d - \gamma \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{4.5}$$

Moreover, since $\gamma \in \sigma(A)$, it satisfies

$$(a - \gamma)(d - \gamma) = bc. \tag{4.6}$$

From the first equation in (4.4), we obtain $(T_p - aI)w = bz$ with $z \in D(T_p)$. Applying $(T_p - \gamma I)$ on both sides of this equation, and taking into account the second equation in (4.4) we obtain:

$$(T_p - \gamma I)(T_p - aI)w = (T_p - \gamma I)bz = bcw + b(d - \gamma)z.$$

Taking into account (4.6) and (4.5) we derive

$$(T_p - \gamma I)(T_p - aI)w = (d - \gamma)[(a - \gamma)w + bz] = (T_p - \gamma I)(d - \gamma)w. \tag{4.7}$$

We observe that, in order to obtain the previous relations, we use the homogeneity of T_p but we cannot commute $T_p - \gamma I$ and $T_p - aI$ because of the non linearity of T_p.

Since $\gamma < 0$, $(T_p - \gamma I)^{-1}$ is well defined, applying it into (4.7) we obtain: $(T_p - aI)w = (d - \gamma)w$, or equivalently

$$T_p w \;=\; (a + d - \gamma)w$$

so that $a + d - \gamma$ is an eigenvalue of T_p. Since the eigenvalues of A are real and equal to

$$(a + d)/2 \pm \sqrt{((a - d)/2)^2 + bc}, \tag{4.8}$$

if $\gamma < 0$ is an eigenvalue of A, the other is $\delta = a + d - \gamma$. Moreover, if $w > 0$, $z > 0$, $T_p w \;=\; \delta w$ implies $\delta = \lambda_1$.

5 The Leray-Schauder degree for the eigenvalue problem

In this section we study the Leray-Schauder degree in terms of the Jordan canonical form of matrices $A \in \mathcal{M}^-$. For this purpose we use the following property: If $A \in \mathcal{M}^-$ and $\sigma(A) \cap \sigma(T_p) = \emptyset$ then $A \notin \mathcal{S}(T_p)$. Therefore (4.1) has only the trivial solution, which comes from Proposition 4.2.(b).

We denote by $\sigma(A) = \{\gamma, \delta\}$ the spectrum of Matrix A , and $\sigma(-\Delta_p) = \sigma(T_p)$ the set of eigenvalues of the operator $-\Delta_p$ with Dirichlet boundary conditions.

Proposition 5.1 *Let $\mathcal{U} \subset (L^{\beta(p)}(\Omega))^2$ be open bounded and $0 \in \mathcal{U}$. Let J be the Jordan canonical form of the matrix A. Assume that $A \in \mathcal{M}^-$ and $\sigma(A) \cap \sigma(-\Delta_p) = \emptyset$. Then*

$$\deg_{LS}(I - T_p^{-1}A, \mathcal{U}, 0) = \deg_{LS}(I - T_p^{-1}J, \mathcal{U}, 0).$$

Moreover, one of the following two conditions is satisfied

1. $J = \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}$ *and* $\deg_{LS}(I - T_p^{-1}J, \mathcal{U}, 0)$
$= \deg_{LS}(I - \gamma T_p^{-1}, \mathcal{U} \cap L^{\beta(p)}(\Omega), 0)\, \deg_{LS}(I - \delta T_p^{-1}, \mathcal{U} \cap L^{\beta(p)}(\Omega), 0)$
2. $J = \begin{pmatrix} \gamma & 0 \\ 1 & \gamma \end{pmatrix}$ *and*
$\deg_{LS}(I - T_p^{-1}J, \mathcal{U}, 0) = [\deg_{LS}(I - \gamma T_p^{-1}, \mathcal{U} \cap L^{\beta(p)}(\Omega), 0)]^2.$

Remark 5.2 This result has been obtained for $p = 2$ in [15, Proposition 2.1]. In this case, the Leray-Schauder degree for compact linear operators applies [11, 20]).

For $p \neq 2$, the question of calculating $\deg_{LS}(I - \gamma T_p^{-1}, \mathcal{U} \cap L^{\beta(p)}(\Omega), 0)$ has been answered in the following cases

- When the spatial dimension $N = 1$ [19, 21].
- With radial symmetry [4, 22].
- Whenever $\gamma < \lambda_1$ or $\lambda_1 < \gamma < \lambda_2$ [3, 22]).

The other cases are still open problems. We consider here the case $\gamma < 0 < \lambda_1$.

Proof of Proposition 5.1 Let P be the invertible matrix such that $A = P^{-1}JP$. Let $M_{2\times 2}(\mathbb{R})$ be the space of 2×2-matrix with real coefficients. Let us consider a continuous function $\mathcal{P} : [0,1] \to M_{2\times 2}(\mathbb{R})$ such that: 1) $\mathcal{P}(t)^{-1}$ exists for all $t \in [0,1]$, 2) $\mathcal{P}(0) = I$, and 3) $\mathcal{P}(1) = P$.

Let us now define the homotopy $h : [0,1] \times (L^{\beta(p)}(\Omega))^2 \to (L^{\alpha(p)}(\Omega))^2$ by

$$h\left(t, \begin{pmatrix} w \\ z \end{pmatrix}\right) = T_p^{-1}\left[\mathcal{P}(t)^{-1}J\mathcal{P}(t)\right]\begin{pmatrix} w \\ z \end{pmatrix},$$

so that

$$h\left(0, \begin{pmatrix} w \\ z \end{pmatrix}\right) = T_p^{-1}J\begin{pmatrix} w \\ z \end{pmatrix}, \quad h\left(1, \begin{pmatrix} w \\ z \end{pmatrix}\right) = T_p^{-1}A\begin{pmatrix} w \\ z \end{pmatrix}$$

If there exists some nontrivial solution of

$$h\left(t, \begin{pmatrix} w \\ z \end{pmatrix}\right) = \begin{pmatrix} w \\ z, \end{pmatrix}$$

then $\left[\mathcal{P}(t)^{-1}J\mathcal{P}(t)\right] \in \mathcal{S}(T_p)$ which is impossible by Proposition 4.2, since $\sigma\left[\mathcal{P}(t)^{-1}J\mathcal{P}(t)\right] = \sigma(J) = \sigma(A)$, $A \in \mathcal{M}^-$ and $\sigma(T_p) \cap \sigma\left[\mathcal{P}(t)^{-1}J\mathcal{P}(t)\right] = \emptyset$. So $h(t, \begin{pmatrix} w \\ z \end{pmatrix}) \neq \begin{pmatrix} w \\ z \end{pmatrix}$ for any $\begin{pmatrix} w \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now the invariance property for homotopies of the Leray-Schauder degree proves that

$$\deg_{LS}(I - T_p^{-1}A, \mathcal{U}, 0) = \deg_{LS}(I - T_p^{-1}J, \mathcal{U}, 0),$$

with $A = P^{-1}JP$.

We consider separately the following two cases:

Case (i)**:** By the product formulae [11, Theorem 8.5], and since $\gamma, \delta \notin \sigma(-\Delta_p)$, we have

$$\deg_{LS}(I - T_p^{-1}J, \mathcal{U}, 0) = \deg_{LS}((I - \gamma T_p^{-1}, I), \mathcal{U}, 0)\deg_{LS}((I, I - \delta T_p^{-1}), K, 0)$$

where $(I - \gamma T_p^{-1}, I)(w,z) = ((I - \gamma T_p^{-1})w, z)$ and K is the connected component of $L^{\beta(p)}(\Omega)^2 \setminus (I - T_p^{-1}J)(\partial\mathcal{U})$ containing zero. The reduction property states that

$$\deg_{LS}((I - \gamma T_p^{-1}, I), \mathcal{U}, 0) = \deg_{LS}(I - \gamma T_p^{-1}, \mathcal{U} \cap L^{\beta(p)}(\Omega), 0)$$

and Part (i) is proved.

Case (ii): $J = \begin{pmatrix} \gamma & 0 \\ 1 & \gamma \end{pmatrix}$. Here $\sigma(A) = \{\gamma, \gamma < 0\}$ and A is a non-diagonalizable matrix.

Let us define the homotopy $H : [0,1] \times (L^{\beta(p)}(\Omega))^2 \to (L^{\alpha(p)}(\Omega))^2$ by

$$H\left(t, \begin{pmatrix} w \\ z \end{pmatrix}\right) = (T_p)^{-1} \begin{pmatrix} \gamma & 0 \\ t & \gamma \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}.$$

We have $\sigma \begin{pmatrix} \gamma & 0 \\ t & \gamma \end{pmatrix} = \sigma \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} = \sigma(A)$. By Proposition 4.2, and due to $A \in \mathcal{M}^-$, if $H(t,.)$ has a non-trivial solution then $\sigma \begin{pmatrix} \gamma & 0 \\ t & \gamma \end{pmatrix} \cap \sigma(-\Delta_p) \neq \emptyset$, which contradicts the hypothesis $\sigma(A) \cap \sigma(-\Delta_p) = \emptyset$. Therefore $\deg_{LS}(I - H(t,.), \mathcal{U}, 0)$ is well defined and independent of $t \in [0,1]$. Moreover by using again the product formulae

$$\begin{aligned} \deg_{LS}(I - (T_p)^{-1}J, \mathcal{U}, 0) &= \deg_{LS}(I - H(1,.), \mathcal{U}, 0) \\ &= \deg_{LS}(I - H(0,.), \mathcal{U}, 0) \\ &= [\deg_{LS}(I - \gamma(T_p)^{-1}, \mathcal{U} \cap L^{\beta(p)}(\Omega), 0)]^2. \end{aligned}$$

6 Existence of Positive Bifurcated Solutions

In this section we study sufficient conditions for the existence of positive solutions bifurcating from $(A_0, (0,0))$ where $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. From Theorem 3.1 we will need that the eigenvalue problem has a (nontrivial) non negative solution, and therefore we will require, from Proposition 4.2, that $\lambda_1 \in \sigma(A_0)$ and therefore $\sigma(T_p) \cap \sigma(A_0) \neq \emptyset$. Another usual requirement is that there is a changement of topological degree (cf. [23],[11, Theorem 28.1], [1], ...). More explicitly, we have the following

Theorem 6.1 *Assume that f, g satisfy (H), that $\lambda_1 \in \sigma(A_0)$ and that $A_0 \in \mathcal{M}$. Then $(A_0, (0,0))$ is a bifurcation point to positive solutions of (1.1) in $\mathbb{R}^4 \times (W_0^{1,p}(\Omega))^2$.*

Moreover, there is a connected component of topological dimension ≥ 4 of the set of nontrivial solutions of (1.1) in $\mathbb{R}^4 \times (W_0^{1,p}(\Omega))^2$ whose closure contains the point $(A_0, (0,0))$.

Remark 6.2 Theorem above is a generalization for systems of the already known situation for one single equation [22, Proposition 2.2].

Proof. Hereafter we denote by $B_E(c, r)$ the ball in some space E with center $c \in E$ and radius r. Suppose that $(A_0, (0,0))$ is not a bifurcation point of (1.1). Since λ_1 is isolated, there are $\epsilon_0 > 0$ and $r_0 > 0$ such that for every

$A \in B_{\mathbb{R}^4}(A_0,\epsilon_0) \subset \mathbb{R}^4$, if $(w,z) \in B_{(W_0^{1,p})^2}((0,0),r_0) \subset (W_0^{1,p})^2$ satisfies (1.1), then $(w,z)=(0,0)$.

Since for any $A \in B_{\mathbb{R}^4}(A_0,\epsilon_0)$ the functions

$$f(a,.,.),\ g(d,.,.) : (L^\alpha)^2 \to L^\beta$$

map bounded sets into bounded sets, the function $F : B_{\mathbb{R}^4}(A_0,\epsilon_0) \times (L^\alpha)^2 \to (L^\alpha)^2$ given by

$$F\left(A, \begin{pmatrix} w \\ z \end{pmatrix}\right) = T_p^{-1}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} f(a,w,z) \\ g(d,w,z) \end{pmatrix}\right]$$

is completely continuous, consequently $\deg_{LS}(I - F(A,.), B_{(L^\alpha)^2}((0,0),r_0),0)$ is well defined and independent of $A \in B_{\mathbb{R}^4}(A_0,\epsilon_0)$. For $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, denote by J_0 its Jordan canonical form. By hypothesis we can always choose two matrices, $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 1,2$, such that

(a) $\sigma(A_i) \cap \sigma(-\Delta_p) = \emptyset$,

(b) $A_i \in \mathcal{M}$

(c) $A_i \in B_{\mathbb{R}^4}(A_0,\epsilon_0)$ and

(d) $\deg_{LS}(I - T_p^{-1}A_1, \mathcal{U}, 0) \neq \deg_{LS}(I - T_p^{-1}A_2, \mathcal{U}, 0)$.

Let us now define the homotopies

$$H_i\left(t, \begin{pmatrix} w \\ z \end{pmatrix}\right) = T_p^{-1}\left[A_i\begin{pmatrix} w \\ z \end{pmatrix} + t\begin{pmatrix} f(a,w,z) \\ g(d,w,z) \end{pmatrix}\right],$$

Next, we show by contradiction that there exists a real number sufficiently small again denoted by r_0 such that

$$H_i\left(t, \begin{pmatrix} w \\ z \end{pmatrix}\right) \neq \begin{pmatrix} w \\ z \end{pmatrix} \quad \text{in} \quad \partial B_{(L^\alpha)^2}((0,0),r_0) \subset (L^\alpha)^2$$

for any $t \in [0,1]$. Assume that for any $n \in \mathbb{N}$ large enough, there exists a sequence

$$\left\{\left(t_n, \begin{pmatrix} w_n \\ z_n \end{pmatrix}\right)\right\} \in [0,1] \times (L^\beta)^2, \quad \left\|\begin{pmatrix} w_n \\ z_n \end{pmatrix}\right\|_{(L^\beta)^2} = 1/n,$$

and

$$T_p\begin{pmatrix} w_n \\ z_n \end{pmatrix} = A_i\begin{pmatrix} w_n \\ z_n \end{pmatrix} + t_n\begin{pmatrix} f(a,w_n,z_n) \\ g(d,w_n,z_n) \end{pmatrix}, \quad \text{in } \Omega,$$

$$w_n = z_n = 0, \quad \text{on } \partial\Omega.$$

Arguing as in the proof of Theorem (3.1), the associated eigenvalue problem has a non-trivial solution which is positive. Hence, by Proposition (4.2),

$$\lambda_1 \in \sigma(A_i) \cap \sigma(-\Delta_p)$$

which contradicts (a). Then $H_i(t,(w,z)) \neq (w,z)$ in $\partial B_{(L^\alpha)^2}((0,0),r_0)$, therefore $\deg_{LS}(I - H_i(t,.), B_{(L^\alpha)^2}((0,0),r_0),0)$ is well defined and independent of t, consequently

$$\begin{aligned} &\deg_{LS}(I - F(A_i,.), B_{(L^\alpha)^2}((0,0),r_0),0) \\ &\quad = \deg_{LS}(I - H_i(1,.), B_{(L^\alpha)^2}((0,0),r_0),0) \\ &\quad = \deg_{LS}(I - H_i(0,.), B_{(L^\alpha)}((0,0),r_0),0) \\ &\quad = \deg_{LS}(I - T_p^{-1}A_i, B_{(L^\alpha)^2}((0,0),r_0),0) \end{aligned}$$

which, jointly with (c) and (d), contradicts the assertion that $\deg_{LS}(I - F(A_i,.), B_{(L^\alpha)^2}((0,0),r_0),0)$ is constant for $A \in B_{\mathbb{R}^4}(A_0,\epsilon_0)$.

Now, we built the nonnegative the matrices A_1, A_2. ¿From the definition of the Jordan's canonical form, there exists an invertible matrix P such that $A_0 = P^{-1}J_0P$. Denote now by $\gamma < 0$ and δ the eigenvalues of A. Assume that $\delta = \lambda_1$. Let us define

$$A_1 := P^{-1}\begin{pmatrix} \delta+\epsilon & 0 \\ 0 & \gamma \end{pmatrix}P, \quad A_2 := P^{-1}\begin{pmatrix} \delta-\epsilon & 0 \\ 0 & \gamma \end{pmatrix}P.$$

Now, the fact that $T_p^{-1} : (L^\beta)^2 \to (W_0^{1,p})^2$ is continuous ensures that $(A_0,(0,0))$ is a bifurcation point of (1.1). The change of the degree and the Theorem of Alexander and Antman [1] complete the present proof.

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