Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 10, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

Oscillation criteria for delay difference equations *

Jianhua Shen & I. P. Stavroulakis

Abstract

This paper is concerned with the oscillation of all solutions of the delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \dots$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer. Some new oscillation conditions are established. These conditions concern the case when none of the well-known oscillation conditions

$$\limsup_{n\to\infty} \sum_{i=0}^{k} p_{n-i} > 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{1}{k} \sum_{i=1}^{k} p_{n-i} > \frac{k^k}{(k+1)^{k+1}}$$

is satisfied.

1 Introduction

In the last few decades the oscillation theory of delay differential equations has been extensively developed. The oscillation theory of discrete analogue of delay differential equations has also attracted growing attention in the recent few years. The reader is referred to [1-5,9,10,15,16,18,20-23]. In particular, the problem of establishing sufficient conditions for the oscillation of all solutions of the delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \dots \tag{1.1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer, has been the subject of many recent investigations. See, for example, [2-7,9,15,16,18,20,21,23] and the references cited therein. Strong interest in (1.1) is motivated by the fact that it represents a discrete analogue of the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad p(t) \geq 0\,,\ \tau > 0\,. \tag{1.2}$$

---

* *Mathematics Subject Classifications:* 39A10.
Key words: Oscillation, non-oscillation, delay difference equation.

Submitted January 9, 2001. Published January 23, 2001.
This work was supported by the State Scholarship Foundation (I.K.Y.), Athens, Greece, for a postdoctoral research, and was done while the first author was visiting the Department of Mathematics, University of Ioannina"

By a solution of (1.1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (1.1) for $n \geq 0$. A solution $\{x_n\}$ of (1.1) is said to be *oscillatory* if the terms x_n of the solution are not eventually positive or eventually negative. Otherwise the solution is called *non-oscillatory*.

In 1989, Erbe and Zhang [9] and Ladas, Philos and Sficas [16] studied the oscillation of (1.1) and proved that all solutions oscillate if

$$\limsup_{n\to\infty} \sum_{i=0}^{k} p_{n-i} > 1, \tag{1.3}$$

or

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{1.4}$$

or

$$\liminf_{n\to\infty} \frac{1}{k}\sum_{i=1}^{k} p_{n-i} > \frac{k^k}{(k+1)^{k+1}}. \tag{1.5}$$

Observe that (1.5) improves (1.4).

It is interesting to establish sufficient conditions for the oscillation of all solutions of (1.1) when (1.3) and (1.5) are not satisfied. (For (1.2), this question has been investigated by many authors, see, for example, [8,11-14,19] and the references cited therein). In 1993, Yu, Zhang and Qian [23] and Lalli and Zhang [18] derived some results in this direction. Unfortunately, the main results in [23,18] are not correct. This is because these results are based on a false discrete version of Koplatadze-Chanturia Lemma (a counterexample is given in [5]).

In 1998 Domshlak [4], studied the oscillation of all solutions and the existence of non-oscillatory solution of (1.1) with r -periodic positive coefficients $\{p_n\}, p_{n+r} = p_n$. It is very important that in the following cases where $\{r = k\}, \{r = k+1\}, \{r = 2\}, \{k = 1, r = 3\}$ and $\{k = 1, r = 4\}$ the results obtained are stated in terms of necessary and sufficient conditions, and their checking is very easy.

Following this historical (and chronological) review we also mention that in the case where

$$\frac{1}{k}\sum_{i=1}^{k} p_{n-i} \geq \frac{k^k}{(k+1)^{k+1}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{k}\sum_{i=1}^{k} p_{n-i} = \frac{k^k}{(k+1)^{k+1}},$$

the oscillation of (1.1) has been studied in 1994 by Domshlak [3] and in 1998 by Tang [21] (see also Tang and Yu [22]). In a case when p_n is asymptotically close to one of the periodic critical states, optimal results about oscillation properties of the equation

$$x_{n+1} - x_n + p_n x_{n-1} = 0$$

were obtained by Domshlak in 1999 [6] and in 2000 [7].

The aim of this paper is to use some new techniques and improve the methods previously used to obtain new oscillation conditions for (1.1). Our results are based on two new lemmas established in section 2.

For convenience, we will assume that inequalities about values of sequences are satisfied eventually for all large n.

2 Some new lemmas

Lemma 2.1 *Let the number $h \geq 0$ be such that for large n,*

$$\frac{1}{k}\sum_{i=1}^{k} p_{n-i} \geq h\,. \tag{2.1}$$

Assume that (1.1) *has an eventually positive solution* $\{x_n\}$*. Then $h \leq k^k/(k+1)^{k+1}$ and*

$$\limsup_{n\to\infty} \frac{x_n}{x_{n-k}} \leq [d(h)]^k, \tag{2.2}$$

where $d(h)$ is the greater real root of the algebraic equation

$$d^{k+1} - d^k + h = 0 \tag{2.3}$$

on the interval [0,1].

Proof. Since (1.5) implies that all solutions of (1.1) oscillate, but (1.1) has an eventually positive solution, from (2.1), it follows that $h \leq k^k/(k+1)^{k+1}$ must hold. We now prove (2.2). To this end, we let

$$w_n = \frac{1}{k}\sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i-1}}. \tag{2.4}$$

and first prove that $\limsup_{n\to\infty} w_n \leq d(h)$. From (1.1), it follows that $\{x_n\}$ is eventually decreasing and so for large n, we have $x_{n-i-1} \geq x_{n-i}$ for $i = 1, 2, \ldots, k$. This implies that

$$w_n = \frac{1}{k}\sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i-1}} \leq 1 := d_1. \tag{2.5}$$

Thus, $\limsup_{n\to\infty} w_n \leq d(h)$ holds for $h = 0$ because of $d(0) = 1$. We now consider the case when $0 < h \leq k^k/(k+1)^{k+1}$. From (1.1), we have

$$x_{n-i-1} = x_{n-i} + p_{n-i-1}x_{n-i-k-1}, \quad i = 1, 2, \ldots, k. \tag{2.6}$$

Using the Arithmetic-Geometric Mean Inequality in (2.5), we have

$$\left(\frac{x_{n-1}}{x_{n-k-1}}\right)^{1/k} \leq d_1,$$

and so

$$\frac{x_{n-i-k-1}}{x_{n-i-1}} \geq d_1^{-k}, \quad i = 1, 2, \ldots, k\,.$$

Dividing both sides of (2.6) by x_{n-i-1} and using the last inequality, we have

$$1 = \frac{x_{n-i}}{x_{n-i-1}} + p_{n-i-1}\frac{x_{n-i-k-1}}{x_{n-i-1}} \geq \frac{x_{n-i}}{x_{n-i-1}} + d_1^{-k} p_{n-i-1}.$$

Summing both sides of the last inequality from $i = 1$ to $i = k$, we obtain

$$\sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i-1}} \leq k - d_1^{-k} \sum_{i=1}^{k} p_{n-i-1}.$$

This, in view of (2.1), leads to

$$w_n \leq 1 - d_1^{-k} \frac{1}{k} \sum_{i=1}^{k} p_{n-i-1} \leq 1 - \frac{h}{d_1^k} := d_2.$$

Using the last inequality and repeating the above arguments, we have

$$w_n \leq 1 - \frac{h}{d_2^k} := d_3.$$

Following this iterative procedure, by induction, we have

$$w_n \leq 1 - \frac{h}{d_m^k} := d_{m+1}, \quad m = 1, 2, \dots \tag{2.7}$$

It is easy to see that $1 = d_1 > d_2 > \cdots > d_m > d_{m+1} > 0, m = 1, 2, \dots$. Therefore, the limit $\lim_{m\to\infty} d_m = d$ exists and satisfies (2.3). Since (2.7) holds for all $m = 1, 2, \dots, \{d_m\}$ is decreasing and $d(h)$ is the greater real root of the equation (2.3), it follows that $\limsup_{n\to\infty} w_n \leq d(h)$ holds. Finally, using the Arithmetic-Geometric Mean Inequality , we have

$$\limsup_{n\to\infty} \left(\frac{x_{n-1}}{x_{n-k-1}}\right)^{1/k} \leq \limsup_{n\to\infty} \frac{1}{k} \sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i-1}} \leq d(h).$$

This implies (2.2). The proof is complete.

We describe by the following proposition and remark the number $d(h)$.

Proposition 2.1 *For* (2.3), *the following statements hold true:*

(i) *If* $h = 0$, *then* (2.3) *has exactly two different real roots* $d_1 = 0$ *and* $d_2 = 1$.

(ii) *If* $0 < h < k^k/(k+1)^{k+1}$, *then* (2.3) *has exactly two different real roots* d_1 *and* d_2 *such that*

$$d_1 \in (0, k/(k+1)), d_2 \in (k/(k+1), 1).$$

(iii) *If* $h = k^k/(k+1)^{k+1}$, *then* (2.3) *has a unique real root* $d = k/(k+1)$.

The proof of this proposition is easy and is omitted.

Remark 2.1 From Proposition 2.1, we see that the number $d(h)$ in Lemma 2.1 satisfies

$$d(h)\text{is}\begin{cases} =1, & h=0 \\ \in (k/(k+1),1), & 0<h<k^k/(k+1)^{k+1} \\ =k/(k+1), & h=k^k/(k+1)^{k+1}. \end{cases}$$

Lemma 2.2 *Let the number $M \geq 0$ be such that for large n,*

$$\sum_{i=1}^{k} p_{n-i} \geq M. \tag{2.8}$$

Assume that (1.1) *has an eventually positive solution $\{x_n\}$. Then $M \leq k^{k+1}/(k+1)^{k+1}$ and*

$$\limsup_{n\to\infty} \frac{x_{n-k}}{x_n} \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} \leq [\overline{d}(M)]^k, \tag{2.9}$$

where $\overline{d}(M)$ is the greater real root of the algebraic equation

$$d^{k+1} - d^k + M^k = 0, \quad \text{on } [0,1]. \tag{2.10}$$

Proof. As in the proof of Lemma 2.1, we have that $M \leq k^{k+1}/(k+1)^{k+1}$ must hold. We now prove (2.9). To this end, we let

$$\overline{w}_n = \frac{1}{k} \sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i+1}} \left(\sum_{j=1}^{k} p_{n-i+j} \right). \tag{2.11}$$

and first prove that

$$\limsup_{n\to\infty} \overline{w}_n \leq \overline{d}(M). \tag{2.12}$$

From (1.1), we have

$$x_{n+j+1} - x_{n+j} + p_{n+j} x_{n+j-k} = 0, \quad j = 0,1,\dots,k-1\,.$$

Summing the above equality from $j=0$ to $j=k-1$, we have

$$x_n = x_{n+k} + \sum_{j=0}^{k-1} p_{n+j} x_{n+j-k}. \tag{2.13}$$

Since $\{x_n\}$ is eventually decreasing, it follows that

$$x_n > \sum_{j=0}^{k-1} p_{n+j} x_{n+j-k} \geq \left(\sum_{j=0}^{k-1} p_{n+j} \right) x_{n-1},$$

and so for $i = 1, 2, \dots, k$, we have

$$\frac{x_{n-i}}{x_{n-i+1}} \left(\sum_{j=1}^{k} p_{n-i+j} \right) < 1.$$

Summing the last inequality from $i = 1$ to $i = k$, we obtain

$$\overline{w}_n = \frac{1}{k} \sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i+1}} \left(\sum_{j=1}^{k} p_{n-i+j} \right) < 1 := d_1. \tag{2.14}$$

Thus (2.12) holds for $M = 0$ because of $\overline{d}(0) = 1$. We now consider the case when $0 < M \le k^{k+1}/(k+1)^{k+1}$. Using (2.8) and the Arithmetic-Geometric Mean Inequality in (2.14), we have

$$M \left(\frac{x_{n-k}}{x_n} \right)^{1/k} < d_1 \text{or} \frac{x_{n-k}}{x_n} < \frac{d_1^k}{M^k}. \tag{2.15}$$

Since $\{x_n\}$ is eventually decreasing, from (2.13), for $i = 1, 2, \dots, k$, we have

$$\begin{aligned} x_{n-i+1} &= x_{n+k-i+1} + \sum_{j=0}^{k-1} p_{n-i+j+1} x_{n-i+j-k+1} \\ &\ge x_{n+k-i+1} + \sum_{j=1}^{k} p_{n-i+j} x_{n-i}, \end{aligned}$$

and so

$$1 \ge \frac{x_{n+k-i+1}}{x_{n-i+1}} + \sum_{j=1}^{k} p_{n-i+j} \frac{x_{n-i}}{x_{n-i+1}}. \tag{2.16}$$

The last inequality, in view of (2.15), yields

$$1 > \frac{M^k}{d_1^k} + \sum_{j=1}^{k} p_{n-i+j} \frac{x_{n-i}}{x_{n-i+1}}.$$

Summing the last inequality from $i = 1$ to $i = k$, we obtain

$$k > \frac{kM^k}{d_1^k} + \sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i+1}} \left(\sum_{j=1}^{k} p_{n-i+j} \right).$$

Thus

$$\overline{w}_n = \frac{1}{k} \sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i+1}} \left(\sum_{j=1}^{k} p_{n-i+j} \right) < 1 - \frac{M^k}{d_1^k} := d_2. \tag{2.17}$$

Using the inequality (2.17) and repeating the above arguments, we have

$$\overline{w}_n < 1 - \frac{M^k}{d_2^k} := d_3.$$

Following this iterative procedure, by induction, we have

$$\overline{w}_n < 1 - \frac{M^k}{d_m^k} := d_{m+1}, m = 1, 2, \dots \tag{2.18}$$

Now (2.12) follows from similar proof as in Lemma 2.1. Next, using the Arithmetic-Geometric Mean Inequality in (2.12) we have

$$\limsup_{n\to\infty} \Big(\frac{x_{n-k}}{x_n} \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j}\Big)^{1/k} \le \limsup_{n\to\infty} \frac{1}{k} \sum_{i=1}^{k} \frac{x_{n-i}}{x_{n-i+1}} \Big(\sum_{j=1}^{k} p_{n-i+j}\Big) \le \overline{d}(M),$$

which leads to (2.9). The proof is complete.

Observe that the number M in Lemma 2.2 satisfies

$$0 \le M^k \le \left(\frac{k^{k+1}}{(k+1)^{k+1}}\right)^k \le \frac{k^k}{(k+1)^{k+1}},$$

and the last equality holds if and only if $k = 1$. Thus, from Proposition 2.1, we have the following conclusion about the equation (2.10).

Proposition 2.2 *For* (2.10)*, the following statements hold true:*

(i) *If $M = 0$, then* (2.10) *has exactly two different real roots $d_1 = 0$ and $d_2 = 1$.*

(ii) *If $k \ne 1$ and $0 < M \le k^{k+1}/(k+1)^{k+1}$, then* (2.10) *has exactly two different real roots d_1 and d_2 which satisfy*

$$d_1 \in (0, k/(k+1)), d_2 \in (k/(k+1), 1).$$

(iii) *If $k = 1$, then* (2.10) *has two real roots of the form*

$$d_1 = \frac{1 - \sqrt{1-4M}}{2} \quad \textit{and} \quad d_2 = \frac{1 + \sqrt{1-4M}}{2}.$$

Remark 2.2 The number $\overline{d}(M)$ in Lemma 2.2 satisfies

$$\overline{d}(M)\text{is} \begin{cases} = 1, & M = 0 \\ \in (k/(k+1), 1), & k \ne 1, 0 < M \le k^{k+1}/(k+1)^{k+1} \\ = (1 + \sqrt{1-4M})/2, & k = 1. \end{cases}$$

This implies that $\overline{d}(M) \le 1$ and the equality holds if and only if $M = 0$. Observe that (2.8) implies

$$\prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} \ge M^k.$$

Thus, from (2.9), we have

$$\liminf_{n\to\infty} \frac{x_n}{x_{n-k}} \ge [\overline{d}(M)]^{-k} M^k.$$

3 Oscillation criteria for Eqn. (1.1)

In this section, by using the results in section 2, we establish new oscillation criteria for (1.1). From section 1, we see that all solutions of (1.1) oscillate if (1.3), or (1.4) or (1.5) is satisfied. Therefore, we establish oscillation conditions for (1.1) in the case when none of these conditions is satisfied. Let

$$\mu = \liminf_{n\to\infty} \frac{1}{k}\sum_{i=1}^{k} p_{n-i}. \tag{3.1}$$

Theorem 3.1 *Assume that $0 \le \mu \le k^k/(k+1)^{k+1}$ and that there exists an integer $l \ge 1$ such that*

$$\limsup_{n\to\infty} \Bigg\{ \sum_{i=1}^{k} p_{n-i} + [\overline{d}(k\mu)]^{-k} \prod_{i=1}^{k}\sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} [d(\mu)]^{-(m+1)k} \sum_{i=1}^{k}\prod_{j=0}^{m+1} p_{n-jk-i} \Bigg\} > 1, \tag{3.2}$$

where $\overline{d}(k\mu)$ and $d(\mu)$ are the greater real roots of the equations

$$d^{k+1} - d^k + (k\mu)^k = 0 \tag{3.3}$$

and

$$d^{k+1} - d^k + \mu = 0, \tag{3.4}$$

respectively. Then all solutions of (1.1) oscillate.

Proof. Assume, for the sake of contradiction, that (1.1) has an eventually positive solution $\{x_n\}$. We consider the two possible cases:

CASE 1. $\mu = 0$. In this case we have $\overline{d}(k\mu) = d(\mu) = 1$. From (1.1), we have

$$x_{n-i} = x_{n-i+1} + p_{n-i}x_{n-k-i}, \quad i = 1, 2, \dots, k\,.$$

Summing both sides of the above equality from $i = 1$ to $i = k$ leads to

$$x_{n-k} = x_n + \sum_{i=1}^{k} p_{n-i}x_{n-k-i}. \tag{3.5}$$

From (1.1), for any positive integer j, we have

$$x_{n-k-j} = x_{n-k-j+1} + p_{n-k-j}x_{n-k-j-k}. \tag{3.6}$$

Substituting (3.6) for $j = i$ into (3.5), we have

$$x_{n-k} = x_n + \sum_{i=1}^{k} p_{n-i}x_{n-k-i+1} + \sum_{i=1}^{k} p_{n-i}p_{n-k-i}x_{n-i-2k}.$$

Substituting (3.6) for $j = i + k$ into the last equality, we have

$$\begin{aligned} x_{n-k} &= x_n + \sum_{i=1}^{k} p_{n-i} x_{n-k-i+1} + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} x_{n-2k-i+1} \\ &\quad + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} p_{n-2k-i} x_{n-i-3k}. \end{aligned}$$

By induction, it is easy to prove that

$$\begin{aligned} x_{n-k} &= x_n + \sum_{i=1}^{k} p_{n-i} x_{n-k-i+1} + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} x_{n-2k-i+1} \\ &\quad + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} p_{n-2k-i} x_{n-3k-i+1} + \cdots \\ &\quad + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} \cdots p_{n-lk-i} x_{n-(l+1)k-i+1} \\ &\quad + \sum_{i=1}^{k} p_{n-i} p_{n-k-i} \cdots p_{n-(l+1)k-i} x_{n-i-(l+2)k}\,. \end{aligned}$$

Removing the last term of the last equality, we have

$$x_{n-k} \geq x_n + \sum_{i=1}^{k} p_{n-i} x_{n-k-i+1} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} x_{n-(m+2)k-i+1} \prod_{j=0}^{m+1} p_{n-jk-i}. \quad (3.7)$$

In the proof of Lemma 2.2, we have (2.14) holds. Using the Arithmetic-Geometric Mean Inequality in (2.14), we have

$$\left(\frac{x_{n-k}}{x_n} \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} \right)^{1/k} < 1,$$

and so

$$x_n > \left(\prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} \right) x_{n-k}. \quad (3.8)$$

Substituting (3.8) into (3.7) and using the fact that $\{x_n\}$ is eventually decreasing, we have

$$x_{n-k} > \left(\sum_{i=1}^{k} p_{n-i} + \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \right) x_{n-k}.$$

Dividing both sides of the last inequality by x_{n-k}, and taking the limit superior as $n \to \infty$, we have

$$1 \geq \limsup_{n\to\infty} \left\{ \sum_{i=1}^{k} p_{n-i} + \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} + \sum_{m=0}^{l-1} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} \right\}.$$

This contradicts (3.2).

CASE 2. $0 < \mu \leq k^k/(k+1)^{k+1}$. In this case, for any $\eta \in (0,\mu)$, we have

$$\frac{1}{k} \sum_{i=1}^{k} p_{n-i} \geq \mu - \eta. \tag{3.9}$$

From (3.7), we have

$$x_{n-k} \geq x_n + \sum_{i=1}^{k} p_{n-i} x_{n-k} + \sum_{m=0}^{l-1} x_{n-(m+2)k} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i}. \tag{3.10}$$

By Lemma 2.2, we have

$$x_n \geq \{[\overline{d}(k(\mu-\eta))]^{-k} - \eta\} \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} x_{n-k}, \tag{3.11}$$

where $\overline{d}(k(\mu-\eta))$ is the greater real root of the equation

$$d^{k+1} - d^k + k^k(\mu-\eta)^k = 0. \tag{3.12}$$

By Lemma 2.1, we have

$$x_{n-(m+2)k} \geq \{[d(\mu-\eta)]^{-(m+1)k} - \eta\} x_{n-k}, \tag{3.13}$$

where $d(\mu-\eta)$ is the greater real root of the equation

$$d^{k+1} - d^k + (\mu-\eta) = 0. \tag{3.14}$$

Now substituting (3.11) and (3.13) into (3.10), we obtain

$$\begin{aligned} x_{n-k} \quad \geq \quad & \sum_{i=1}^{k} p_{n-i} x_{n-k} + \{[\overline{d}(k(\mu-\eta))]^{-k} - \eta\} \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} x_{n-k} \\ & + \sum_{m=0}^{l-1} \{[d(\mu-\eta)]^{-(m+1)k} - \eta\} \sum_{i=1}^{k} \prod_{j=0}^{m+1} p_{n-jk-i} x_{n-k}. \end{aligned}$$

Dividing both sides of the last inequality by x_{n-k} then taking the limit superior as $n \to \infty$, we have

$$1 \geq \limsup_{n\to\infty} \left\{ \sum_{i=1}^{k} p_{n-i} + \{[\overline{d}(k(\mu-\eta))]^{-k} - \eta\} \prod_{i=1}^{k} \sum_{j=1}^{k} p_{n-i+j} \right.$$

$$+\sum_{m=0}^{l-1}\{[d(\mu-\eta)]^{-(m+1)k}-\eta\}\sum_{i=1}^{k}\prod_{j=0}^{m+1}p_{n-jk-i}\Bigg\}.$$

Letting $\eta \to 0$, we have $\overline{d}(k(\mu-\eta)) \to \overline{d}(k\mu)$ and $d(\mu-\eta) \to d(\mu)$, so that the last inequality contradicts (3.2). The proof is now complete.

Notice that when $k=1$, from Remark 2.1 and Remark 2.2, we have $d(\mu) = \overline{d}(\mu) = (1+\sqrt{1-4\mu})/2$, so condition (3.2) reduces to

$$\limsup_{n\to\infty}\left\{Cp_n + p_{n-1} + \sum_{m=0}^{l-1} C^{m+1}\prod_{j=0}^{m+1}p_{n-j-1}\right\} > 1, \tag{3.15}$$

where $C = 2/(1+\sqrt{1-4\mu}), \mu = \liminf_{n\to\infty} p_n$. Therefore, from Theorem 3.1, we have the following corollary.

Corollary 3.1 *Assume that* $0 \le \mu \le 1/4$ *and that* (3.15) *holds. Then all solutions of the equation*

$$x_{n+1} - x_n + p_n x_{n-1} = 0 \tag{3.16}$$

oscillate.

A condition obtained from (3.15) and whose checking is more easy is given in next corollary.

Corollary 3.2 *Assume that* $0 \le \mu \le 1/4$ *and that*

$$\limsup_{n\to\infty} p_n > \left(\frac{1+\sqrt{1-4\mu}}{2}\right)^2. \tag{3.17}$$

Then all solutions of (3.16) *oscillate.*

Proof. When $\mu = 0$, by condition (1.3), all solutions of (3.17) oscillate. For the case when $0 < \mu \le 1/4$, by Theorem 3.1, it suffices to prove that (3.17) implies (3.15). Notice

$$\frac{1+\sqrt{1-4\mu}}{2} = 1 - \frac{\mu}{1-C\mu},$$

by (3.17) and $\mu = \liminf_{n\to\infty} p_n$, there exists $\varepsilon \in (0,\mu)$ such that $p_n \ge \mu - \varepsilon$ and

$$C\limsup_{n\to\infty} p_n > 1 - \frac{\mu-\varepsilon}{1-C(\mu-\varepsilon)}.$$

The last inequality, in view of the fact that $[C(\mu-\varepsilon)]^m \to 0$ as $m \to \infty$, implies that for some sufficiently large integer $l > 1$

$$\begin{aligned} C\limsup_{n\to\infty} p_n &> 1 - \frac{(\mu-\varepsilon)\{1-[C(\mu-\varepsilon)]^{l+1}\}}{1-C(\mu-\varepsilon)} \\ &= 1-(\mu-\varepsilon) - C(\mu-\varepsilon)^2 - \cdots - C^l(\mu-\varepsilon)^{l+1}, \end{aligned}$$

which leads to (3.15), because

$$p_{n-1}+\sum_{m=0}^{l-1} C^{m+1}\prod_{j=0}^{m+1} p_{n-j-1}\geq (\mu-\varepsilon)+C(\mu-\varepsilon)^2+\cdots+C^l(\mu-\varepsilon)^{l+1}.$$

The proof is complete.

Observe that when $\mu=1/4$, condition (3.17) reduces to $\limsup_{n\to\infty} p_n > 1/4$, which can not be improved in the sense that the lower bound $1/4$ can not be replaced by a smaller number. Indeed, by Theorem 2.3 in [9], we see that (3.16) has a non-oscillatory solution if $p_n\leq 1/4$ for large n. Note, however, that even in the critical state $\lim_{n\to\infty} p_n = 1/4$ (3.16) can be either oscillatory or non-oscillatory. For example, if $p_n=\frac{1}{4}+\frac{c}{n^2}$ then (3.16) will be oscillatory in case $c>1/4$ and non-oscillatory in case $c<1/4$ (the Kneser-like theorem, [3]).

Example. Consider the equation

$$x_{n+1}-x_n+\left(\frac{1}{4}+a\sin^4\frac{n\pi}{8}\right)x_{n-1}=0,$$

where $a>0$ is a constant. It is easy to see that

$$\liminf_{n\to\infty} p_n=\liminf_{n\to\infty}\left(\frac{1}{4}+a\sin^4\frac{n\pi}{8}\right)=\frac{1}{4},$$

$$\limsup_{n\to\infty} p_n=\limsup_{n\to\infty}\left(\frac{1}{4}+a\sin^4\frac{n\pi}{8}\right)=\frac{1}{4}+a.$$

Therefore, by Corollary 3.2, all solutions of the equation oscillate. However, none of the conditions (1.3)-(1.5) and those appear in [4,20,23] is satisfied.

The following corollary concerns the case when $k>1$.

Corollary 3.3 *Assume that $0\leq\mu\leq k^k/(k+1)^{k+1}$ and that*

$$\limsup_{n\to\infty}\sum_{i=1}^{k} p_{n-i} > 1-[\overline{d}(k\mu)]^{-k}(k\mu)^k-\frac{k[d(\mu)]^{-k}\mu_*^2}{1-[d(\mu)]^{-k}\mu_*}, \tag{3.18}$$

where $\mu_*=\liminf_{n\to\infty} p_n$, and $\overline{d}(k\mu), d(\mu)$ are as in Theorem 3.1. Then all solutions of (1.1) oscillate.

Proof. If $\mu=0$ (then $\mu_*=0$), then, by (1.3), all solutions of (1.1) oscillate. If $\mu_*=0, \mu>0$, then (3.18) reduces to

$$\limsup_{n\to\infty}\sum_{i=1}^{k} p_{n-i} > 1-[\overline{d}(k\mu)]^{-k}(k\mu)^k. \tag{3.19}$$

From (3.1) and (3.19), for some sufficiently small $\eta\in(0,\mu)$ we have

$$\frac{1}{k}\sum_{i=1}^{k} p_{n-i}\geq\mu-\eta,\quad \limsup_{n\to\infty}\sum_{i=1}^{k} p_{n-i} > 1-[\overline{d}(k\mu)]^{-k}(k(\mu-\eta))^k. \tag{3.20}$$

Thus, we obtain

$$[\overline{d}(k\mu)]^{-k}\prod_{i=1}^{k}\sum_{j=1}^{k}p_{n-i+j} \ge [\overline{d}(k\mu)]^{-k}(k(\mu-\eta))^k.$$

From this and the second inequality of (3.20), we see that (3.2) holds. By Theorem 3.1, all solutions of (1.1) oscillate. We now consider the case when $0 < \mu_* \le k^k/(k+1)^{k+1}$. By Theorem 3.1, it suffices to prove that condition (3.18) implies condition (3.2). From (3.18), it follows that, for some sufficiently small $\eta \in (0, \mu_*)$ we have

$$\limsup_{n\to\infty}\sum_{i=1}^{k}p_{n-i} > 1-[\overline{d}(k\mu)]^{-k}(k(\mu-\eta))^k - \frac{k[d(\mu)]^{-k}(\mu_*-\eta)^2}{1-[d(\mu)]^{-k}(\mu_*-\eta)}.$$

This, in view of the fact that $[[d(\mu)]^{-k}(\mu_*-\eta)]^m \to 0$ as $m\to\infty$, implies that for some sufficiently large integer $l > 1$

$$\begin{aligned}\limsup_{n\to\infty}\sum_{i=1}^{k}p_{n-i} \quad &> \quad 1-[\overline{d}(k\mu)]^{-k}(k(\mu-\eta))^k \\ &\quad -\frac{k(\mu_*-\eta)^2[d(\mu)]^{-k}\{1-[[d(\mu)]^{-k}(\mu_*-\eta)]^l\}}{1-[d(\mu)]^{-k}(\mu_*-\eta)} \\ &= \quad 1-[\overline{d}(k\mu)]^{-k}(k(\mu-\eta))^k - k(\mu_*-\eta)^2[d(\mu)]^{-k} \\ &\quad \times\{1+[d(\mu)]^{-k}(\mu_*-\eta)+[d(\mu)]^{-2k}(\mu_*-\eta)^2 \\ &\quad +\cdots+[d(\mu)]^{-(l-1)k}(\mu_*-\eta)^{l-1}\}.\end{aligned}$$

This leads to (3.2) because

$$\begin{aligned}&[\overline{d}(k\mu)]^{-k}\prod_{i=1}^{k}\sum_{j=1}^{k}p_{n-i+j}+\sum_{m=0}^{l-1}[d(\mu)]^{-(m+1)k}\sum_{i=1}^{k}\prod_{j=0}^{m+1}p_{n-jk-i} \\ &\ge \quad [\overline{d}(k\mu)]^{-k}(k(\mu-\eta))^k + k(\mu_*-\eta)^2[d(\mu)]^{-k}+k(\mu_*-\eta)^3[d(\mu)]^{-2k} \\ &\quad +\cdots+k(\mu_*-\eta)^{l+1}[d(\mu)]^{-lk}.\end{aligned}$$

The proof is complete.

References

[1] R.P. Agarwal and P.J.Y. Wong, Advanced Topics in Difference Equations, Kluwer Academic Publishers, 1997.

[2] Sui Sun Cheng and Guang Zhang, "Virus" in several discrete oscillation theorems, *Applied Math. Letters*, **13** (2000), 9-13.

[3] Y. Domshlak, Sturmian comparison method in oscillation study for discrete difference equations, I, *Differential and Integral Equations*, **7** (1994), 571-582.

[4] Y. Domshlak, Delay-difference equations with periodic coefficients: sharp results in oscillation theory, *Math. Inequal. Appl.*, **1** (1998), 403-422.

[5] Y. Domshlak, What should be a discrete version of the Chanturia-Koplatadze Lemma? *Functional Differential Equations* (to appear).

[6] Y. Domshlak, Riccati Difference Equations with almost periodic coefficients in the critical state, *Dynamic Systems Appl.*, **8** (1999), 389-399.

[7] Y. Domshlak, The Riccati Difference Equations near "extremal" critical states, *J. Difference Equations Appl.*, **6** (2000), 387-416.

[8] Á. Elbert and I.P. Stavroulakis, Oscillations of first order differential equations with deviating arguments, Univ. of Ioannina, T. R. N^0 172 1990, *Recent trends in differential equations* 163-178, *World Sci. Ser. Appl. Anal.*, World Sci. Publishing Co. (1992).

[9] L. Erbe and B.G. Zhang, Oscillation of discrete analogues of delay equations, *Differential and Integral Equations*, **2** (1989), 300-309.

[10] I. Györi and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[11] J. Jaroš and I.P. Stavroulakis, Oscillation tests for delay equations, *Rocky Mountain J. Math.*, **29** (1999), 197-207.

[12] M. Kon, Y.G. Sficas and I.P. Stavroulakis, Oscillation criteria for delay equations, *Proc. Amer. Math. Soc.*, **128** (2000), 2989-2997.

[13] R. Koplatadze and T. Chanturia, On oscillatory and monotonic solutions of first order delay differential equations with deviating arguments, *Differential'nye Uravnenija*, **18** (1982), 1463-1465 (Russian).

[14] M.K. Kwong, Oscillation of first-order delay equations, *J. Math. Anal. Appl.*, **156** (1991), 274-286.

[15] G. Ladas, Recent developments in the oscillation of delay difference equations, In *International Conference on Differential Equations, Stability and Control*, Dekker, New York, 1990.

[16] G. Ladas, C. Philos and Y. Sficas, Sharp conditions for the oscillation of delay difference equations, *J. Appl. Math. Simulation*, **2** (1989), 101-112.

[17] G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.

[18] B. Lalli and B.G. Zhang, Oscillation of difference equations, *Colloquium Math.*, **65** (1993), 25-32.

[19] Ch.G. Philos and Y.G. Sficas, An oscillation criterion for first order linear delay differential equations, *Canad. Math. Bull.*, **41** (1998), 207-213.

[20] I.P. Stavroulakis, Oscillation of delay difference equations, *Computers Math. Applic.*, **29** (1995), 83-88.

[21] X.H. Tang, Oscillations of delay difference equations with variable coefficients, (Chinese), *J. Central South Univ. of Technology*, **29** (1998), 287-288.

[22] X.H. Tang and J.S. Yu, Oscillation of delay difference equations, *Computers Math. Applic.*, **37** (1999), 11-20.

[23] J.S. Yu, B.G. Zhang and X.Z. Qian, Oscillations of delay difference equations with oscillating coefficients, *J. Math. Anal. Appl.*, **177** (1993), 432-444.

Jianhua Shen
Department of Mathematics, Hunan Normal University
Changsha, Hunan 410081, China
e-mail: jhsh@public.cs.hn.cn

Ioannis P. Stavroulakis
Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
e-mail: ipstav@cc.uoi.gr