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# Existence of positive solutions to a superlinear elliptic problem *

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*Dedicated to Professor J. V. Goncalves*

**Abstract**

We study the existence of positive solutions to the semilinear elliptic problem

$$-\epsilon^2\Delta u + V(z)u = f(u)$$

in $\mathbb{R}^N$ ($N \geq 2$), where the function $f$ has superlinear growth at infinity without any restriction from aboveon its growth.

## 1 Introduction

We are concerned with the existence of positive solutions to the semilinear elliptic problem

$$-\epsilon^2\Delta u + V(z)u = f(u), \quad \text{in } \mathbb{R}^N \ (N \geq 2), \tag{1.1}$$

where $\epsilon$ is a positive parameter, $V : \mathbb{R}^N \to [0, +\infty)$ and $f : [0, +\infty) \to [0, +\infty)$ are non-negative continuous functions. We study here the superlinear problem, that is, when the nonlinearity $f$ satisfies the conditions

**F1:** $\lim_{t\to\infty} \frac{f(t)}{t} = +\infty$.

**F2:** The Ambrosetti-Rabinowitz growth condition: There exists $\theta > 2$ such that

$$0 \leq \theta F(t) = \theta \int_0^t f(s)\, ds \leq tf(t), \quad t \in \mathbb{R}.$$

There are many papers that study (1.1) under several assumptions on the potential $V$ and on the growth of $f$. It is well known that solvability of (1.1) depends on the rate of growth of $f$ at infinity and that the cases $N \geq 3$ and

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$N = 2$ are strikingly different. We can divide these studies in three cases as defined below, where we use the convention

$$2^* := \frac{2N}{N-2}\,.$$

**Subcritical growth:** $\lim_{t\to+\infty} \frac{|f(t)|}{|t|^{2^*}} = 0$, if $N \geq 3$; and $\lim_{t\to+\infty} \frac{|f(t)|}{\exp(\alpha t^2)} = 0$, for all $\alpha$, if $N = 2$.

**Critical growth:** $\lim_{t\to+\infty} \frac{|f(t)|}{|t|^{2^*}} = L$ with $L > 0$, if $N \geq 3$; and for $N = 2$, there exists $\alpha_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{|f(t)|}{\exp(\alpha t^2)} = 0 \quad \forall \alpha > \alpha_0, \quad \lim_{t\to+\infty} \frac{|f(t)|}{\exp(\alpha t^2)} = +\infty \quad \forall \alpha < \alpha_0\,.$$

**Supercritical growth:** $\lim_{t\to+\infty} \frac{|f(t)|}{|t|^{2^*}} = +\infty$, if $N \geq 3$; and $\lim_{t\to+\infty} \frac{|f(t)|}{\exp(\alpha t^2)} = +\infty$ for all $\alpha$, if $N = 2$.

We begin by recalling some results for subcritical growth case. For $(N \geq 3)$, Rabinowitz [14] has found a solution with minimal energy for all small $\epsilon$, when

$$\liminf_{|z|\to\infty} V(z) > \inf_{z\in\mathbb{R}^N} V(z) \equiv V_0 > 0\,.$$

In the case $N = 1$ and $p = 3$, Floer and Weinstein [10], still imposing a global condition on $V$, have shown that the solution concentrates around of the critical point of $V$, as $\epsilon \to 0$. This result was extended by Oh [12, 13] and by Wang [17] for higher dimensions $N \geq 3$. In the case $N \geq 3$, Ambrosetti-Badiale and Cingolani [6], basead on the Lyapunov-Schmidt reduction, showed a similar result with the concentration involving a local maximun of $V$. Del Pino and Felmer [8] assume only that $V$ has a local minima in a bounded set $\Lambda \subset \mathbb{R}^N$ with

$$\inf_{\bar{V}} V < \inf_{\partial\Lambda} V$$

and some additional hypotheses on $f$. They use local variational techniques without any global restriction involving the minimun of $V$ to concluded that the solutions of (1.1) with $N \geq 3$ concentrate around local minima of $V$. Ren and Wei [15] also studied the behavior of solutions to (1.1) on $\mathbb{R}^2$ with $\epsilon = 1$ and $f(u) = u^\tau$, as $\tau \to \infty$.

For the critical case the first author and Souto [2] have considered (1.1) with $N \geq 3$ and $V$ having same global property given in [14] but with $f(u) := \lambda u^q + u^{2^*-1}$ where $\lambda > 0$ and $1 < q < 2^* - 1$, and they proved that the solutions also concentrate in the global minima of $V$. Later, the first author together with do Ó and Souto [1] using the same arguments explored in [8] showed that similar fenomena holds for local minima of $V$ when $f$ has the growth found in [2]. For the case involving critical growth in $N = 2$, we cite the paper by do Ó and Souto [9] that worked with local minima of $V$ studying also the concentration of solutions. Imposing among others assumption on $f$ and $V$, for instance that $V$ is a nonconstant function having a finite limit at infinity, Cao [7] proved some existence result for (1.1).

For the situation involving supercritical growth when $N \geq 3$, we cite the work of the first author [3], where he studied problem (1.1) assuming that $f(u) = u^p (p > 1)$ without any hypothesis on $p$ besides supposing that $V$ is radial and satisfies the following condition:

There exist positive constants $R_1 < r_1 < r_2 < R_2$ such that

**V1:** $V(z) = 0$ in the set $\Omega = \{z \in \mathbb{R}^N : r_1 < |z| < r_2\}$

**V2:** $V(z) \geq V_0 > 0$ in $\Lambda^c = B^c_{R_2} \cup B_{R_1}$.

In [3], the author does not study the concentration phenomena, there the result obtained involves only the existence of positive solutions to (1.1) for $\epsilon$ sufficiently small. Here we shall study problem (1.1) with $N \geq 2$ and show the existence of positive solutions imposing assumptions on the function $f$. We will explore the geometric conditions V1 and V2 in order to conclude that growth of $f$ can be made in some sense "free". We will show that in dimension $N \geq 3$, if such conditions on $V$ hold the function $f$ can have an exponential growth. The main fact is that the geometry of $V$ implies that we do not need any additional restrictions from above on growth of $f$. Similarly, for $N = 2$ the function $f$ can have the behavior like $\exp(\beta u^s)$ with $\beta > 0$ and $s \geq 2$, which is known in the literature as supercritical growth in $\mathbb{R}^2$. Thus, the growth above implies that (1.1) can not be solved directly by applying the usual variational methods, because in this case the energy functional related to problem (1.1) is not well defined on the suitable Sobolev spaces $H^1(\mathbb{R}^N)$ or $H^1_{\text{rad}}(\mathbb{R}^N)$.

To show the main result, we use similar arguments to those used in [8] and [3]. The strategy consists of exploring the special deformation on the nonlinearity $f$ and some properties on the radial functions.

Before to write our main result, we fix the hypotheses on $f$. In our work we assume that the function $f$ is continuous and verifies the following conditions

**F3:** $\dfrac{f(t)}{t}$ is non-decreasing with respect to $t$, for $t > 0$

**F4:** $\displaystyle\lim_{t\to 0} \frac{f(t)}{t} = 0$.

**Theorem 1.1** *Assume Conditions F1-F4, V1, V2. Then, there exists $\epsilon_o > 0$ such that for all $\epsilon \in (0, \epsilon_o)$, problem (1.1) has a classical solution $u_\epsilon \in H^1(\mathbb{R}^N)$ with*

$$u_\epsilon(z) \to 0, \quad \text{as } |z| \to \infty\,.$$

**Remark:** Theorem 1.1 improves and complements the results showed in [3] and [7] respectively, because in our work we study the behavior on other nonlinearities and our approach treats at same time the cases $N \geq 3$ and $N = 2$.

Hereafter, $\int_U f$ represents $\int_U f(z)dz$ and

$$H^1_{\mathrm{rad}} = H^1_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : \ u \text{ is radially symmetric}\}.$$

## 2 Preliminaries

In this section, we prove some auxiliary results for the proof of Theorem 1.1. Since we are concerned with positive solutions, we can assume in the sequel that $f(t) = 0$ for $t \leq 0$.

**Lemma 2.1** *Let $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous and radially symmetric function, that is, $g(z,u) = g(|z|,u)$, for all $z \in \mathbb{R}^N$ and $\in \mathbb{R}$. Given positive constants $a$ and $b$, let*

$$A = \{z \in \mathbb{R}^N : a < |z| < b\} \quad \text{and} \quad G(z,t) := \int_0^t g(z,s)\,ds\,.$$

*If $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{rad}}$, then*

$$\int_A g(z,u_n)u_n \to \int_A g(z,u)u \quad \text{and} \quad \int_A G(z,u_n) \to \int_A G(z,u), \ \text{as} \to \infty\,.$$

**Proof.** Since $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{rad}}$, there exists a positive constant $C$, such that $\|u_n\| \leq C$. Using Straus's inequality (see [11] or [16]),

$$|u_n(z)| \leq \frac{2\pi\|u_n\|}{|z\,|^{1/2}}, \ \forall z \in \mathbb{R}^N \setminus \{0\} \tag{2.1}$$

we obtain

$$|u(z)| \leq \frac{2\pi C}{a^{1/2}} \equiv \bar{a} \in L^1(A), \ \forall z \in \mathbb{R}^N \setminus \{0\}.$$

From this, we have

$$|g(z,u_n)u_n| \leq \max_{(z,t)\in A\times[-\bar{a},\bar{a}]} g(z,t)\bar{a} \ \equiv \ \bar{c} \in L^1(A), \ \forall z \in \mathbb{R}^N \setminus \{0\}.$$

Similarly,

$$|G(z,u_n)| \leq \hat{c} \in L^1(A), \ \forall z \in \mathbb{R}^N \setminus \{0\}.$$

Then from the Lebesgue dominated convergence theorem, we conclude the present proof. ♢

Let

$$g(z,t) = \chi_\Lambda(z)f(t) + (1-\chi_\Lambda)(z)\bar{f}(t),$$

where $\chi_\Lambda$ denotes the characteristic function on $\Lambda$,

$$\bar{f}(t) = \begin{cases} f(t) & t \le a\,, \\ \frac{V_0 t}{k} & t > a, \end{cases}$$

and $a$ is a positive constant so that $\frac{f(a)}{a} = \frac{V_0}{k}$ with $k > \max\{\frac{\theta}{\theta-2}, 2\}$.

It is easy to see that $g$ satisfies not only the condition F2, with $f$ replaced by $g$, but also the following conditions

**G2:** $0 \le \theta G(z,t) \le g(z,t)t$ for all $z \in \Lambda$, $t \in \mathbb{R}$.

**G3:** $0 \le 2G(z,t) \le g(z,t)t \le \frac{V(z)t^2}{k}$ for $z \in \Lambda^c$, $t \in \mathbb{R}$.

In the sequel, we denote by G1, the condition F2 with $f$ replaced by $g$. Now we shall state the crucial auxiliary result.

**Theorem 2.2** *Assume Conditions V1, V2, and G1–G3. Then the problem*

$$-\Delta u + V(z)u = g(z,u), \quad \text{in } \mathbb{R}^N \tag{2.2}$$

*admits a positive solution.*

To prove this theorem, we first fix notation and prove some technical results. We work in the Hilbert space

$$E = \{u \in H^1_{\rm rad}(\mathbb{R}^N) : \int_{\mathbb{R}^N} Vu^2 < \infty\}$$

endowed by the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u\,|^2 + Vu^2)\Big)^{1/2}.$$

We shall find critical points on $E$ of the $C^1$ functional

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\,\nabla u\,|^2 + Vu^2) - \int_{\mathbb{R}^N} G(z,u)$$

whose Fréchet derivative is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + Vuv - g(z,u)v), \quad u, v \in E\,.$$

Next, we shall prove some lemmas related to this functional.

**Lemma 2.3** *I satisfies the following conditions*

**(i)** *There exist $\rho, \beta > 0$ such that $I(u) \ge \beta$ for $\|u\| = \rho$*

**(ii)** *There exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.*

**Proof.** Part (i): From F4, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(t) \le \frac{\epsilon t^2}{2}, \quad |t| \le \delta.$$

Thus

$$\int_\Lambda F(u) \le \frac{\epsilon}{2}\int_\Lambda u^2, \text{ as } ||u|| \le \rho, \ \rho \text{ small enough} \tag{2.3}$$

Now, using condition G3 and (2.3), we have

$$\begin{aligned} I(u) &= (\int_\Lambda + \int_{\Lambda^c})(\frac{1}{2}(|\nabla u|^2 + V(z)u^2) - G(z,u))dz \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + Vu^2) - \int_\Lambda F(u) - \frac{1}{2k}\int_{\Lambda^c} Vu^2 \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{1}{2}(1-\frac{1}{k})\int_{\mathbb{R}^N} Vu^2 - \int_\Lambda F(u) \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + (1-\frac{1}{k})Vu^2) - \frac{\epsilon}{2}\int_\Lambda u^2 \\ &\ge C_1\|u\|^2 - \frac{\epsilon}{2}\int_\Lambda u^2 . \end{aligned} \tag{2.4}$$

Recalling that

$$\int_\Lambda u^2 \le C\int_{\mathbb{R}^N}(|\nabla u|^2 + Vu^2),$$

from (2.4) we have

$$I(u) \ge C_2\|u\|^2, \quad \text{for } ||u|| = \rho.$$

The proof of part $(i)$ is complete.

Verification of part (ii): Choose $\psi \in C_0^\infty(\Lambda)$, so that $\psi > \psi_0 > 0$ for all $x \in \mathrm{K} \subset \operatorname{supp}\psi$. Then, by condition F2 there exists a positive constant $C_1$, such that

$$F(t\psi) \ge C(t\psi)^\theta, \quad t \ge t_0, \ \forall z \in K, \quad t_0 > 0.$$

Using this inequality, we get

$$\begin{aligned} I(t\psi) &= \frac{t^2}{2}\|\psi\|^2 - \int_\Lambda G(z,t\psi) \\ &\le \frac{t^2}{2}\|\psi\|^2 - \int_K F(t\psi) \\ &\le \frac{t^2}{2}\|\psi\|^2 - C_1 t^\theta, \text{ for } t \ge t_0. \end{aligned} \tag{2.5}$$

This proves $(ii)$ and it completes the proof of Lemma 2.3. ♢

Now, by using Ambrosetti and Rabinowitz Mountain Pass Theorem [5], there exists a $(PS)_c$ sequence $\{u_n\}$; that is,

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0,$$

where $c = \inf_{h\in\Gamma} \max_{t\in[0,1]} I(h(t))$ and

$$\Gamma = \{h \in C([0,1], E) : h(0) = 0, h(1) = e\}.$$

**Lemma 2.4** *The functional $I$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.*

**Proof:** Firstly, from Conditions G2 and G3, we have

$$\begin{aligned} &\|u_n\| + M \\ &\geq I(u_n) - \frac{1}{\theta} I'(u_n)u_n \\ &= (\frac{1}{2} - \frac{1}{\theta}) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V u_n^2) + (\int_\Lambda + \int_{\Lambda^c})(\frac{g(z,u_n)u_n}{\theta} - G(z,u_n)) \\ &\geq (\frac{1}{2} - \frac{1}{\theta}) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V u_n^2) + \int_{\Lambda^c} (\frac{g(z,u_n)u_n}{\theta} - G(z,u_n)) \\ &\geq (\frac{1}{2} - \frac{1}{\theta})(\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V u_n^2) - \int_{\Lambda^c} g(z,u_n)u_n) \\ &\geq (\frac{1}{2} - \frac{1}{\theta})(\int_{\mathbb{R}^N} |\nabla u_n|^2 + (1 - \frac{1}{k}) \int_{\mathbb{R}^N} V u_n^2). \end{aligned}$$

By this inequality, there exists a constant $C > 0$ such that $\|u_n\| + M \geq C\|u_n\|^2$, which implies that $\{u_n\}$ is bounded in E. Therefore, up to subsequence, there exists $u \in E$ such that

$$u_n \rightharpoonup u \text{ weakly in } E, \quad \text{and} \quad u_n \to u, \text{ a.e. in } \mathbb{R}^N.$$

Now we state the following

**Claim 1** Given $\epsilon > 0$, there exists a $R > 4R_2$ such that

$$\limsup_{n\to\infty} \int_{|z|>R} (|\nabla u_n|^2 + V u_n^2) < \epsilon.$$

Proof of claim 1: Arguing as in [3] and [8], from Conditions G2 and G3, and taking a cut-off function $\eta_R \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\eta_R = 0 \text{ in } B_{R/2}, \quad \eta_R = 1, \text{ in } B_R^c \quad \text{and} \quad |\nabla \eta_R| \leq \frac{C}{R},$$

we obtain

$$\begin{aligned} &I'(u_n)(u_n\eta_R) \\ &= \int_{B_{R/2}^c} (|\nabla u_n|^2 + V u_n^2)\eta_R + \int_{B_R \setminus B_{R/2}} u_n |\nabla u_n| \nabla \eta_R - \int_{B_{R/2}^c} g(z,u_n)u_n\eta_R \\ &\geq \int_{B_{R/2}^c} (|\nabla u_n|^2 + V u_n^2)\eta_R - |u_n|_2 |\nabla u_n|_2 \frac{C}{R} - \frac{1}{k} \int_{B_{R/2}^c} V u_n^2 \eta_R + r(n). \end{aligned}$$

where $r(n)$ is an $o(1)$-function as $n$ approaches $+\infty$. Since $I'(u_n)(u_n\eta_R) = o(1)$, we have

$$\begin{aligned}(1-\frac{1}{k})\int_{B_R^c}(|\nabla u_n|^2+Vu_n^2)\eta_R &\leq (1-\frac{1}{k})\int_{B_{R/2}^c}(|\nabla u_n|^2+Vu_n^2)\eta_R\\ &\leq \frac{C}{R}(|u_n|_2|\nabla u_n|_2)+o(1),\\ &\leq \frac{C_1}{R}+o(1).\end{aligned}$$

So that the proof of Claim 1 follows by choosing $R > C_1/\epsilon$.

**Claim 2:**

**(i)** $\int_{\mathbb{R}^N} g(z,u_n)u_n \to \int_{\mathbb{R}^N} g(z,u)u$,

**(ii)** $u$ is a critical point of $I$, that is, $I'(u)v = 0$ for all $v \in E$.

Assuming Claim 2, from $I'(u_n)u_n = o(1)$, it follows that

$$\begin{aligned}\|u_n\|^2 &= \int_{\mathbb{R}^N} g(z,u_n)u_n + o(1)\\ &= \int_{\mathbb{R}^N} g(z,u)u + o(1)\\ &= \|u\|^2 + o(1)\,.\end{aligned}$$

Therefore, $u_n \to u$ strongly in $E$.

Proof of Claim 2 Part i): Note that

$$\begin{aligned}\int_{\mathbb{R}^2}(g(z,u_n)u_n - g(z,u)u) &= (\int_{B_{R_1}} + \int_{B_R\setminus B_{R_1}} + \int_{B_R^c})(g(z,u_n)u_n - g(z,u)u)\\ &= I_1 + I_2 + I_3.\end{aligned}$$

We shall prove that each of these terms approaches zero as $n \to \infty$. From the boundedness of $B_{R_1} \subset \Lambda^c$, we have $u_n \to u$, in $L^2(B_{R_1})$. By Condition G3 it follows that $I_1 \to 0$. From Lemma 2.1, we conclude that $I_2 \to 0$. Finally, combining Claim 1 and condition G3, we get $I_3 \to 0$. Then $(i)$ holds.

Proof of Claim 2 Part (ii): Since $I'(u_n)v = o(1)$, it suffices to prove the following

$$\int_{\mathbb{R}^N} g(z,u_n)v \to \int_{\mathbb{R}^N} g(z,u)v, \text{ as } \to \infty.$$

Arguing as before, splitting the integral in two,we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}(g(z,u_n) - g(z,u))v &= (\int_{\Lambda} + \int_{\Lambda^c})(g(z,u_n) - g(z,u))v\\ &= J_1 + J_2.\end{aligned}$$

From the behaviour of $u_n$, that is by (2.1), we have

$$|u_n(x)| \leq \frac{C}{R_1^{1/2}} \equiv a \tag{2.6}$$

and since $g$ is a bounded function on $\Lambda$, applying Lebesgue's Dominated Convergence Theorem follows that $J_1 \to 0$, as $n \to \infty$. Now, from (2.6) and Conditions G3, we get

$$\int_{\Lambda^c} (g(z,u_n) - g(z,u))^2 \leq \int_{\Lambda^c} (\frac{V_0(|u_n| + |u\ |)}{k})^2 \leq \int_{\Lambda^c} C(|u_n\ |^2 + |u\ |^2) \leq C_1,$$

for some positive constant $C_1$. Now, using a Lemma from Brezis and Lieb [11], it follows that $J_2 \to 0$. This completes the proof of Lemma 2.4. ♢

**Proof of Theorem 2.2** From Lemmas 2.3 and 2.4, problem (2.2) has at least one positive weak solution $u \in E$. Similarly, for each $\epsilon > 0$, there exists $u_\epsilon \in E$ weak positive solution of (2.2), satisfying

$$I'_\epsilon(u_\epsilon)v = 0, \quad \forall v \in E,$$

where

$$I_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(\epsilon^2|\nabla u\ |^2 + Vu^2) - \int_{\mathbb{R}^N} G(z,u).$$

# 3 Proof of Theorem 1.1

Let $\{u_\epsilon\}$ be the sequence of positive weak solutions of (2.2) obtained in the previous section. The crucial result for this section is the following.

**Lemma 3.1** $\|u_\epsilon\|_{H^1} \to 0$ *as* $\epsilon \to 0$.

**Proof.** Note that $u_\epsilon$ satisfies

$$I_\epsilon(u_\epsilon) = c_\epsilon \quad \text{and} \quad I'_\epsilon(u_\epsilon)v = 0,\ \forall v \in E_\epsilon,$$

where $c_\epsilon = \inf\limits_{\psi \in E_\epsilon} \max\limits_{t \geq 0} I_\epsilon(t\psi)$ and

$$E_\epsilon = \{u \in H^1_{\text{rad}} : \int_{\mathbb{R}^2} \frac{1}{2}(\epsilon^N|\nabla u\ |^2 + Vu^2) < \infty\}.$$

Taking $\psi \in C^\infty_{o,\text{rad}}(\Omega)$, a nonnegative function with $\operatorname{supp}\psi \subset \Omega$, there is an unique $t_\epsilon \in \mathbb{R}^+$ such that

$$I_\epsilon(t_\epsilon\psi) = \max_{t \leq 0} I_\epsilon(t\psi),$$

so

$$0 \leq c_\epsilon \leq I_\epsilon(t_\epsilon\psi) \leq \frac{t_\epsilon^2}{2} \int_\Omega \epsilon^2|\nabla\psi|^2 - \int_\Omega F(t_\epsilon\psi).$$

On the other hand, we know that

$$\epsilon^2 \int_\Omega |\nabla\psi|^2 = \int_\Omega \frac{f(t_\epsilon\psi)}{t_\epsilon}\psi, \tag{3.1}$$

choosing $\Omega_1 \subset \Omega$ such that $\psi(z) \geq \psi_0 > 0\ \forall z \in \Omega_1$, it follows

$$\epsilon^2 \int_\Omega |\nabla\psi|^2 \geq \int_{\Omega_1} \frac{f(t_\epsilon\psi)}{t_\epsilon}\psi \geq \psi_0^2 \int_{\Omega_1} \frac{f(t_\epsilon\psi)}{t_\epsilon\psi}, \tag{3.2}$$

thus from (3.2) and Conditions F1–F3 that $t_\epsilon \to 0$ as $\epsilon \to 0$. Now, remarking that

$$c_\epsilon \leq I_\epsilon(t_\epsilon\psi) = (t_\epsilon^2/2)||\psi||^2 - \int_{\mathbb{R}^N} F(t_\epsilon\psi) \leq (t_\epsilon^2/2)||\psi||^2 \tag{3.3}$$

and arguing as in the proof of Lemma 2.4, we obtain

$$\begin{aligned} I_\epsilon(u_\epsilon) &= I_\epsilon(u_\epsilon) - \frac{1}{\theta} I'_\epsilon(u_\epsilon)u_\epsilon \\ &\geq (\frac{1}{2} - \frac{1}{\theta})(\int_{\mathbb{R}^N} (\epsilon^2|\nabla u_\epsilon\ |^2 + (1 - \frac{1}{k})Vu_\epsilon^2) \\ &\geq C\epsilon^2 \int_{\mathbb{R}^N} (|\nabla u_\epsilon\ |^2 + Vu_\epsilon^2). \end{aligned}$$

Hence, combining this last inequality with (3.3), we have

$$C\epsilon^2 \int_{\mathbb{R}^N} |\nabla u_\epsilon\ |^2 + Vu_\epsilon^2 \leq I_\epsilon(u_\epsilon) \leq \frac{t_\epsilon^2\epsilon^2}{2} \int_\Omega |\nabla\psi\ |^2,$$

that is,

$$\|u_\epsilon\|_{H^1}^2 \leq C\|u_\epsilon\|^2 \leq \frac{t_\epsilon^2}{2} \int_\Omega |\nabla\psi\ |^2\ .$$

Therefore, the proof of Lemma 3.1 is complete. ♢

Next, using an argument similar to those used in [8], we will prove that $u_\epsilon$ is a solution of (1.1). For each $\epsilon > 0$, from (2.1) we have

$$m_\epsilon^1 = \max_{\partial B_{R_1}} u_\epsilon(z) \to 0, \quad \text{as } \epsilon \to 0, \tag{3.4}$$

and

$$m_\epsilon^2 = \max_{\partial B_{R_2}} u_\epsilon(z)\ \to\ 0, \quad \text{as } \epsilon \to 0. \tag{3.5}$$

Combining (3.4) and (3.5), there exists $\epsilon_o > 0$ such that

$$m_\epsilon^i < a, \quad \forall\epsilon \in (0, \epsilon_o),\ i = 1, 2.$$

Now, since $(u_\epsilon - a)_+ \in E_\epsilon$, we have

$$\int_{\mathbb{R}^N\setminus\bar\Lambda} \epsilon^2 \mid \nabla(u_\epsilon - a)_+ \mid^2 + Vu_\epsilon(u_\epsilon - a)_+ = \int_{\mathbb{R}^N\setminus\bar\Lambda} (g(z, u_\epsilon)u_\epsilon(u_\epsilon - a)_+. \tag{3.6}$$

On the other hand, from $G3$, we obtain

$$Vu_\epsilon(u_\epsilon - a)_+ - g(z,u_\epsilon)u_\epsilon(u_\epsilon - a)_+ \geq 0, \quad \forall z \in \Lambda^c,$$

which together with (3.6), we have

$$\int_{\mathbb{R}^N\setminus\bar{\Lambda}} \epsilon^2|\nabla(u_\epsilon - a)_+ |^2 = 0.$$

Therefore, $u_\epsilon(z) \leq a$ for all $z \in \mathbb{R}^N \setminus \bar{\Lambda}$. Using this, we conclude that

$$g(z,u_\epsilon(z)) = f(u_\epsilon(z)), \quad \forall z \in \mathbb{R}^N \setminus \bar{\Lambda}\,.$$

So, for all $\epsilon \in (0,\epsilon_o)$, $u_\epsilon$ satisfies

$$\int_{\mathbb{R}^N}(\epsilon^2\nabla u_\epsilon\nabla\eta + Vu_\epsilon\eta) = \int_{\mathbb{R}^N} f(u_\epsilon)\eta,\ \forall\eta \in E_\epsilon.$$

Thus, we infer that $f(u_\epsilon) \in L^1_{loc}(\mathbb{R}^N)$.

On the other hand, using a result by Alves, de Moraes Filho and Souto (see [4, Lemma1]), we can conclude that $u_\epsilon$ satisfies (1.1) in $D'(\mathbb{R}^N)$ and by the elliptic regularity (see e.g. [4]), we have that $u_\epsilon \in C^2(\mathbb{R}^N)$. This completes the proof of Theorem 1.1.

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