Electronic Journal of Differential Equations, Vol. 2001(2001), No. 12, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF A CLASS OF SECOND ORDER DIFFERENTIAL SYSTEMS

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Abstract. In the present paper it is proved that for any solution $x_1(t)$ of the system $M\ddot{x} + \dot{x} = f(t,x)$, for which $\lim_{t\to\infty} \|\dot{x}_1(t)\| = 0$, there exists a solution $x_2(t)$ of the system $\dot{x} = f(t,x)$ such that $\lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0$. Some generalizations of this result are also presented. The case $f(t,x) = -\nabla U(x)$ has been investigated explicitly.

1. Statements and main results

We consider the following two n-dimensional systems

$$M\ddot{x} + \dot{x} = f(t,x) \tag{1}$$

and

$$\dot{x} = f(t,x), \tag{2}$$

where $M \geq 0$ is a constant; "$\cdot$" denotes differentiation with respect to t; $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$; $\mathbb{R}_+ \equiv [0,\infty)$; Ω is a domain in $\mathbb{R}^n$; $\mathbb{R}^n$ is the n-dimensional Euclidean space with Euclidean scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$.

Let $(x_{11}, x_{12}) \in \Omega \times \mathbb{R}^n$ and $x_2 \in \Omega$ be fixed initial points and $t_0 \in \mathbb{R}_+$ be a fixed initial moment; $x_1(t; t_0, x_{11}, x_{12})$ and $x_2(t; t_0, x_2)$ denote the solutions of the systems (1) and (2) with initial conditions

$$x_1(t_0; t_0, x_{11}, x_{12}) = x_{11}, \quad \dot{x}_1(t_0; t_0, x_{11}, x_{12}) = x_{12} \tag{3}$$

and

$$x_2(t_0; t_0, x_2) = x_2, \tag{4}$$

respectively.

We introduce the following hypotheses (H1):

(H1.1) Ω is a bounded domain in $\mathbb{R}^n$; $f \in \mathbf{C}(\mathbb{R}_+ \times \Omega, \mathbb{R}^n)$.
(H1.2) The function f is Lipschitz with respect to the second argument with Lipschitz constant $L \geq 0$.

Mathematics Subject Classification. 34D05, 34D10, 34E05.
Key words. asymptotic behaviour, gradient systems, T. Wazewski's theorem.

Submitted April 4, 2000; December 28, 2000. Published January 30, 2001.

(H1.3) For arbitrary initial conditions $(t_0, x_{11}, x_{12}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, $(t_0, x_2) \in \mathbb{R}_+ \times \Omega$, the Cauchy problems (1), (3) and (2), (4) have unique solutions $x_1(t; t_0, x_{11}, x_{12})$ and $x_2(t; t_0, x_2)$, respectively, defined on t-interval $\mathbb{R}_+$. Moreover

$$\{x_1(t; t_0, x_{11}, x_{12}) : t \in \mathbb{R}_+\} \subset \Omega, \quad \{x_2(t; t_0, x_2) : t \in \mathbb{R}_+\} \subset \Omega.$$

In this article, the following theorem contains one of the basic results.

Theorem 1. *Assume the following conditions hold:*

1. *The hypothesis (H1) holds.*
2. $(t_0, x_{11}, x_{12}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ *is the fixed initial condition for* (1), (3).
3.
$$\lim_{t\to\infty} \|\dot{x}_1(t; t_0, x_{11}, x_{12})\| = 0\,. \tag{5}$$

Then there exists at least one initial condition $x_2 \in \Omega$ *for the system* (2) *such that*

$$\lim_{t\to\infty} \|x_1(t; t_0, x_{11}, x_{12}) - x_2(t; t_0, x_2)\| = 0. \tag{6}$$

Theorem 1 is proved in subsection 4.1.

Example 1. Let us consider the differential equations:

$$\ddot{x} + \dot{x} = t, \tag{7}$$

and

$$\dot{x} = t. \tag{8}$$

An immediate integration of these equations yields:

$$x_1(t; 0, x_{11}, x_{12}) = 1 + x_{12} + x_{11} - t + \frac{t^2}{2} - e^{-t}(1 + x_{12})$$

and

$$x_2(t; 0, x_2) = x_2 + \frac{t^2}{2}.$$

It is not difficult to check that for any three points $x_{11}, x_{12}, x_2 \in \mathbb{R}$ we have

$$\lim_{t\to\infty} |x_1(t; 0, x_{11}, x_{12}) - x_2(t; 0, x_2)| \neq 0.$$

Moreover, in this example, for any initial conditions $x_{11}, x_{12} \in \mathbb{R}$ of the problem (7), (3) we have

$$\lim_{t\to\infty} |\dot{x}_1(t; 0, x_{11}, x_{12})| = \lim_{t\to\infty} |-1 + t + e^{-t}(1 + x_{12})| = \infty.$$

From the above equality it follows that (5) is not true and, as we have shown, equality (6) is not valid. Thus (5) is an essential condition.

The following result is derived similarly to the proof of Theorem 1 (see subsection 4.2).

Theorem 2. *Assume the following conditions are fulfilled:*

1. *The hypothesis (H1) holds.*
2. $(t_0, x_{11}, x_{12}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ *is a fixed initial condition for* (1), (3).
3.
$$\lim_{t\to\infty} \|\ddot{x}_1(t; t_0, x_{11}, x_{12})\| = 0. \tag{9}$$

Then there exists at least one initial condition $x_2 \in \Omega$ *of system* (2) *such that the equality* (6) *is valid.*

2. An application: Second order gradient systems

In the present section we shall discuss some asymptotic properties of the solutions of the following system

$$M\ddot{x} + \dot{x} = -\nabla U(x), \tag{10}$$

where $M \geq 0$ is a constant; $U \in \mathbf{C^1}(\Omega, \mathbb{R})$; Ω is a domain in $\mathbb{R}^n$ such that any solution of (10) starting in Ω remains in Ω.

First, let us write (10) as a first order system in $\mathbb{R}^{2n}$:

$$\dot{x} = y, \quad \dot{y} = M^{-1}\left(-y - \nabla U(x)\right). \tag{11}$$

Setting

$$L(x, y) = \frac{M}{2}y^2 + U(x),$$

it is not difficult to see that

$$L'(x, y) = My\dot{y} + \nabla U(x)y = (M\dot{y} + \nabla U(x))y = -y^2 \leq 0, \tag{12}$$

for any $(x, y) \in \Omega \times \mathbb{R}^n$. Therefore, if $U(x) \geq 0$, $x \in \Omega$ then $L(x, y)$ is a Liapunov function (i.e. a continuous non-negative function which satisfies locally a Lipschitz condition) for (11).

Let $\mathcal{M} = \{(x, y) : L'(x, y) = 0\} = \{(x, 0) : x \in \Omega\}$ and let $\mathcal{M}_1$ be the union of all points (x_0, y_0) of all orbits $(x(t; x_0, y_0), y(t; x_0, y_0))$ such that

$$\{(x(t; x_0, y_0), y(t; x_0, y_0)) : t \in \mathbb{R}\} \subset \mathcal{M}.$$

Theorem 3. *Suppose that $U(x) \geq 0$ for all $x \in \Omega$, and $\lim_{\|x\| \to \infty} U(x) = \infty$.*

Then:

1. *All solutions of* (11) *are bounded.*
2. *Every solution of* (11) *approaches $\mathcal{M}_1$ as $t \to \infty$, i.e. for any $(x_0, y_0) \in \Omega \times \mathbb{R}^n$ we have*

$$(x(t; x_0, y_0), y(t; x_0, y_0)) \to \mathcal{M}_1, \quad \textit{as} \quad t \to \infty.$$

3. *For any $(x_0, y_0) \in \Omega \times \mathbb{R}^n$,*

$$\lim_{t \to \infty} \dot{x}(t; x_0, y_0) = \lim_{t \to \infty} y(t; x_0, y_0) = 0\,.$$

Proof. Obviously, $\lim_{\|x\|^2 + \|y\|^2 \to \infty} L(x, y) = \infty$. Then the first statement of Theorem 3 follows from [4, Theorem 10.1].

The second statement follows immediately from results in [2], see also [4, Theorem 14.4, Theorem 14.7].

The third statement follows from second one and implication $\mathcal{M}_1 \subset \mathcal{M} = \{(x, 0) : x \in \Omega\}$. □

The following result follows immediately from Theorem 1 and Theorem 3.

Theorem 4. *Let the following conditions hold true:*

1. *Ω is a domain in $\mathbb{R}^n$; $U \in \mathbf{C^1}(\Omega, \mathbb{R}^n)$; the hypothesis (H 1.3) is valid, where $f(t, x) = -\nabla U(x)$.*
2. *$(t_0, x_{11}, x_{12}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ is a fixed initial condition for the initial-value problem* (10), (3).

Then there exists at least one initial condition $x_2 \in \Omega$ for the system $\dot{x} = -\nabla U(x)$ such that

$$\lim_{t\to\infty} \|x_1(t;t_0,x_{11},x_{12}) - x_2(t;t_0,x_2)\| = 0. \tag{13}$$

It is not difficult to derive some properties of the solutions of systems (10) or (11): the ω-limit set of a solution of system (11) consists of critical points only; if there are two critical points in the ω-limit set then there are infinitely many critical points in the same ω-limit set; there is no non-trivial periodic solutions of (11), etc. The proofs of these facts follow from Theorem 4 and results in [3, Chapter 1, §1].

3. A Topological Principle

In the present Section we shall deduce the Topological Principle in the theory of autonomous dynamical systems or the so-called T. Wazewski's Theorem. The Topological Principle is related to the initial value-problem

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \tag{14}$$

where $f \in \mathbf{C}(\mathcal{E}, R^n)$; $\mathcal{E}$ is an open (t,x)-set in $\mathbb{R}\times\mathbb{R}^n$; $(t_0,x_0)\in\mathcal{E}$. Let $\mathcal{E}_0$ be a non-empty open subset in $\mathcal{E}$.

We recall the following definitions.

Definition 1. *The point $(t_0,x_0)\in\mathcal{E}\cap\partial\mathcal{E}_0$ is said to be:*

1. *an egress point of $\mathcal{E}_0$ with respect to the system (14) if, for every solution $x(t;t_0,x_0)$ of (14) there exists $\theta>0$ such that $\{(t,x(t;t_0,x_0)) : t\in[t_0-\theta,t_0)\}\subset\mathcal{E}_0$.*
2. *an strict egress point of $\mathcal{E}_0$ with respect to the system (14) if, (t_0,x_0) is an egress point of $\mathcal{E}_0$ and $\{(t,x(t;t_0,x_0)) : t\in[t_0,t_0+\theta]\}\subset\mathcal{E}\setminus\mathcal{E}_0$ for sufficiently small $\theta>0$. See Figure 1.*

In the following, $\mathcal{E}_0^e$ ($\mathcal{E}_0^{se}$) denotes the set of all egress (strict egress) points of $\mathcal{E}_0$. It is clear that $\mathcal{E}_0^{se}\subset\mathcal{E}_0^e$.

Definition 2. *The open subset $\mathcal{E}_0$ in $\mathcal{E}$ is said to be an open $[U,V]$-subset in $\mathcal{E}$ with respect to the system (14) if:*

1. *There exist integers $p,q\geq 1$ and continuous functions $U_j:\mathcal{E}\to\mathbb{R}$, $j=1,\dots,p$ and $V_k:\mathcal{E}\to\mathbb{R}$, $k=1,\dots,q$ such that*
$$\mathcal{E}_0=\{(t,x) : U_j(t,x)<0 \;\; \textit{and} \;\; V_k(t,x)<0,\ 1\leq j\leq p,\ 1\leq k\leq q\}.$$
2. *If for any two indexes $\alpha=1,\dots,p$ and $\beta=1,\dots,q$ we denote*
$$\begin{aligned}\mathcal{U}_\alpha &= \big\{(t,x) : U_\alpha(t,x)=0,\ U_j(t,x)\leq 0 \textit{ and } V_k(t,x)<0,\\ &\quad 1\leq j\leq p,\ j\neq\alpha,\ 1\leq k\leq q\big\},\\ \mathcal{V}_\beta &= \big\{(t,x) : U_j(t,x)<0,\ V_\beta(t,x)=0 \textit{ and } V_k(t,x)\leq 0,\\ &\quad 1\leq j\leq p,\ 1\leq k\leq q,\ k\neq\beta\big\},\end{aligned}$$
then the trajectory derivatives
$$U'_\alpha(t_0,x_0)=\frac{dU_\alpha(t,x(t;t_0,x_0))}{dt}\Big|_{t=t_0},$$
$$V'_\beta(t_0,x_0)=\frac{dV_\beta(t,x(t;t_0,x_0))}{dt}\Big|_{t=t_0}$$

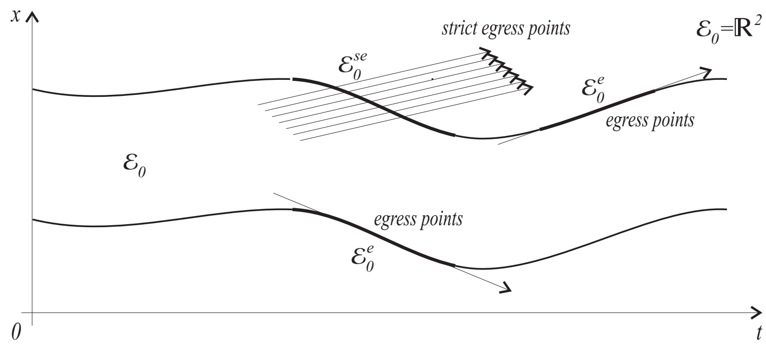

FIGURE 1. Egress points and strict egress points

exist, satisfying the inequalities

$$U'_\alpha(t_0,x_0) > 0, \quad \textit{for any point } \ (t_0,x_0)\in\mathcal{U}_\alpha,$$

$$V'_\beta(t_0,x_0) < 0, \quad \textit{for any point } \ (t_0,x_0)\in\mathcal{V}_\beta,$$

along all solutions of (14) *through* (t_0,x_0).

The theorem (T. Wazewski's Theorem) is known also as the Topological Principle in the theory of autonomous dynamical systems.

Theorem 5. *Assume the following conditions:*

1. $\mathcal{E}$ *is an open* (t,x)*-set in* $\mathbb{R}\times\mathbb{R}^n$*;* $f\in\mathbf{C}(\mathcal{E},R^n)$.
2. *The initial-value problem* (14) *has a unique solution through every point of* $\mathcal{E}$*, and these solutions depend continuously on initial values.*
3. $\mathcal{E}_0$ *is an open subset in* $\mathcal{E}$.
4. *All egress points of the set* $\mathcal{E}_0$ *are strict egress points, i.e.* $\mathcal{E}_0^e=\mathcal{E}_0^{se}$.
5. $\mathcal{W}$ *is a non-empty subset in* $\mathcal{E}_0\cup\mathcal{E}_0^e$ *such that* $\mathcal{W}\cap\mathcal{E}_0^e$ *is a retract of* $\mathcal{E}_0^e$*, but is not a retract of* $\mathcal{W}$.

Then there exists at least one point $(t_0,x_0)\in\mathcal{W}\cap\mathcal{E}_0$ *such that* $x(t;t_0,x_0)\in\mathcal{E}_0$ *for any* t *in the right-maximal interval of existence of* $x(t;t_0,x_0)$.

An useful tool for checking the validity of condition 4 of Theorem 5 is the following lemma.

Lemma 1. *Assume the following conditions:*

1. *The conditions 1 and 2 of Theorem 5 hold.*
2. $\mathcal{E}_0$ *is an open* $[U,V]$*-subset of* $\mathcal{E}$ *with respect to the system* (14).

Then

$$\mathcal{E}_0^e=\mathcal{E}_0^{se}=\bigcup_{\alpha=1}^{p}\mathcal{U}_\alpha \ \setminus \ \bigcup_{\beta=1}^{q}\mathcal{V}_\beta,$$

where $\mathcal{U}_\alpha$ *and* $\mathcal{V}_\beta$ *are the sets introduced in Definition 2.*

One may find the circumstantial explanation and proofs of all results in this Section in [1, Chapter X, §3].

4. Proofs

4.1. **Proof of Theorem 1.** Let $(x_{11}, x_{12}) \in \Omega \times \mathbb{R}^n$ be a fixed initial condition for the system (1). For simplification of notations we suppose that $t_0 = 0$. Further, we shall use the notation $x_1(t) = x_1(t; 0, x_{11}, x_{12})$.

We set

$$g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}, \quad g(t,u) = -\frac{6}{5}Lu^2 - \frac{6}{5}LM\sqrt[3]{u}\|\dot{x}_1(t)\|.$$

For any initial condition $u_0 \in \mathbb{R}_+$, the differential inequality

$$2u\dot{u} < g(t,u) \tag{15}$$

has a solution $u(t; u_0)$ for which:

$$u(0; u_0) = u_0, \tag{16}$$

and

$$\lim_{t\to\infty} u(t; u_0) = 0. \tag{17}$$

To prove these facts, it is sufficient to see that the initial-value problem

$$2u\dot{u} = -2Lu^2 - \frac{6}{5}LM\sqrt[3]{u}\|\dot{x}_1(t)\|, \quad u(0; u_0) = u_0 \tag{18}$$

has solution for which (16) and (17) hold true. The mentioned solution is

$$u(t, u_0) = e^{-Lt}\left(u_0^{\frac{5}{3}} - LM\int_0^t e^{\frac{5Ls}{3}}\|\dot{x}_1(s)\|\,ds\right)^{\frac{3}{5}}.$$

Below, we shall use the notation $u(t) = u(t; u_0)$.

We set:

$$\begin{aligned} &U : \mathbb{R}_+ \times \Omega \to \mathbb{R}, && U(t,x) = \|M\dot{x}_1(t) + x_1(t) - x\|^{\frac{6}{5}} - u^2(t); \\ &V : \mathbb{R}_+ \to \mathbb{R}_- \equiv (-\infty, 0], && V(t) = -t; \end{aligned}$$

$$\begin{aligned} \mathcal{U} &= \{(t,x) \in \mathbb{R}_+ \times \Omega : U(t,x) = 0 \text{ and } V(t) < 0\}; \\ \mathcal{V} &= \{(t,x) \in \mathbb{R}_+ \times \Omega : U(t,x) < 0 \text{ and } V(t) = 0\}; \\ \mathcal{E}_0 &= \{(t,x) \in \mathbb{R}_+ \times \Omega : U(t,x) < 0 \text{ and } V(t) < 0\}; \qquad \mathcal{E} = \mathbb{R}_+ \times \Omega. \end{aligned}$$

Our goal is to show that there exists at least one initial condition $\xi_2 \in \Omega$ and initial moment $\tau > 0$ for the problem (2), (4) such that

$$\{(t, x_2(t; \tau, \xi_2)) : t > \tau\} \subset \mathcal{E}_0. \tag{19}$$

First, we shall prove if

$$(t_*, x_*) \in \mathcal{U}, \text{ then } U'(t_*, x_*) > 0, \tag{20}$$

where $'$ denotes the derivative of function $U(t,x)$ along the trajectories of system (2), i.e. $U'(t_*, x_*) = \frac{d}{dt}U(t, x_2(t; t_*, x_*))|_{t=t_*}$.

Let $(t_*, x_*) \in \mathcal{U}$ be a fixed point. We set

$$m : \mathbb{R}_+ \to \mathbb{R}_+, \quad m(t) = \|M\dot{x}_1(t) + x_1(t) - x_2(t; t_*, x_*)\|^2.$$

The definition of the set $\mathcal{U}$ implies

$$\begin{aligned} 0 = U(t_*, x_*) &= \|M\dot{x}_1(t_*) + x_1(t_*) - x_*\|^{\frac{6}{5}} - u^2(t_*) \\ &= \|M\dot{x}_1(t_*) + x_1(t_*) - x_2(t_*; t_*, x_*)\|^{\frac{6}{5}} - u^2(t_*) \\ &= m^{\frac{3}{5}}(t_*) - u^2(t_*) \end{aligned}$$

or $m^{\frac{3}{5}}(t_*) = u^2(t_*)$. On the other hand, if $h_0 > 0$ is sufficiently small and if $h \in (-h_0, h_0)$ then

$$\begin{aligned} m(t_* + h) &= m(t_*) + 2h\Big\langle M\dot{x}_1(t_*) + x_1(t_*) - x_2(t_*; t_*, x_*), \\ &\qquad M\ddot{x}_1(t_*) + \dot{x}_1(t_*) - \dot{x}_2(t_*; t_*, x_*)\Big\rangle + \varepsilon_1(h), \end{aligned}$$

where $\varepsilon_1 : (-h_0, h_0) \to \mathbb{R}$ and

$$\lim_{h\to 0} \frac{\varepsilon_1(h)}{h} = 0. \tag{21}$$

The equalities $M\ddot{x}_1(t) + \dot{x}_1(t) = f(t, x_1(t))$ and $\dot{x}_2(t; t_*, x_*) = f(t, x_2(t; t_*, x_*))$ imply

$$\begin{aligned} &\|M\ddot{x}_1(t) + \dot{x}_1(t) - \dot{x}_2(t; t_*, x_*)\| = \|f(t, x_1(t)) - f(t, x_2(t; t_*, x_*))\| \le \\ &\le L\|x_1(t) - x_2(t; t_*, x_*)\| = L\|M\dot{x}_1(t) + x_1(t) - x_2(t; t_*, x_*) - M\dot{x}_1(t)\| \le \\ &\qquad \le L\sqrt{m(t)} + LM\|\dot{x}_1(t)\|. \end{aligned} \tag{22}$$

From (21) and (22) at $t = t_*$ we obtain

$$m(t_* + h) \le m(t_*) + 2|h| \Big(Lm(t_*) + LM\sqrt{m(t_*)}\|\dot{x}_1(t_*)\|\Big) + \varepsilon_1(h). \tag{23}$$

The formula (23) yields

$$\begin{aligned} \tfrac{m(t_*+h)-m(t_*)}{h} &\le 2Lm(t_*) + 2LM\sqrt{m(t_*)}\|\dot{x}_1(t_*)\| + \tfrac{\varepsilon_1(h)}{h}, \quad \text{for } h > 0, \\ \tfrac{m(t_*+h)-m(t_*)}{h} &\ge -2Lm(t_*) - 2LM\sqrt{m(t_*)}\|\dot{x}_1(t_*)\| + \tfrac{\varepsilon_1(h)}{h}, \quad \text{for } h < 0. \end{aligned} \tag{24}$$

From the definition of the function $m(t)$ it follows that $m(t)$ is $\mathbf{C^1}$-smooth. Letting $h \to \pm 0$ in the inequalities (24) and using (21) we obtain the following estimates for the derivative of function $m(t)$ at $t = t_*$

$$-2Lm(t_*) - 2LM\sqrt{m(t_*)}\|\dot{x}_1(t_*)\| \le \dot{m}(t_*) \le 2Lm(t_*) + 2LM\sqrt{m(t_*)}\|\dot{x}_1(t_*)\|. \tag{25}$$

Therefore, from definitions of functions $U(t, x)$, $u(t)$, (21) and left hand-side of (25) it follows that

$$\begin{aligned} U'(t_*, x_*) = \; & \tfrac{d}{dt}\Big(m^{\frac{3}{5}}(t_*) - u^2(t_*)\Big) = \tfrac{3}{5}m^{-\frac{2}{5}}(t_*)\dot{m}(t_*) - 2u\dot{u}(t_*) \\ \ge \; & \tfrac{3}{5}m^{-\frac{2}{5}}(t_*)\Big(-2Lm(t_*) - 2LM\sqrt{m(t_*)}\|\dot{x}_1(t_*)\|\Big) - 2u(t_*)\dot{u}(t_*) \\ = \; & -\tfrac{6}{5}Lm^{\frac{3}{5}}(t_*) - \tfrac{6}{5}LMm^{\frac{1}{10}}(t_*) - 2u(t_*)\dot{u}(t_*) \\ = \; & -\tfrac{6}{5}Lu^2(t_*) - \tfrac{6}{5}LMu^{\frac{1}{3}}(t_*) - 2u(t_*)\dot{u}(t_*) \\ = \; & g(t_*, u(t_*)) - 2u(t_*)\dot{u}(t_*) > 0. \end{aligned}$$

The last inequality prove the implication (20). Immediately, the definition of function $V(t)$ yields

$$\text{if } (t_*, x_*) \in \mathcal{V}, \text{ then } V'(t_*, x_*) = -1 < 0. \tag{26}$$

From (20) and (26) it follows, $\mathcal{E}_0$ is an open $[U, V]$-subset in $\mathcal{E}$ with respect to the system (2). Therefore, using Lemma 1 we conclude

$$\mathcal{E}_0^e = \mathcal{E}_0^{se} = \mathcal{U} \setminus \mathcal{V} = \mathcal{U}. \tag{27}$$

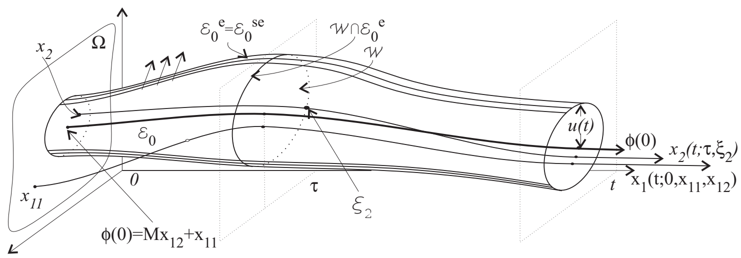

FIGURE 2

Now, from the definitions of the sets $\mathcal{U}$, $\mathcal{V}$ and equality (27) it is not difficult to conclude (see. Figure 4.1)

$$\mathcal{E}_0^e = \{(t,x) \in \mathbb{R} \times \Omega : t > 0 \text{ and } \|\phi(t) - x\| = u(t)\}, \tag{28}$$

where $\phi(t) = M\dot{x}_1(t) + x_1(t)$.

Let $\tau > 0$ be a fixed number. Setting

$$\mathcal{W} = \{(t,x) \in \mathbb{R}_+ \times \Omega : t = \tau \text{ and } \|\phi(\tau) - x\| \le u(\tau)\} \subset \mathcal{E}_0 \cup \mathcal{E}_0^e.$$

we obtain that $\mathcal{W}$ is a ball in $\mathbb{R}^n$, and

$$\mathcal{W} \cap \mathcal{E}_0^e = \{(t,x) \in \mathbb{R}_+ \times \Omega : t = \tau \text{ and } \|\phi(\tau) - x\| = u(\tau)\}. \tag{29}$$

Obviously, the boundary $\partial\mathcal{W}$ of the set $\mathcal{W}$ is not a retract of $\mathcal{W}$, i.e. the set $\mathcal{W} \cap \mathcal{E}_0^e$ is not a retract of $\mathcal{W}$. We shall show that $\mathcal{W} \cap \mathcal{E}_0^e$ is a retract of $\mathcal{E}_0^e$. For this purpose we introduce the map

$$\pi : \mathcal{E}_0^e \to \mathbb{R}^{1+n}, \quad \pi(t,x) = (\tau, \pi_2(t,x)),$$

where

$$\pi_2(t,x) = \phi(\tau) + (x - \phi(t))\frac{u(\tau)}{u(t)}.$$

Obviously π is a continuous map. Moreover, if $(\widetilde{t}, \widetilde{x}) \in \mathcal{E}_0^e$, then

$$\|\phi(\widetilde{t}) - \widetilde{x}\| = u(\widetilde{t}).$$

That is why

$$\|\phi(\tau) - \pi_2(\widetilde{t}, \widetilde{x})\| = \|\phi(\widetilde{t}) - \widetilde{x}\|\frac{u(\tau)}{u(\widetilde{t})} = u(\tau),$$

or $\pi : \mathcal{E}_0^e \to \mathcal{W} \cap \mathcal{E}_0^e$. For $(\tau, \widetilde{x}) \in \mathcal{W} \cap \mathcal{E}_0^e$, we have

$$\pi(\tau, \widetilde{x}) = (\tau, \pi_2(\tau, \widetilde{x})) = (\tau, \phi(\tau) + (\widetilde{x} - \phi(\tau))) = (\tau, \widetilde{x}).$$

Therefore, π is a retraction.

From the Wazewski's Theorem (see Theorem 5) it follows that there exists at least one point $(\tau, \xi_2) \in \mathcal{W} \cap \mathcal{E}_0^e$, such that (19) holds true.

The definition of set $\mathcal{E}_0$ yields

$$\|M\dot{x}_1(t) + x_1(t) - x_2(t;\tau,\xi_2)\| < u^{\frac{5}{3}}(t) \text{ for } t > \tau. \tag{30}$$

From (30) and (17) we conclude that

$$\lim_{t\to\infty} \|M\dot{x}_1(t) + x_1(t) - x_2(t;\tau,\xi_2)\| \le \lim_{t\to\infty} u^{\frac{5}{3}}(t) = 0. \tag{31}$$

Therefore, (31) and (5) imply

$$\lim_{t\to\infty} \|x_1(t) - x_2(t;\tau,\xi_2)\| = \lim_{t\to\infty} \|M\dot{x}_1(t) + x_1(t) - x_2(t;\tau,\xi_2) - M\dot{x}_1(t)\| \le$$
$$\le \lim_{t\to\infty} \|M\dot{x}_1(t) + x_1(t) - x_2(t;\tau,\xi_2)\| + M \lim_{t\to\infty} \|\dot{x}_1(t)\| = 0.$$

To complete the proof of Theorem 1 it is enough to set $x_2 = x_2(0;\tau,\xi_2)$.

4.2. **Proof of Theorem 2.** The proof of the Theorem 2 is analogous to the proof of Theorem 1. We shall present only the appropriate settings:

$$g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, \quad g(t,u) = -\frac{6}{5}Lu^2 - \frac{6}{5}LMu^{\frac{1}{3}}\|\ddot{x}_1(t)\|$$

and

$$U : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \quad U(t,x) = \|x_1(t) - x\|^{\frac{6}{5}} - u^2(t).$$

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