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# Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces *

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**Abstract**

In this paper, we study the existence of global solutions to Schrödinger equations in one space dimension with a derivative in a nonlinear term. For the Cauchy problem we assume that the data belongs to a Sobolev space weaker than the finite energy space $H^1$. Global existence for $H^1$ data follows from the local existence and the use of a conserved quantity. For $H^s$ data with $s < 1$, the main idea is to use a conservation law and a frequency decomposition of the Cauchy data then follow the method introduced by Bourgain [3]. Our proof relies on a generalization of the tri-linear estimates associated with the Fourier restriction norm method used in [1, 25].

## 1 Introduction

In this paper, we study the well-posedness for the Cauchy problem associated with the Schrödinger equation

$$iu_t + u_{xx} = i\lambda(|u|^2u)_x, \quad u(0) = u_0, \quad (t, x) \in \mathbb{R}^2, \tag{1.1}$$

where the unknown function $u$ is complex valued with arguments $(t, x) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$. Equation (1.1) is a model of the propagation of circularly polarized Alfvèn waves in magnetized plasma with a constant external magnetic field [22, 23]. When $\lambda = 0$, the above equation is called the free equation.

Many results are known for the Cauchy problem in the energy space $H^1$ [10, 11, 12, 24]. When looking for solutions of (1.1), we meet with a derivative loss stemming from the derivative in the nonlinear term. In [10, 11, 12, 24], it was proved that for small data $u_0 \in H^1$ the Cauchy problem (1.1) is globally well-posed. The proof of existence of solutions was obtained by using gauge transformations, which reduces the original equation (1.1) to a system of nonlinear Schrödnger equations with no derivative in the nonlinearity. Then the

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result for nonlinear Schrödinger equations can be combined with energy conservation laws to show the existence of global solutions in $H^1$. Let us consider data $u_0$ in classical Sobolev spaces $H^s$ of low order. In [25], it was proved that the Cauchy problem (1.1) is locally well-posed in $H^s$ for $s \geq \frac{1}{2}$. The aim of the paper is to present the extension of that solution to a global solution. We shall sketch the proof of [25] briefly, which is convenient to pursue our result. The result for $s \geq \frac{1}{2}$ was proved by using the Fourier restriction norm method, in addition to the gauge transformation. The Fourier restriction norm method was first introduced by J. Bourgain [1], and was simplified by C. E. Kenig, G. Ponce and L. Vega [14, 16]. The Fourier restriction norm associated with the free solutions, is defined as follows.

**Definition 1.1** For $s, b \in \mathbb{R}$, we define the space $X_{s,b}$ to be the completion of the Schwarz function space on $\mathbb{R}^2$ with respect to the norm

$$\|f\|_{X_{s,b}} = \left( \iint_{\mathbb{R}^2} \langle \xi \rangle^{2s} \langle \tau + \xi^2 \rangle^{2b} |\widehat{f}(\tau, \xi)|^2 \, d\tau d\xi \right)^{1/2},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. We denote the Fourier transform in $t$ and $x$ of $f$ by $\widehat{f}$, and often abbreviate $\|f\|_{X_{s,b}}$ by $\|f\|_{s,b}$.

Via the transformation $v(t,x) = e^{-i\lambda \int_{-\infty}^{x} |u(t,y)|^2 \, dy} u(t,x)$ used in [25, 10, 11, 12, 24]), (1.1) is formally rewritten as the Cauchy problem

$$\begin{gathered} iv_t + v_{xx} = -i\lambda v^2 \overline{v}_x - \frac{\lambda^2}{2} |v|^4 v, \\ v(0,x) = v_0(x), \end{gathered} \tag{1.2}$$

where $v_0(x) = e^{-i\lambda \int_{-\infty}^{x} |u_0(y)|^2 \, dy} u_0(x)$. The Cauchy problem (1.2) is interesting because of the derivative in the nonlinearity has been removed: $|u|^2 u_x$ in (1.1) has been replaced by the quintic nonlinearity $|v|^4 v$ in (1.2). The Strichartz estimate can control the nonlinearity $|v|^4 v$ easy (e.g., [9, 29]). In [25], it is proved that the contraction argument provides the local well-posedness, once the following estimate holds for some $b \in \mathbb{R}$,

$$\|uv\overline{w}_x\|_{s,b-1} \lesssim \|u\|_{s,b} \|v\|_{s,b} \|w\|_{s,b}. \tag{1.3}$$

In fact, whenever $s \geq \frac{1}{2}$, the estimate (1.3) holds, and then successfully this is relevant to the local well-posedness in $H^s$.

In this paper, we shall prove the global well-posedness for the Cauchy problem (1.2), as stated in the following theorem.

**Theorem 1.1** *Let $\frac{32}{33} < s < 1$ and let $b$ be a positive constant $b > \frac{1}{2}$ and close enough to $\frac{1}{2}$. We impose $\|v_0\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}$ for data $v_0 \in H^s$. Let $T > 0$, there exists a unique solution $v$ of (1.2) on the time interval $(-T, T)$ such that*

$$\psi_T v(t) \in C((-T,T) \,:\, H^s) \cap X_{s,b},$$

*where* $\psi_T$ *is a smooth time cut off function of* $\psi_T(t) = \psi(\frac{t}{T}),\ \psi \in C^\infty,\ \psi(t) = 1$ *for* $|t| \leq 1$ *and* $\psi(t) = 0$ *for* $|t| \geq 2$*. Moreover the solution* $v$ *satisfies*

$$v(t) - e^{it\partial_x^2} v_0 \in H^1.$$

As a consequence, the standard argument with the corresponding inverse transformation; $u(t,x) = e^{i\lambda \int_{-\infty}^{x} |v(t,y)|^2\,dy} v(t,x)$ [25] exhibits the global well-posedness result for the Cauchy problem (1.1).

**Theorem 1.2** *The Cauchy problem (1.1) is globally well-posed in* $H^s,\ s > \frac{32}{33}$*, assuming* $\|u_0\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}$*.*

One may expect the local solution to be global by making the iteration process of local well-posedness. But iteration method can not by itself yield the global well-posedness. Usually, the proof of global well-posedness relies on providing the a priori estimate of solution, besides the proof of the local well-posedness. We know that the conserved quantity is of use in the ingredient of the a priori estimate for the solution. The $H^1$ conservation low is, in actually employed to extend the local solutions at infinitely. If there was the conserved estimate for solutions in $H^s,\ \frac{1}{2} \leq s < 1$, we would immediately show the global well-posedness.

The proof of Theorem 1.1 was the argument due to Bourgain [3] (see also [6]), where the global well-posedness was shown for the two dimensional nonlinear Schrödinger equation in weaker spaces than the space needed by the conservation law directly. Let $S_t$ and $S(t)$ denote the nonlinear flow map and the linear flow map associated with the Cauchy problem of the nonlinear Schrödinger equation, respectively. We let $X$ and $Y$ be Banach spaces such that $X \subsetneq Y$, where the space $X$ is the conserved space of equations, while the space $Y$ is the initial data space. The strategy of [3] is that if

$$(S_t - S(t))u_0 \in X, \tag{1.4}$$

whereas $u \in Y$, we have the global well-posedness in $Y$. It is noted that $S_t u_0$ never belong to $X$ for $u_0 \notin X$. The statement (1.4) mentions that the nonlinear part is regular than data, where the proof estimates, roughly speaking, the high Sobolev norm of solution by low Sobolev norm, which aims to control the transportation of energy between the low frequency and the high frequency. Thus, this performance presents the a priori estimate of solution.

In ordinary way, we seek the solution to be the integral equation associated with (1.2)

$$v(t) = e^{it\partial_x^2} v_0 - \int_0^t e^{it(t-s)\partial_x^2} (\lambda v^2 \overline{v}_x + i\frac{\lambda^2}{2}|v|^4 v)(s)\,ds.$$

In (1.4), for $v(t) \in H^s$ we will show

$$\int_0^t e^{it(t-s)\partial_x^2} (\lambda v^2 \overline{v}_x + i\frac{\lambda^2}{2}|v|^4 v)(s) ds \in H^1. \tag{1.5}$$

The above estimate has to really recover more than one derivative loss, since the estimate (1.4) controls the $H^1$ norm for $v \in H^s$, and there is one space derivative in (1.5). This is quite different from the case of the nonlinear Schrödinger equation. In the present paper, we generalize the estimate (1.3) used in [25] to prove (1.5). Then we combine (1.5) with the argument of [3] to show Theorem 1.1.

**Remark 1.1** It is said in section 7 that the local well-posedness in $H^{1/2}$ is the sharp result. Note that the exponent $s > \frac{32}{33}$ of Theorem 1.1 is far from the above critical exponent. For more rough data, we do not consider here.

**Notation.** Throughout the paper we write $a \lesssim b$ (resp. $a \gtrsim b$) to denote $a \le cb$ (resp. $ca \ge b$) for some constant $c > 0$. We also write $a \sim b$ to denote both $a \lesssim b$ and $a \gtrsim b$. We denote $\widehat{f}$ as the Fourier transform of $f$ with respect to the time-space variables, while $\mathcal{F}^{-1}$ denotes the inverse operator of Fourier transformation in the time-space variables. Let $\mathcal{F}_x f$ denote the Fourier transform in $x$ of $f$. Let $\|f\|_{L_t^q L_x^p}$ (resp. $\|f\|_{L_x^p L_t^q}$) denote the mixed space-time norm as $\|f\|_{L_t^q L_x^p} = \big\|\|f\|_{L_x^p}\big\|_{L_t^q}$ (resp. $\|f\|_{L_x^p L_t^q} = \big\|\|f\|_{L_t^q}\big\|_{L_x^p}$). We denote $\|f\|_{L^p}$ as $\|f\|_{L^p} = \|f\|_{L_t^p L_x^p}$. The Riesz and the Bessel potentials of order $-s$ are denoted by $D^s = (-\partial^2)^{\frac{s}{2}}$ and $J^s = (1-\partial^2)^{\frac{s}{2}}$, respectively. We use notation $a\pm$ as $a \pm \epsilon$ for sufficiently small $\epsilon > 0$, respectively. We let $a_+ = \max\{a, 0\}$.

The rest of this paper is organized as follows. In section 2, we improve the estimates developed in [25]. In sections 3 and 4, we prove the estimates by results in section 2 to use in section 5. In section 5, we consider the evolution of the initial value problems with data restricted to low and high frequencies. In section 6, we show Theorem 1.1 by results in section 4 and section 5. In section 7, we show that the data-map fails in $H^s$ for $s < \frac{1}{2}$.

**Remark 1.2** We may relax the condition of the nonlinearity for the equation (1.2). More precisely, instead of (1.2), there seems a chance to show the result similar to Theorem 1.1 for the equation with more general nonlinearity. However, we shall not consider this problem in this paper for simplicity.

## 2 Preliminary estimates

We start this section by stating the variant Strichartz estimates.

**Lemma 2.1** *For $\frac{2}{q} = \frac{1}{2} - \frac{1}{p}$, $2 \le p \le \infty$, $b > \frac{1}{2}$, we have*

$$\|u\|_{L_t^q L_x^p} \lesssim \|u\|_{0,b}, \tag{2.1}$$

$$\|D_x^{1/2} u\|_{L_x^\infty L_t^2} \lesssim \|u\|_{0,b}, \tag{2.2}$$

$$\|u\|_{L_x^4 L_t^\infty} \lesssim \|D_x^{\frac{1}{4}} u\|_{0,b}. \tag{2.3}$$

**Proof.** We estimate (2.1) first, because the proof for (2.2) and (2.3) follows in the same way. We have the classical version of the Strichartz inequality for the Schrödinger equation:

$$\|e^{it\partial_x^2}u_0\|_{L_t^qL_x^p} \lesssim \|u_0\|_{L^2}, \tag{2.4}$$

for $u_0 \in L^2$, where $\frac{2}{q} = \frac{1}{2} - \frac{1}{p}$, $2 \le p \le \infty$ (e.g., [9, 29]). The estimate (2.1) follows from the argument of [14, Lemma 3.3], once we obtain (2.4). In a similar way to above, the following two estimates imply (2.2) and (2.3), respectively (e.g., [13] for the estimate in (2.5)):

$$\|D_x^{1/2}e^{it\partial_x^2}u_0\|_{L_x^\infty L_t^2} \lesssim \|u_0\|_{L^2}, \quad \|e^{it\partial_x^2}u_0\|_{L_x^4L_t^\infty} \lesssim \|D_x^{\frac{1}{4}}u_0\|_{L^2}. \tag{2.5}$$

$\square$

**Remark 2.1** It is noted $\|\overline{u}\|_{s,b} = |||u|||_{s,b}$ where $|||\cdot|||_{s,b}$ is defined as

$$|||u|||_{s,b} = \left(\iint_{\mathbb{R}^2} \langle\xi\rangle^{2s}\langle\tau-\xi^2\rangle^{2b}|\widehat{u}(\tau,\xi)|^2\,d\tau d\xi\right)^{1/2}.$$

Therefore, by (2.1), (2.2), (2.3), the following estimates hold for same numbers $q, p, b$ of Lemma 2.1

$$\|u\|_{L_t^qL_x^p} \lesssim |||u|||_{0,b}, \quad \|D_x^{1/2}u\|_{L_x^\infty L_t^2} \lesssim |||u|||_{0,b}, \quad \|u\|_{L_x^4L_t^\infty} \lesssim |||D_x^{\frac{1}{4}}u|||_{0,b}. \tag{2.6}$$

Let us introduce some variables

$$\sigma = \tau + \xi^2, \quad \sigma_1 = \tau_1 + \xi_1^2, \quad \sigma_2 = \tau_2 + \xi_2^2, \quad \sigma_3 = \tau - \xi_3^2.$$

We write $\int_*$ to denote the convolution integral $\int_{\substack{\sigma=\sigma_1+\sigma_2+\sigma_3 \\ \xi=\xi_1+\xi_2+\xi_3}} d\tau_1 d\tau_2 d\tau_3 d\xi_1 d\xi_2 d\xi_3$ throughout this paper. We assume that the functions $d$, $c_1$, $c_2$, $c_3$ are non negative functions on $\mathbb{R}^2$.

Using Lemma 2.1, we obtain the following lemma.

**Lemma 2.2** *For* $0 \le a \le 1-b$, $b' > \frac{1}{2}$, $b'-b > a-\frac{1}{2}$, $a - b' \le 0$, *we have*

$$\int_* \max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\sigma_3|\}^a \frac{d(\tau,\xi)}{\langle\sigma\rangle^{1-b}} \prod_{j=1}^3 \frac{c_j(\tau_j,\xi_j)}{\langle\sigma_j\rangle^{b'}} \lesssim \|d\|_{L^2}\prod_{j=1}^3\|c_j\|_{L^2}. \tag{2.7}$$

**Proof.** We estimate (2.7) by dividing the domain of integration into subcases. When $|\sigma|$ dominates in (2.7) which means that the $|\sigma|$ takes the maximum in (2.7), the Plancherel identity, (2.1) and (2.6) yield that the contribution of the above region to the left hand side of (2.7) is bounded by

$$\|\mathcal{F}^{-1}d\|_{L^2}\prod_{j=1}^3\|\mathcal{F}^{-1}(\langle\sigma_j\rangle^{-b'}c_j)\|_{L^6} \lesssim \|d\|_{L^2}\prod_{j=1}^3\|c_j\|_{L^2}.$$

In the other cases, if $\sigma_1$ or $\sigma_2$ or $\sigma_3$ dominates, we can assume that $|\sigma_1| \geq \max\{|\sigma_1|, |\sigma_2|, |\sigma_3|\}$ by symmetry. By $|\sigma_1|^a \leq |\sigma_1|^{b'}|\sigma|^{a-b'}$, by taking $\mathcal{F}^{-1}\frac{d}{\langle\sigma\rangle^{b'+1-a-b}}$, $\mathcal{F}^{-1}\frac{c_j}{\langle\sigma_j\rangle^{b'}}$, $j = 2, 3$, in $L^6$ and $\mathcal{F}^{-1}c_1$ in $L^2$, in a similar way, we deduce that the contribution of the above region to the left hand side of (2.7) is bounded by

$$\|\mathcal{F}^{-1}(\langle\sigma\rangle^{a-b'+b-1}d)\|_{L^6}\|\mathcal{F}^{-1}c_1\|_{L^2}\prod_{j=2}^{3}\|\mathcal{F}^{-1}(\langle\sigma_j\rangle^{-b'}c_j)\|_{L^6} \lesssim \|d\|_{L^2}\prod_{j=1}^{3}\|c_j\|_{L^2},$$

for $b' - b > a - \frac{1}{2}$. Then we have the desired estimate. $\square$

**Lemma 2.3** *Let us define*

$$\begin{aligned}
A_1 &= \{(\tau, \xi, \tau_1, \xi_1, \tau_2, \xi_2, \tau_3, \xi_3) | \max\{|\sigma|, |\sigma_3|\} \geq \max\{|\sigma_1|, |\sigma_2|\}\},\\
A_2 &= \{(\tau, \xi, \tau_1, \xi_1, \tau_2, \xi_2, \tau_3, \xi_3) | \max\{|\sigma_1|, |\sigma_2|\} > \max\{|\sigma|, |\sigma_3|\}\},\\
F_1(\xi, \xi_1, \xi_2, \xi_3) &= \frac{\min\{|\xi|, |\xi_3|\}^{1/2}}{|\xi_1|^{\frac{1}{4}}|\xi_2|^{\frac{1}{4}}},\\
F_2(\xi, \xi_1, \xi_2, \xi_3) &= \frac{\max\{|\xi|, |\xi_3|\}^{1/2}}{\min\{|\xi|, |\xi_3|\}^{\frac{1}{4}}\max\{|\xi_1|, |\xi_2|\}^{\frac{1}{4}}},\\
M(\tau, \xi, \tau_1, \xi_1, \tau_2, \xi_2, \tau_3, \xi_3) &= F_1(\xi, \xi_1, \xi_2, \xi_3)\chi_{A_1} + F_2(\xi, \xi_1, \xi_2, \xi_3)\chi_{A_2},
\end{aligned}$$

*where $\chi_{A_j}$, $j = 1, 2$ denote the characteristic function on $A_j$, $j = 1, 2$, respectively. Then for $b' > \frac{1}{2}$, $0 \leq a \leq 1 - b$, $b' - b > a - \frac{1}{2}$, $a - b' \leq 0$, we have*

$$\begin{aligned}
&\int_* M(\tau, \xi, \tau_1, \xi_1, \tau_2, \xi_2, \tau_3, \xi_3)\max\{|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|\}^a\frac{d(\tau, \xi)}{\langle\sigma\rangle^{1-b}}\prod_{j=1}^{3}\frac{c_j(\tau_j, \xi_j)}{\langle\sigma_j\rangle^{b'}}\\
&\lesssim \|d\|_{L^2}\prod_{j=1}^{3}\|c_j\|_{L^2}. \qquad (2.8)
\end{aligned}$$

**Proof.** First of all, we observe that when $|\sigma_i|$ dominates, it follows that $|\sigma_i|^a \leq |\sigma_i|^{b'}|\sigma|^{a-b'}$. If $|\sigma|$ or $|\sigma_3|$ dominates, namely $\chi_{A_2} = 0$, we take $\mathcal{F}^{-1}d$ in $L^2$ and $D_x^{1/2}\mathcal{F}^{-1}\frac{c_3}{\langle\sigma_3\rangle^{b'}}$ in $L_x^\infty L_t^2$, or $D_x^{1/2}\mathcal{F}^{-1}\frac{d}{\langle\sigma\rangle^{b'+1-a-b}}$ in $L_x^\infty L_t^2$ and $\mathcal{F}^{-1}c_3$ in $L^2$, respectively, and $D_x^{-\frac{1}{4}}\mathcal{F}^{-1}\frac{c_j}{\langle\sigma_j\rangle^{b'}}$, $j = 1, 2$, in $L_x^4L_t^\infty$, so that we have that the contribution of the above region to the left hand side of (2.8) is bounded by

$$\begin{aligned}
&\int_*\Big(d(\tau, \xi)\frac{|\xi_3|^{1/2}c_3(\tau_3, \xi_3)}{\langle\sigma_3\rangle^{b'}} + \frac{|\xi|^{1/2}d(\tau, \xi)}{\langle\sigma\rangle^{1-b-a+b'}}c_3(\tau_3, \xi_3)\Big)\prod_{j=1}^{2}\frac{c_j(\tau_j, \xi_j)}{|\xi_j|^{1/4}\langle\sigma_j\rangle^{b'}}\\
&\lesssim \Big(\|\mathcal{F}^{-1}d\|_{L^2}\|D_x^{1/2}\mathcal{F}^{-1}\frac{c_3}{\langle\sigma_3\rangle^{b'}}\|_{L_x^\infty L_t^2}\\
&\quad + \|D_x^{1/2}\mathcal{F}^{-1}\frac{d}{\langle\sigma\rangle^{b'+1-b-a}}\|_{L_x^\infty L_t^2}\|\mathcal{F}^{-1}c_3\|_{L^2}\Big)\prod_{j=1}^{2}\|D_x^{-\frac{1}{4}}\mathcal{F}^{-1}\frac{c_j}{\langle\sigma_j\rangle^{b'}}\|_{L_x^4L_t^\infty}
\end{aligned}$$

$$\lesssim \quad \|d\|_{L^2}\prod_{j=1}^{3}\|c_j\|_{L^2},$$

where we use (2.2), (2.3) and (2.6). In the case that $|\sigma_1|$ or $|\sigma_2|$ dominates, we use the estimates (2.2) and (2.3), as we just used. We can assume $|\sigma_1| \geq |\sigma_2|$ by symmetry. In a similar way to above, we arrive the estimate of (2.8) in

$$\begin{aligned}
&\int_* c_1(\tau_1,\xi_1)\frac{c_2(\tau_2,\xi_2)}{|\xi_2|^{\frac14}\langle\sigma_2\rangle^{b'}}\\
&\times\Big(\frac{|\xi|^{1/2}d(\tau,\xi)}{\langle\sigma\rangle^{1-b-a+b'}}\frac{c_3(\tau_3,\xi_3)}{|\xi_3|^{\frac14}\langle\sigma_3\rangle^{b'}}+\frac{d(\tau,\xi)}{|\xi|^{\frac14}\langle\sigma\rangle^{1-b-a+b'}}\frac{|\xi_3|^{1/2}c_3(\tau_3,\xi_3)}{\langle\sigma_3\rangle^{b'}}\Big)\\
&\quad\lesssim\quad\Big(\|D_x^{1/2}\mathcal{F}^{-1}\frac{d}{\langle\sigma\rangle^{b'+1-a-b}}\|_{L_x^\infty L_t^2}\|D_x^{-\frac14}\mathcal{F}^{-1}\frac{c_3}{\langle\sigma_3\rangle^{b'}}\|_{L_x^4L_t^\infty}\\
&\qquad\quad+\|D_x^{-\frac14}\mathcal{F}^{-1}\frac{d}{\langle\sigma\rangle^{b'+1-a-b}}\|_{L_x^4L_t^\infty}\|D_x^{1/2}\mathcal{F}^{-1}\frac{c_3}{\langle\sigma_3\rangle^{b'}}\|_{L_x^\infty L_t^2}\Big)\\
&\qquad\quad\times\|\mathcal{F}^{-1}c_1\|_{L^2}\|D_x^{-\frac14}\mathcal{F}^{-1}\frac{c_2}{\langle\sigma_2\rangle^{b'}}\|_{L_x^4L_t^\infty},
\end{aligned}$$

which is bounded by $c\|d\|_{L^2}\prod_{j=1}^{3}\|c_j\|_{L^2}$. This completes the proof. □

Let us introduce the operator $A(v_1,v_2)$ defined by

$$\mathcal{F}_xA(v_1,v_2)(\xi)=\int_{\xi=\xi_1+\xi_2}\chi_{|\xi_1|\geq|\xi_2|}\mathcal{F}_xv_1(\xi_1)\mathcal{F}_xv_2(\xi_2)d\xi_1,$$

which easily gives $v_1v_2 = A(v_1,v_2)+A(v_2,v_1)$.

**Lemma 2.4** *Let* $0\leq s\leq 1$, $\frac12<b\leq\frac58$, $b'>\frac12$. *Then*

$$\|A(v_1,v_2)(\overline{v}_3)_x\|_{s,b-1}\lesssim\sum\|v_1\|_{s_1,b'}\|v_2\|_{s_2,b'}\|v_3\|_{s_3,b'},\tag{2.9}$$

*where the summation is taken by choosing non-negative different numbers* $(s_1,s_2,s_3)$ *in the different cases (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), such that*

$$s_1+s_2+s_3\geq s+1,\tag{2.10}$$
$$s_1\geq(s+b-1)_+,\quad s_2\geq 0,\quad s_3\geq 0,\tag{2.11}$$
$$s_1\geq s+b-1+(1-s_3-\min\{s_2,1-b\})_+\tag{2.12}$$
$$s_1+s_3\geq\tfrac34,\quad s_2\geq 0,\tag{2.13}$$
$$s_1+s_3\geq b+(s-\min\{s_2,1-b\})_+,\tag{2.14}$$
$$s_3\geq s+2b-\tfrac54+\big(\tfrac14-s_1\big)_+,\quad s_2\geq 0.\tag{2.15}$$

**Remark 2.2** Lemma 2.4 includes estimate (1.3). Namely, to recover such an estimate, one should take $s=s_1=s_2=s_3\geq\frac12$, $\frac12<b=b'\leq\frac58$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15).

**Proof.** By duality and the Plancherel identity, it suffices to show that

$$\begin{aligned}&\int_* \chi_{|\xi_1|\geq|\xi_2|}\langle\xi\rangle^s|\xi_3|\frac{d(\tau,\xi)}{\langle\sigma\rangle^{1-b}}|\widehat{v}_1(\tau_1,\xi_1)||\widehat{v}_2(\tau_2,\xi_2)||\widehat{\overline{v}}_3(\tau_3,\xi_3)| \\ &\lesssim \|d\|_{L^2}\sum\prod_{j=1}^{3}\|v_j\|_{s_j,b'}, \end{aligned}\tag{2.16}$$

for $d\in L^2$, $d\geq 0$. One let

$$\begin{aligned} K(\xi,\xi_1,\xi_2,\xi_3) &= \chi_{|\xi_1|\geq|\xi_2|}\frac{\langle\xi\rangle^s|\xi_3|}{\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\langle\xi_3\rangle^{s_3}}, \\ c_j(\tau_j,\xi_j) &= \langle\xi_j\rangle^{s_j}\langle\sigma_j\rangle^{b'}\times\begin{cases}|\widehat{v}_j(\tau_j,\xi_j)|, & j=1,2,\\ |\widehat{\overline{v}}_j(\tau_j,\xi_j)|, & j=3.\end{cases}\end{aligned}$$

We easily see that $\|c_j\|_{L^2}=\|v_j\|_{s_j,b'}$. Then introduce the identity $\sigma-\sigma_1-\sigma_2-\sigma_3=2(\xi-\xi_1)(\xi-\xi_2)$ which implies that, at least, one factor among $|\sigma|,|\sigma_1|,|\sigma_2|,|\sigma_3|$ is bigger than $\frac{1}{2}|\xi-\xi_1||\xi-\xi_2|$, namely

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|,|\sigma_3|\}\geq\frac{1}{2}|\xi-\xi_1||\xi-\xi_2|. \tag{2.17}$$

Thus, in the case of $|\xi-\xi_1|>1$ and $|\xi-\xi_2|>1$, we make use of (2.17) similar to the KdV equation case [3, 14, 16].

We estimate (2.16) with the bounds $\|d\|_{L^2}\prod_{j=1}^3\|c_j\|_{L^2}$ for choosing a pair of non negative different numbers $(s_1,s_2,s_3)$, by separating the domain of integration into several subdomains. The different cases correspond to the different cases of (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), respectively.

**Case $|\xi|\leq 2$.** If $|\xi-\xi_1|\leq 1$ or $|\xi-\xi_2|\leq 1$, we easy see that $|\xi_3|\lesssim\max\{\langle\xi_1\rangle,\langle\xi_2\rangle\}$, then

$$K(\xi,\xi_1,\xi_2,\xi_3)\lesssim\frac{|\xi_3|^{1/2}}{|\xi_1|^{\frac14}|\xi_2|^{\frac14}\langle\xi_1\rangle^{s_1}\langle\xi_3\rangle^{s_3}}\langle\xi_3\rangle^{\frac34}\lesssim\frac{|\xi_3|^{1/2}}{|\xi_1|^{\frac14}|\xi_2|^{\frac14}},$$

for $s_1+s_3\geq\frac34$. Therefore, since by (2.13) and by the Plancherel identity, the contribution of the above region to (2.16) is bounded by

$$\|\mathcal{F}^{-1}d\|_{L^2}\|D_x^{1/2}\mathcal{F}^{-1}\frac{c_3}{\langle\sigma_3\rangle^{b'}}\|_{L_x^\infty L_t^2}\prod_{j=1}^{2}\|D_x^{-\frac14}\mathcal{F}^{-1}\frac{c_j}{\langle\sigma_j\rangle^{b'}}\|_{L_x^4L_t^\infty}\lesssim\|d\|_{L^2}\prod_{j=1}^{3}\|c_j\|_{L^2},$$

where we use (2.2), (2.3) and (2.6). In the subdomain of $|\xi-\xi_1|>1$ and $|\xi-\xi_2|>1$, it follows that $\langle\xi-\xi_1\rangle\sim\langle\xi_1\rangle,\langle\xi-\xi_2\rangle\sim\langle\xi_2\rangle$, then

$$\frac{K(\xi,\xi_1,\xi_2,\xi_3)}{\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}}\lesssim\frac{\langle\xi_3\rangle^{1-s_3}}{\langle\xi_1\rangle^{s_1+1-b}\langle\xi_2\rangle^{s_2+1-b}},$$

which is bounded by a constant since by (2.13). Hence we obtain that the contribution of this region to (2.16) is bounded by $c\|d\|_{L^2}\prod_{j=1}^3\|c_j\|_{L^2}$, since by Lemma 2.2 and (2.17).

**Case $|\xi| > 2$ and $|\xi_3| \le 2$.** When $|\xi - \xi_1| \le 1$ or $|\xi - \xi_2| \le 1$, we have that

$$K(\xi, \xi_1, \xi_2, \xi_3) \lesssim \frac{\max(|\xi_1|, |\xi_2|)^{1/2}}{\min(|\xi_1|, |\xi_2|)^{\frac{1}{4}} |\xi_3|^{\frac{1}{4}}} \frac{\max(|\xi_1|, |\xi_2|)^{s-\frac{1}{2}}}{\langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2}}.$$

Then, as in the case above, for $|\xi| < 2$, and $|\xi - \xi_1| \le 1$ or $|\xi - \xi_2| \le 1$, we have the contribution of this region to (2.16) is bounded by $c\|d\|_{L^2} \prod_{j=1}^{3} \|c_j\|_{L^2}$, since by $s_1 \ge (s + b - 1)_+ > s - \frac{1}{2}$ of (2.11).

In the domain of $|\xi - \xi_1| > 1$ and $|\xi - \xi_2| > 1$, we easy see that by $|\xi_3| < 2$, $\langle \xi - \xi_1 \rangle \sim \langle \xi_2 \rangle$, $\langle \xi - \xi_2 \rangle \sim \langle \xi_1 \rangle$ and

$$\frac{K(\xi, \xi_1, \xi_2, \xi_3)}{\langle \xi - \xi_1 \rangle^{1-b} \langle \xi - \xi_2 \rangle^{1-b}} \lesssim \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^{s_1+1-b} \langle \xi_2 \rangle^{s_2+1-b}} \le c,$$

provided $s_1 \ge s + b - 1$ of (2.11). The argument of the case $|\xi| < 2$, $|\xi - \xi_1| > 1$, $|\xi - \xi_2| > 1$ is applied to this case. Then by Lemma 2.2 and (2.17), we have that the contribution of the above region to (2.16) is bounded by $c\|d\|_{L^2} \prod_{j=1}^{3} \|c_j\|_{L^2}$.

**Case $|\xi| > 2$ and $|\xi_3| > 2$.** We separate the domain of integration into four subdomains:
**case a** $|\xi - \xi_1| \le 1$ or $|\xi - \xi_2| \le 1$,
**case b** $|\xi - \xi_1| > 1$, $|\xi - \xi_2| > 1$, $|\xi| \ll |\xi_3|$,
**case c** $|\xi - \xi_1| > 1$, $|\xi - \xi_2| > 1$, $|\xi| \sim |\xi_3|$,
**case d** $|\xi - \xi_1| > 1$, $|\xi - \xi_2| > 1$, $|\xi| \gg |\xi_3|$.
For the points of case a, it follows

$$K(\xi, \xi_1, \xi_2, \xi_3) \lesssim \frac{|\xi_1|^{1/2}}{|\xi_2|^{\frac{1}{4}} |\xi_3|^{\frac{1}{4}}} \times \begin{cases} \langle \xi_1 \rangle^{s-s_1-\frac{1}{2}} \langle \xi_2 \rangle^{\frac{3}{2}-s_2-s_3}, & \text{if } |\xi - \xi_1| \le 1, \\ \langle \xi_1 \rangle^{\frac{3}{4}-s_1-s_3} \langle \xi_2 \rangle^{s+\frac{1}{4}-s_2}, & \text{if } |\xi - \xi_2| \le 1, \end{cases}$$

which is bounded by $\frac{|\xi_1|^{1/2}}{|\xi_2|^{\frac{1}{4}} |\xi_3|^{\frac{1}{4}}}$ provided (2.10) or (2.12), and (2.10) or (2.13), respectively. Hence we use (2.2), (2.3) and (2.6) again, then we have the contribution of the case a to (2.16) is bounded,

$$\begin{aligned} &\|\mathcal{F}^{-1} d\|_{L^2} \|D_x^{1/2} \mathcal{F}^{-1} \frac{c_1}{\langle \sigma_1 \rangle^{b'}}\|_{L_x^\infty L_t^2} \prod_{j=2}^{3} \|D_x^{-\frac{1}{4}} \mathcal{F}^{-1} \frac{c_j}{\langle \sigma_j \rangle^{b'}}\|_{L_x^4 L_t^\infty} \\ \lesssim \;& \|d\|_{L^2} \prod_{j=1}^{3} \|c_j\|_{L^2}. \end{aligned}$$

In both cases b and d, we get

$$\max\{|\xi|, |\xi_3|\} \sim |\xi_1 + \xi_2| \lesssim \max\{|\xi_1|, |\xi_2|\} \lesssim \max\{|\xi - \xi_1|, |\xi - \xi_2|\}, \quad (2.18)$$
$$\min\{|\xi|, |\xi_3|\} \lesssim \max\{\min\{|\xi_1|, |\xi_2|\}, \min\{|\xi - \xi_1|, |\xi - \xi_2|\}\}, \quad (2.19)$$

which follows from the fact $\xi-\xi_3=\xi_1+\xi_2,\ (\xi-\xi_1)-(\xi-\xi_2)=\xi_2-\xi_1$. The conditions of (2.18) and (2.19) yield the bound

$$\frac{K(\xi,\xi_1,\xi_2,\xi_3)}{\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}}\lesssim\begin{cases}\langle\xi\rangle^{s-\min\{1-b,s_2\}}\langle\xi_3\rangle^{b-s_1-s_3}, & \text{case b},\\ \langle\xi_3\rangle^{1-s_3-\min\{1-b,s_2\}}\langle\xi\rangle^{s-s_1+b-1}, & \text{case d},\end{cases}$$

which is bounded by a constant, because by (2.14), (2.12), respectively. Thereby, applying Lemma 2.2 again with (2.17) to these cases, we obtain the desired estimate for cases b and d in an analogous argument to above.

In case c, we have the estimate either

$$\min\{|\xi_1|,|\xi_2|\}\gtrsim|\xi| \tag{2.20}$$

or

$$\min\{|\xi-\xi_1|,|\xi-\xi_2|\}\gtrsim|\xi|\gg\min\{|\xi_1|,|\xi_2|\}. \tag{2.21}$$

The condition (2.20) leads the bound of $K(\xi,\xi_1,\xi_2,\xi_3)$ by a constant provided (2.10). Then we use Lemma 2.2 again with (2.17) and we have the desired estimate. The proof is very similar to above, so that we omit the detail.

On the other hand, in the case of (2.21), $\frac{K(\xi,\xi_1,\xi_2,\xi_3)}{\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}}$ is bounded by

$$\begin{aligned}&\min\Big\{F_1(\xi,\xi_1,\xi_2,\xi_3)\frac{\langle\xi_1\rangle^{\frac14-s_1}\langle\xi_2\rangle^{\frac14-s_2}\langle\xi\rangle^{s-s_3+\frac12}}{\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}},\\ &F_2(\xi,\xi_1,\xi_2,\xi_3)\frac{\langle\xi\rangle^{s-s_3+\frac34}}{\langle\xi_1\rangle^{s_1-\frac14}\langle\xi_2\rangle^{s_2}\langle\xi-\xi_1\rangle^{1-b}\langle\xi-\xi_2\rangle^{1-b}}\Big\}\\ &\lesssim\ \min\{F_1(\xi,\xi_1,\xi_2,\xi_3),F_2(\xi,\xi_1,\xi_2,\xi_3)\}\le M(\tau,\xi,\tau_1,\xi_1,\tau_2,\xi_2,\tau_3,\xi_3),\end{aligned}$$

provided for (2.15). We apply Lemma 2.3 with (2.17) and obtain the desired estimate, because by (2.15). This completes the proof of Lemma 2.4. □

**Lemma 2.5** *Let $b,q_k^j,p_k^j,\ 1\le k,j\le5$, be such that $\frac12<b<\frac34,\ 4\le q_k^j\le\infty,\ 2\le p_k^j\le\infty$ for $1\le k,j\le5$ and $\sum_{k=1}^5\frac{1}{p_k^j}=2b-\frac12,\ \sum_{k=1}^5\frac{1}{q_k^j}=\frac32-b$ for $1\le j\le5$. Let $s,s_k^j,\ 1\le k,j\le5$ be such that $0\le s\le1,\ \sum_{k=1}^5 s_k^j=s+2b-1$ for $1\le j\le5$, and $s\le s_j^j<s+\frac{1}{p_j^j}$ if $p_j^j<\infty$, while $s_j^j=s$ if $p_j^j=\infty$ and $0\le s_k^j<\frac{1}{p_k^j}$ if $p_k^j<\infty$ for $k\ne j$, while $s_k^j=0$ if $p_k^j=\infty$ for $k\ne j$. Then the following estimate holds*

$$\|D_x^s(v_1v_2v_3v_4v_5)\|_{0,b-1}\lesssim\sum_{j=1}^5\prod_{k=1}^5\|v_k\|_{L_t^{q_k^j}\dot W_x^{s_k^j,p_k^j}}. \tag{2.22}$$

**Proof.** With Plancherel identity (c.f., Leibniz rule for fractional power), it suffices to show the case of $s=0$, namely we show the following inequality

$$\|v_1v_2v_3v_4v_5\|_{0,b-1}\lesssim\prod_{j=1}^5\|v_j\|_{L_t^{q_j}\dot W_x^{s_j,p_j}}, \tag{2.23}$$

for $2 \le p_j \le \infty,\ 4 \le q_j \le \infty,\ 0 \le s_j < \frac{1}{p_j}$ if $p_j < \infty$, while $s_j = 0$ if $p_j = \infty$ for $1 \le j \le 5$ such that $\sum_{j=1}^5 \frac{1}{p_j} = 2b - \frac{1}{2},\ \sum_{j=1}^5 \frac{1}{q_j} = \frac{3}{2} - b,\ \sum_{j=1}^5 s_j = 2b - 1$. In a similar way to [25, Lemma 3.4], the Hölder inequality, the Sobolev embedding theorem with respect to the time variable and Minkowski's inequality show that the left hand side of (2.23) is bounded by

$$\|v_1 v_2 v_3 v_4 v_5\|_{L_t^{\frac{1}{\frac{3}{2}-b}} L_x^2} \lesssim \prod_{j=1}^5 \|v_j\|_{L_t^{q_j} L_x^{\overline{p}_j}},$$

where $\frac{1}{\overline{p}_j} = \frac{1}{p_j} - s_j$, which is bounded by the right hand side of (2.23), since by Sobolev inequality.

# 3 Nonlinear estimates I

As a consequence of Lemma 2.4, we obtain the following lemma, which play a role for the proof of the local well-posedness.

**Lemma 3.1** *For $0 \le s \le 1,\ \frac{1}{2} < b \le \frac{5}{8},\ b' > \frac{1}{2}$, we have*

$$\|v^2 \overline{v}_x\|_{0,b-1} \lesssim \|v\|_{0,b'}^2 \|v\|_{1,b'}, \tag{3.1}$$

$$\|(v^2 \overline{v}_x)_x\|_{0,b-1} \lesssim \|v\|_{0,b'} \|v\|_{1,b'}^2, \tag{3.2}$$

$$\|vw\overline{v}_x\|_{s,b-1} \lesssim \|v\|_{0,b'} \|v\|_{1,b'} \|w\|_{s,b'} + \|v\|_{s,b'} \|v\|_{b,b'} \|w\|_{0,b'}, \tag{3.3}$$

$$\|w^2 \overline{v}_x\|_{s,b-1} \lesssim \|v\|_{1,b'} \|w\|_{0,b'} \|w\|_{s,b'}, \tag{3.4}$$

$$\|v^2 \overline{w}_x\|_{s,b-1} \lesssim \|v\|_{0,b'} \|v\|_{1,b'} \|w\|_{s,b'}, \tag{3.5}$$

$$\|vw\overline{w}_x\|_{s,b-1} \lesssim \|v\|_{1,b'} \|w\|_{\max\{\frac{1}{2},(\frac{3}{4}-s)_+\},b'} \|w\|_{s,b'}, \tag{3.6}$$

$$\|w^2 \overline{w}_x\|_{s,b-1} \lesssim \|w\|_{\frac{1}{2},b'}^2 \|w\|_{s,b'}. \tag{3.7}$$

**Proof.** We use Lemma 2.4 by taking different variables $(s_1, s_2, s_3)$ corresponding to the different cases in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), respectively.

For (3.1), we choose the numbers $s = s_1 = s_2 = 0,\ s_3 = 1$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), respectively. Then by $v^2 = 2A(v,v)$, we have the desired estimate.

For (3.2), in a similar way to above, we take $s = s_1 = s_3 = 1,\ s_2 = 0$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), which shows the estimate (3.2).

For (3.3), we first note that $vw = A(v,w) + A(w,v)$, then

$$\|vw\overline{v}_x\|_{s,b-1} \le \|A(v,w)\overline{v}_x\|_{s,b-1} + \|A(w,v)\overline{v}_x\|_{s,b-1}. \tag{3.8}$$

For the treatment of first term of (3.8), we take $s = s_2,\ s_1 = 1,\ s_3 = 0$ in (2.10), (2.11), (2.13). In (2.12), (2.14), (2.15), we put $s = s_3,\ s_1 = b,\ s_2 = 0$. Such a choice shows

$$\|A(v,w)\overline{v}_x\|_{s,b-1} \lesssim \|v\|_{0,b'} \|v\|_{1,b'} \|w\|_{s,b'} + \|v\|_{s,b'} \|v\|_{b,b'} \|w\|_{0,b'}. \tag{3.9}$$

On the other hand, for the second term of (3.8), we choose $s = s_1,\ s_2 = 0,\ s_3 = 1$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), which yields

$$\|A(w,v)\overline{v}_x\|_{s,b-1} \lesssim \|v\|_{0,b'}\|v\|_{1,b'}\|w\|_{s,b'}. \tag{3.10}$$

As a consequence, the estimate (3.3) follows immediately from (3.9) and (3.10).

For (3.4), we put $s = s_1,\ s_2 = 0,\ s_3 = 1$ in Lemma 2.4. Then in a similar way to above we have (3.4).

For (3.5), $s_1 = 1,\ s_2 = 0,\ s_3 = s$ are taken in Lemma 2.4. We omit the detail because the proof is very similar to above.

For (3.6), we follow the same argument as the proof of (3.3). We take $s_1 = 1,\ s_2 = \frac{1}{2},\ s_3 = s$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) for the treatment of $\|A(v,w)\overline{w}_x\|_{s,b-1}$, which yields

$$\|A(v,w)\overline{w}_x\|_{s,b-1} \lesssim \|v\|_{1,b'}\|w\|_{\frac{1}{2},b'}\|w\|_{s,b'}. \tag{3.11}$$

For $\|A(w,v)\overline{w}_x\|_{s,b-1}$, we shall take $s = s_1,\ s_2 = 1,\ s_3 = \frac{1}{2}$ in (2.10), (2.11), (2.12), and we take $s_1 = \frac{1}{2},\ s_2 = 1,\ s_3 = s$ in (2.15). In (2.13), (2.14), we put $s_1 = s,\ s_2 = 1,\ s_3 = \frac{1}{2}$ if $s \geq 1-b$, while $s_1 = (\frac{3}{4}-s)_+,\ s_2 = 1,\ s_3 = s$ if $s < 1-b$. Such a choice shows

$$\|A(w,v)\overline{w}_x\|_{s,b-1} \lesssim \|v\|_{1,b'}\|w\|_{\frac{1}{2},b'}\|w\|_{s,b'} + \|v\|_{1,b'}\|w\|_{(\frac{3}{4}-s)_+,b'}\|w\|_{s,b'}. \tag{3.12}$$

The estimates (3.11) and (3.12) give (3.6).

For (3.7), we choose $s_1, s_2, s_3$ as follows; $s_1 = s,\ s_2 = s_3 = \frac{1}{2}$ in (2.10), (2.11), (2.12), and $s_1 = s_3 = \frac{1}{2},\ s_2 = s$ in (2.13), and $s_1 = s_2 = \frac{1}{2},\ s_3 = s$ in (2.15). In (2.14), we choose $s_1 = s_3 = \frac{1}{2},\ s_2 = s$ if $s \leq 1-b$, while $s_1 = s,\ s_2 = s_3 = \frac{1}{2}$ if $s > 1-b$. Such a choice shows (3.7).

This completes the proof of Lemma 3.1. □

# 4 Nonlinear estimates II

In this section, we prove the estimates needed for the proof of Theorem 1.1. In section 6, the following lemma is used to show (1.5).

**Lemma 4.1** *Let $b > \frac{1}{2}$ be close enough to $\frac{1}{2}$. For $b' > \frac{1}{2}$, we have*

$$\|(vw\overline{v}_x)_x\|_{0,b-1} \lesssim \|v\|_{1,b'}^2\|w\|_{\frac{1}{2}+,b'}, \tag{4.1}$$

$$\|(w^2\overline{v}_x)_x\|_{0,b-1} \lesssim \|v\|_{1,b'}\|w\|_{\frac{1}{2}+,b'}^2, \tag{4.2}$$

$$\|(v^2\overline{w}_x)_x\|_{0,b-1} \lesssim \|v\|_{1,b'}^2\|w\|_{0+,b'} + \|v\|_{0,b'}\|v\|_{\frac{1}{4},b'}\|w\|_{\frac{3}{4}+,b'}, \tag{4.3}$$

$$\|(vw\overline{w}_x)_x\|_{0,b-1} \lesssim \|v\|_{1,b'}\|w\|_{\frac{1}{2}+,b'}\|w\|_{\frac{3}{4}+,b'}, \tag{4.4}$$

$$\|(w^2\overline{w}_x)_x\|_{0,b-1} \lesssim \|w\|_{\frac{1}{2},b'}^2\|w\|_{\frac{3}{4}+,b'}. \tag{4.5}$$

**Proof.** We use Lemma 2.4 again for choosing $s = 1$, $b > \frac{1}{2}$ and let $b$ be close enough to $\frac{1}{2}$.

For (4.1), in a similar way to the proof of Lemma 3.1, we have

$$\|(vw\overline{v}_x)_x\|_{0,b-1} \le \|(A(v,w)\overline{v}_x)_x\|_{0,b-1} + \|(A(w,v)\overline{v}_x)_x\|_{0,b-1}. \tag{4.6}$$

We take $s = s_1 = s_3 = 1,\ s_2 = \frac{1}{2}+$ for the first term of (4.6), and $s = s_2 = s_3 = 1,\ s_1 = \frac{1}{2}+$ for the second term of (4.6), respectively, in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), which yields the result.

For (4.2), we put $s = s_3 = 1,\ s_1 = s_2 = \frac{1}{2}+$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15). Then we have the desired estimate.

For (4.3), $s = s_1 = s_2 = 1,\ s_3 = 0+$ are taken in (2.10), (2.11), (2.12), (2.13), (2.14) which yields the estimate $\|v\|_{1,b'}^2\|w\|_{0+,b'}$. In (2.15), we put $s = 1,\ s_1 = \frac{1}{4},\ s_2 = 0,\ s_3 = \frac{3}{4}+$, which yields the estimate $\|v\|_{0,b'}\|v\|_{\frac{1}{4},b'}\|w\|_{\frac{3}{4}+,b'}$. The combination of these estimates implies (4.3).

For (4.4), we choose $s = s_1 = 1,\ s_2 = \frac{1}{2}+,\ s_3 = \frac{3}{4}+$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), which yields

$$\|(A(v,w)\overline{w}_x)_x\|_{0,b-1} \lesssim \|v\|_{1,b'}\|w\|_{\frac{1}{2}+,b'}\|w\|_{\frac{3}{4}+,b'}. \tag{4.7}$$

Similarly, we have

$$\|(A(w,v)\overline{w}_x)_x\|_{0,b-1} \lesssim \|v\|_{1,b'}\|w\|_{\frac{1}{2}+,b'}\|w\|_{\frac{3}{4}+,b'}, \tag{4.8}$$

by choosing $s = s_2 = 1,\ s_1 = \frac{1}{2}+,\ s_3 = \frac{3}{4}+$. The estimate (4.4) is obtained by (4.7) and (4.8).

For (4.5), we put $s = 1,\ s_1 = s_2 = \frac{1}{2}+,\ s_3 = \frac{3}{4}+$ in (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) to obtain (4.4).

This completes the proof of Lemma 4.1. □

# 5 Preliminaries for the proof of Theorem 1.1

In this section we summarize some results needed for the proof of Theorem 1.1. We begin by stating some facts of the space $X_{s,b}$.

**Lemma 5.1** *For $s \in \mathbb{R},\ \frac{1}{2} < b < b' < 1,\ 0 < \delta < 1$, we have*

$$\|\psi_\delta(t)e^{it\partial_x^2}u_0\|_{s,b} \lesssim \delta^{\frac{1}{2}-b}\|u_0\|_{H^s}, \tag{5.1}$$

$$\|\psi_\delta(t)\int_0^t e^{i(t-s)\partial_x^2}f(s)\,ds\|_{s,b} \lesssim \delta^{\frac{1}{2}-b}\|f\|_{s,b-1}, \tag{5.2}$$

$$\|\psi(t)\int_0^t e^{i(t-s)\partial_x^2}f(s)\,ds\|_{L_t^\infty H_x^s} \lesssim \|f\|_{s,b-1}, \tag{5.3}$$

$$\|\psi_\delta(t)f\|_{s,b-1} \lesssim \delta^{b'-b}\|f\|_{s,b'-1}. \tag{5.4}$$

For the proof of Lemma 5.1, see [14, 16].

Let us consider the following Cauchy problem

$$iv_t + v_{xx} = -i\lambda v^2\overline{v}_x - \frac{\lambda^2}{2}|v|^4v, \quad v(0) = v_0 \in H^1. \tag{5.5}$$

**Lemma 5.2** *Let $v_0 \in H^1$. For $T_0 > 0$ such that $T_0^{\frac{1}{8}-}(1 + \|v_0\|_{H^1}) \ll 1$, there exists a unique solution $v$ of the Cauchy problem (5.5) in the time interval $[-T_0, T_0]$ such that*

$$\psi_{T_0} v \in C([-T_0, T_0] \,:\, H^1) \cap X_{1,\frac{1}{2}+}.$$

*Moreover we have*

$$\|\psi_{T_0} v\|_{0,\frac{1}{2}+} \lesssim T_0^{0-}\|v_0\|_{L^2}, \quad \|\psi_{T_0} v_x\|_{0,\frac{1}{2}+} \lesssim T_0^{0-}\|v_0\|_{H^1}.$$

**Proof.** For $b > \frac{1}{2}$, we define

$$\mathcal{B}_1 = \left\{ v \in X_{1,b} \mid T_0^{b-\frac{1}{2}}\|v\|_{0,b} \le 2c\|v_0\|_{L^2}, \quad T_0^{b-\frac{1}{2}}\|v_x\|_{0,b} \le 2c\|v_0\|_{H^1} \right\},$$
$$\Phi(v)(t) = \psi_{T_0}(t) e^{it\partial_x^2} v_0 + \psi_{T_0}(t) \textstyle\int_0^t e^{i(t-s)\partial_x^2} N_1(v)(s)\, ds,$$

where the constant $c$ is taken large enough relatively to $\|v_0\|_{H^1}$, and $N_1(v)$ is the nonlinear term of (5.5) defined as follows:

$$N_1(v) = -\lambda v^2 \overline{v}_x + i\frac{\lambda^2}{2}|v|^4 v.$$

We look for the solution of the integral equation $v = \Phi(v)$. We choose $b$ close enough to $\frac{1}{2}$. By Lemma 2.5, (2.1) and (2.6), we have

$$\||v|^4 v\|_{\gamma,b'-1} \lesssim \|v\|_{\gamma,b}\|v\|_{2b'-1,b}\|v\|_{0,b}^3, \tag{5.6}$$

for $\frac{1}{2} < b' < \frac{3}{4}$ and $0 \le \gamma \le 1$, where we choose numbers $q_k^j, p_k^j, s_k^j$ in Lemma 2.5 such that $s_j^j = \gamma$, $s_{k_0}^j = 2b' - 1$, $p_j^j = \infty$, $q_j^j = 4$, $\frac{1}{p_{k_0}^j} = 2b' - 1+$, $\frac{2}{q_{k_0}^j} = \frac{1}{2} - \frac{1}{p_{k_0}^j}$ for choosing some $k_0$ among $k \ne j$, and $s_k^j = 0$, $\frac{3}{p_k^j} = 2b' - \frac{1}{2} - \frac{1}{p_{k_0}^j}$, $\frac{2}{q_k^j} = \frac{1}{2} - \frac{1}{p_k^j}$ for $k \ne j, k_0$. We only deal with the homogeneous spaces in Lemma 2.5. However, again, applying the similar proof of Lemma 2.5 to the inhomogeneous space, we can justify (5.6), since by the trivial embedding; $\|u\|_{s_1,b_1} \le \|u\|_{s_2,b_2}$ for $s_1 \le s_2$, $b_1 \le b_2$.

Therefore, by (3.1), (5.1), (5.2), (5.4) and (5.6), we have

$$\begin{aligned} & T_0^{b-\frac{1}{2}}\|\Phi(v)\|_{0,b-1} \\ \lesssim\ & \|v_0\|_{L^2} + T_0^{\frac{5}{8}-b}\|N_1(v)\|_{0,\frac{5}{8}-1} \\ \lesssim\ & \|v_0\|_{L^2} + T_0^{\frac{1}{8}-}\|v\|_{0,b}^2\|v\|_{1,b} + T_0^{\frac{1}{8}-}\|v\|_{0,b}^4\|v\|_{1,b} \le 2c\|v_0\|_{L^2}, \end{aligned}$$

for $v \in \mathcal{B}_1$, since by $T_0^{b-\frac{1}{2}}\|v\|_{0,b} \le 2c\|v_0\|_{L^2}$ and $T_0^{\frac{1}{8}-}\|v\|_{1,b} \ll 1$, where we put $b' = \frac{5}{8}$ in (5.4). Similarly, for $v \in \mathcal{B}_1$, by (3.2), (5.1), (5.2), (5.4) and (5.6), we have

$$T_0^{b-\frac{1}{2}}\|\Phi(v)_x\|_{0,b-1} \lesssim \|v_0\|_{\dot{H}^1} + T_0^{\frac{1}{8}-}\|v\|_{0,b}\|v\|_{1,b}^2 + T_0^{\frac{1}{8}-}\|v\|_{0,b}^3\|v\|_{1,b}^2 \le 2c\|v_0\|_{H^1}.$$

In an analogous way to above, we obtain

$$\|\Phi(v_1)-\Phi(v_2)\|_{1,b}\le \frac{1}{2}\|v_1-v_2\|_{1,b},$$

for $v_1, v_2 \in \mathcal{B}_1$. Then we conclude that $\Phi$ is a contraction map. Thus we obtain the unique local existence result in $X_{1,b}$ of (5.5) by the contraction argument. This completes the proof. $\square$

Let us assume $\frac{3}{4}\le s<1$. For $v$ obtained in Lemma 5.2, let us consider the following Cauchy problem with the variable coefficient depending on $v$

$$iw_t+w_{xx}=iN_2(v,w),\quad w(0)=w_0\in H^s, \tag{5.7}$$

where the nonlinearity $N_2(v,w)$ is defined by

$$N_2(v,w)=-\lambda\sum_{\substack{v_1,v_2,v_3=v\ or\ w\\(v_1,v_2,v_3)\neq(v,v,v)}} v_1v_2(\overline{v}_3)_x+i\frac{\lambda^2}{2}\sum_{\substack{v_1,v_2,v_3,v_4,v_5=v\ or\ w\\(v_1,v_2,v_3,v_4,v_5)\neq(v,v,v,v,v)}} v_1\overline{v}_2v_3\overline{v}_4v_5, \tag{5.8}$$

which is equivalent to $N_2(v,w)=N_1(v+w)-N_1(v)$. We put the first term and the second term of the right hand side of (5.8) in $N_{21}(v,w)$ and $N_{22}(v,w)$, respectively.

Let $N\gg 1$ be a large number. We next show the well-posedness of the Cauchy problem (5.7).

**Lemma 5.3** *Let $\frac{3}{4}\le s<1$. Assume that $\|w_0\|_{H^\gamma}\le cN^{\gamma-s}\|w_0\|_{H^s}$ for $0\le\gamma\le s$. Moreover assume $\|v_0\|_{\dot H^1}\le cN^{1-s}$. There exists a unique solution $w$ of the Cauchy problem (5.7) in the time interval $[-T_0,T_0]$ such that*

$$\psi_{T_0}w\in C([-T_0,T_0]\,:\,H^s)\cap X_{s,\frac12+},$$

*where $T_0$ is the same as in Lemma 5.2. Moreover the solution $w$ satisfies*

$$\|\psi_{T_0}w\|_{\gamma,\frac12+}\le cN^{\gamma-s+}. \tag{5.9}$$

**Proof.** Let $b>\frac12$ be close enough to $\frac12$. We define

$$\mathcal{B}_2=\left\{w\in X_{s,b}\,:\,T_0^{b-\frac12}\|w\|_{0,b}\le 2cN^{-s}\|w_0\|_{H^s},\ T_0^{b-\frac12}\|w\|_{s,b}\le 2c\|w_0\|_{H^s}\right\},$$

$$\Psi(w)(t)=\psi_{T_0}(t)e^{it\partial_x^2}w_0+\psi_{T_0}(t)\int_0^t e^{i(t-s)\partial_x^2}N_2(v,w)(s)\,ds, \tag{5.10}$$

where the constant $c$ is taken large enough relatively to $\|w_0\|_{H^s}$. In a similar way to the proof of Lemma 5.2, by Lemma 2.5, (2.1), we have

$$\|N_{22}(v,w)\|_{\gamma',b'-1}\lesssim(\|v\|_{0,b}+\|w\|_{0,b})^3(\|v\|_{1,b}+\|w\|_{\frac12,b})\|w\|_{\gamma',b}, \tag{5.11}$$

for $\frac12<b'<\frac34$ and $0\le\gamma'\le s$, where we choose $s_j^j=\gamma'$ in the case for taking the summation with respect to $v_j=w$ in Lemma 2.5, where $p_j^j=\infty$, $q_j^j=$

4, $s^j_{k_0}=2b'-1$, $\frac{1}{p^j_{k_0}}=2b'-1+$, $\frac{2}{q^j_{k_0}}=\frac12-\frac{1}{p^j_{k_0}}$ for choosing some $k_0$ among $k\neq j$, and $s^j_k=0$, $\frac{3}{p^j_k}=2b'-\frac12-\frac{1}{p^j_{k_0}}$, $\frac{2}{q^j_k}=\frac12-\frac{1}{p^j_k}$ for $k\neq k_0,j$, while $s^j_j=\min\{1,\gamma'+2b'-1\}$ in the case of $v_j=v$ where we take in Lemma 2.5 as follows; $\frac{1}{p^j_j}=2b'-1+$, $\frac{2}{q^j_j}=\frac12-\frac{1}{p^j_j}$, $s^j_{k_0}=(\gamma'+2b'-2)_+$, $p^j_{k_0}=2+$, $\frac{2}{q^j_{k_0}}=\frac12-\frac{1}{p^j_{k_0}}$ for choosing some $k_0$ among $k\neq j$ of $v_k=w$, and $s^j_k=0$, $\frac{1}{p^j_k}=\frac13\left(2b'-\frac12-\frac{1}{p^j_j}-\frac{1}{p^j_{k_0}}\right)\in(0,\frac12)$, $\frac{2}{q^j_k}=\frac12-\frac{1}{p^j_k}$ for $k\neq k_0,j$. We remark that by interpolation,

$$T_0^{\frac18-}\|v\|_{\gamma',b}\|v\|_{\frac58,b}\lesssim T_0^{\frac18-}\|v\|_{1,b}^{\gamma'+\frac58}\|v\|_{0,b}^{\frac{11}{8}-\gamma'}\lesssim N^{\gamma'-}, \tag{5.12}$$

for $w\in\mathcal{B}_2$ and $0\le\gamma'\le s$, because by $T_0^{\frac18-}\|v\|_{1,b}^{\gamma'+\frac58}\lesssim\|v\|_{1,b}^{\gamma'-\frac38}\lesssim N^{(1-s)(\gamma'-\frac38)}\lesssim N^{\gamma'-}$ and $\|v\|_{0,b}\lesssim 1$ by Lemma 5.2.

Then by (3.3), (3.4), (3.5), (3.6), (3.7), (5.1), (5.2), (5.4), (5.11), we obtain the estimates

$$\begin{aligned}T_0^{b-\frac12}\|\Psi(w)\|_{0,b} &\lesssim \|w_0\|_{L^2}+T_0^{\frac18-}(\|v\|_{0,b}+\|w\|_{\frac12,b})(\|v\|_{1,b}+\|w\|_{\frac12,b})\|w\|_{0,b}\\ &\quad +T_0^{\frac18-}\|v\|_{1,b}\|w\|_{\frac34,b}\|w\|_{0,b}\\ &\quad +T_0^{\frac18-}(\|v\|_{0,b}+\|w\|_{0,b})^3(\|v\|_{1,b}+\|w\|_{\frac12,b})\|w\|_{0,b}\\ &\le 2cN^{-s}\|w_0\|_{H^s},\end{aligned}$$

and

$$\begin{aligned}&T_0^{b-\frac12}\|\Psi(w)\|_{s,b}\\ &\lesssim \|w_0\|_{H^s}+T_0^{\frac18-}(\|v\|_{0,b}+\|w\|_{\frac12,b})(\|v\|_{1,b}+\|w\|_{\frac12,b})\|w\|_{s,b}\\ &\quad +T_0^{\frac18-}\|v\|_{s,b}\|v\|_{\frac58,b}\|w\|_{0,b}+T_0^{\frac18-}(\|v\|_{0,b}+\|w\|_{0,b})^3(\|v\|_{1,b}+\|w\|_{\frac12,b})\|w\|_{s,b}\\ &\le 2c\|w_0\|_{H^s},\end{aligned}$$

for all $w\in\mathcal{B}_2$, where one uses in (5.4) with $b'=\frac58$ to make the appearance $T_0^{\frac18-}$, and we use (5.12) to estimate $T_0^{\frac18-}\|v\|_{s,b}\|v\|_{\frac58,b}\|w\|_{0,b}$. Similarly, it follows

$$\begin{aligned}&(N^s\|\Psi(w_1)-\Psi(w_2)\|_{0,b}+\|\Psi(w_1)-\Psi(w_2)\|_{s,b})\\ &\quad\le\frac12\left(N^s\|w_1-w_2\|_{0,b}+\|w_1-w_2\|_{s,b}\right),\end{aligned}$$

for $w_1,w_2\in\mathcal{B}_2$. Therefore, by the Picard's iteration argument, we obtain the unique local existence result in $X_{s,b}$.

We now turn to show $T_0^{b-\frac12}\|w\|_{\gamma,b}\le cN^{\gamma-s}$ for $0\le\gamma\le s$. By Lemma 2.5, (5.10) of $w=\Psi(w)$, (3.3), (3.4), (3.5), (3.6), (3.7), (5.11), we have

$$T_0^{b-\frac12}\|w\|_{\gamma,b}\lesssim\|w_0\|_{H^\gamma}+T_0^{\frac18-}(\|v\|_{0,b}+\|w\|_{\frac12,b})(\|v\|_{1,b}+\|w\|_{\frac34,b})\|w\|_{\gamma,b}$$

$$
\begin{aligned}
& +T_0^{\frac{1}{8}-}\|v\|_{1,b}\|w\|_{\frac{3}{4},b}\|w\|_{\gamma,b}+T_0^{\frac{1}{8}-}\|v\|_{\gamma,b}\|v\|_{\frac{5}{8},b}\|w\|_{0,b} \\
& +T_0^{\frac{1}{8}-}(\|v\|_{0,b}+\|w\|_{0,b})^3(\|v\|_{1,b}+\|w\|_{\frac{1}{2},b})\|w\|_{\gamma,b} \\
\lesssim\ & \|w_0\|_{H^\gamma}+T_0^{0+}\|w\|_{\gamma,b}+T_0^{\frac{1}{8}-}\|v\|_{\gamma,b}\|v\|_{\frac{5}{8},b}\|w\|_{0,b} \\
\lesssim\ & \|w_0\|_{H^\gamma}+T_0^{0+}\|w\|_{\gamma,b}+N^{\gamma-s},
\end{aligned}
$$

which shows (5.9), where we use $(\frac{3}{4}-\gamma)_+ \leq \frac{3}{4}$ in (3.6), and (5.12) for the treatment of $T_0^{\frac{1}{8}-}\|v\|_{\gamma,b}\|v\|_{\frac{5}{8},b}\|w\|_{0,b}$. □

We note that $\psi_{T_0}v$ and $\psi_{T_0}w$ given by Lemmas 5.2 and 5.3 satisfy the integral equations corresponding to (5.5) and (5.7), respectively. For simplicity, we abbreviate $\psi_{T_0}v$ and $\psi_{T_0}w$ to $v$ and $w$, respectively, hereafter.

# 6 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by using the results in sections 4 and 5. Let us consider the following Cauchy problem

$$
iu_t+u_{xx}=-i\lambda u^2\overline{u}_x-\frac{\lambda^2}{2}|u|^4u, \quad u(0)=u_0\in H^s. \tag{6.1}
$$

We break data $u_0$ into two pieces; low frequency and high frequency, as follows; $u_0=v_0+w_0$,

$$
\mathcal{F}_xv_0(\xi)=\mathcal{F}_xu_0(\xi)\big|_{|\xi|\leq N}, \quad \mathcal{F}_xw_0(\xi)=\mathcal{F}_xu_0(\xi)\big|_{|\xi|>N}, \tag{6.2}
$$

where $N\gg 1$ is to be determined later. We try to solve the Cauchy problems (5.5) and (5.7) for data $v_0$ and $w_0$, respectively. If this is accomplished, we solve the Cauchy problem (6.1), since the solution $u$ of (6.1) is written as $u(t)=v(t)+w(t)$. From Lemmas 5.2 and 5.3, the Cauchy problems (5.5) and (5.7) are locally well-posed in $H^1$ and $H^s$, respectively, on the time interval $[-T_0,T_0]$ if $T_0^{\frac{1}{8}-}\|v_0\|_{H^1}\ll 1$. The definition (6.2) implies $\|v_0\|_{\dot{H}^1}\lesssim N^{1-s}$. We can put $T_0\sim N^{8(s-1)-}$. Without loss of generality, it is sufficient to treat the nonnegative time, because the case of $t<0$ is similar.

In order to extend the existence time of (6.1) up to any time $T>0$, we make the iteration scheme as follows; At the time $t=T_0$, we choose $v_1$ and $w_1$ as the corresponding $v_0$ and $w_0$, respectively,

$$
v_1=v(T_0)+\nu(T_0), \quad w_1=e^{iT_0\partial_x^2}w_0,
$$

where

$$
\nu(t)=\int_0^t e^{i(t-s)\partial_x^2}N_2(v,w)\,ds.
$$

We want to continue the above process of $T/T_0$ steps to obtain $v_{T/T_0}$ and $w_{T/T_0}$ with the Cauchy problems (5.5) and (5.7), which concludes the proof of Theorem 1.1. To show this, we have only to ensure $v_j\in H^1$ and $w_j\in H^s$ on each step.

The unitarity of $e^{it\partial_x^2}$ shows $w_j \in H^s$ and $\|w_j\|_{H^s} = \|w_0\|_{H^s}$. Then if the growth order of $v_j$ is the same as $v_0$ on each step, namely, $\|v_j\|_{L^2} \sim \|v_0\|_{L^2} \sim 1$ and $\|v_j\|_{\dot H^1} \sim \|v_0\|_{\dot H^1} \lesssim N^{1-s}$ for $1 \le j \le T/T_0$, we have the solution of (6.1) until $t = T$, since by the argument of section 5. Then we shall check it.

**Lemma 6.1** *Assume $\frac{6}{7} < s < 1$. Let $v, w$ be given as above. We have*

$$\|N_2(v,w)_x\|_{0,-\frac12+} \lesssim N^{1-\frac{5s}{4}+}. \tag{6.3}$$

**Proof.** We use Lemma 4.1 for the estimate of trilinear terms $N_{21}(v,w)$. By (4.1), (4.2), (4.3), (4.4), (4.5), respectively, we have

$$\begin{aligned}\|N_{21}(v,w)_x\|_{0,-\frac12+} &\lesssim N^{2(1-s)+\frac12-s+} + N^{1-s+2(\frac12-s)+} + N^{2(1-s)-s+} \\ &\quad + N^{\frac{1-s}{4}+\frac34-s+} + N^{1-s+\frac12-s+\frac34-s+} + N^{2(\frac12-s)+\frac34-s+} \\ &\lesssim N^{\frac{1-s}{4}+\frac34-s+} = N^{1-\frac{5s}{4}+},\end{aligned}$$

where $\|v\|_{\gamma_1,\frac12+} \lesssim N^{\gamma_1(1-s)+}$ and $\|w\|_{\gamma_2,b} \lesssim N^{\gamma_2-s+}$ for $0 \le \gamma_1 \le 1$ and $0 \le \gamma_2 \le s$ obtained in Lemmas 5.2 and 5.3 are used.

For the quintic nonlinearity $N_{22}(v,w)$, by (2.2), (2.3) and the Sobolev inequality, we have

$$\begin{aligned}&\|N_{22}(v,w)_x\|_{L^2} \\ &\lesssim \sum_{\substack{u_1,u_2,u_3,u_4,u_5=v \text{ or } w \\ (u_1,u_2,u_3,u_4,u_5)\neq(v,v,v,v,v)}} \|(u_1)_x\|_{L_x^\infty L_t^2}\|u_2\|_{L_x^4L_t^\infty}\|u_3\|_{L_x^4L_t^\infty}\|u_4\|_{L^\infty}\|u_5\|_{L^\infty} \\ &\lesssim \sum_{\substack{u_1,u_2,u_3,u_4,u_5=v \text{ or } w \\ (u_1,u_2,u_3,u_4,u_5)\neq(v,v,v,v,v)}} \|u_1\|_{\frac12,\frac12+}\|u_2\|_{\frac14,\frac12+}\|u_3\|_{\frac14,\frac12+}\|u_4\|_{\frac12+,\frac12+}\|u_5\|_{\frac12+,\frac12+} \\ &\lesssim N^{\frac{1-s}{2}+\frac{1-s}{4}+\frac{1-s}{4}+\frac{1-s}{2}+\frac12-s+} = N^{2-\frac{5s}{2}+},\end{aligned}$$

where the last order $N^{2-\frac{5s}{2}+}$ in the right hand side of (6.4) appears in the order for $u_j = v,\ j = 1,2,3,4$, and $u_5 = w$. As combining these results, we have the estimate (6.3). □

Let us come back the proof of Theorem 1.1. The conservation of $L^2$ norm for the Cauchy problem (6.1) gives

$$\|v_1\|_{L^2} = \|u(T_0) - e^{iT_0\partial_x^2}w_0\|_{L^2} \le \|u_0\|_{L^2} + \|w_0\|_{L^2} \le \|u_0\|_{L^2} + cN^{-s}. \tag{6.4}$$

We want to estimate $\|v_1\|_{\dot H^1}$. Let us define

$$E(v) = \|v_x\|_{L^2}^2 - \frac{\lambda}{2}\mathrm{Im}\langle |v|^2 v, v_x\rangle_{L^2}.$$

The simple calculation shows that $E(v)$ is the conserved quantity for the Cauchy problem (5.5); $E(v(t)) = E(v_0)$ for all $t \in \mathbb{R}$ (see [10, Lemma 2.3], [24, Proposition 3.1]). Moreover the following estimates hold (e.g., [10, Lemma 2.4], [24, Proposition 3.2])

$$\|v_x\|_{L^2} \lesssim (1+\|v\|_{L^2}^2)\|(Gv)_x\|_{L^2}, \tag{6.5}$$

$$E(v) \geq \left(1-\tfrac{\lambda^2}{4\pi^2}\|v\|_{L^2}^4\right)\|(Gv)_x\|_{L^2}^2, \tag{6.6}$$

where $Gv$ is defined as

$$(Gv)(x) = \exp\left(-i\frac{\lambda}{4}\int_{-\infty}^{x}|v(y)|^2\,dy\right).$$

By (5.3), (5.4), Lemma 6.1 and (6.4), we have

$$E(v_1) = E(v(T_0)+\nu(T_0)) - E(v(T_0)) + E(v_0) \leq E(v_0) + cN^{2-\frac{9s}{4}+}, \tag{6.7}$$

where one uses that $E$ is conserved for solutions to (5.5), the order $N^{2-\frac{9s}{4}+}$ corresponds to the order of $\|v(T_0)\|_{\dot H^1}\|\nu(T_0)\|_{\dot H^1} \lesssim N^{1-s}N^{1-\frac{5s}{4}+}$. Hence by (6.4), (6.5), we have

$$\|v_1\|_{L^2} < \sqrt{\tfrac{2\pi}{|\lambda|}}, \quad \|v_1\|_{\dot H^1} \lesssim N^{1-s},$$
$$E(v_1) \leq E(v_0) + cN^{2-\frac{9s}{4}+} \lesssim N^{2(1-s)},$$

provided $\|v_0\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}$ for (6.6) and sufficiently large $N \gg 1$, where the second inequality comes from the third inequality, since by (6.5) and (6.6). Here we use the fact $\|v_0\|_{L^2} \leq \|u_0\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}$ assumed in Theorem 1.1 to be (6.6) positive.

We want to take this process $T/T_0$ steps. Let $j_0$ such that $1 < j_0 < T/T_0$. As long as $\|v_j\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}$ and $\|v_j\|_{\dot H^1} \lesssim N^{1-s}$ for $1 \leq j \leq j_0$, after iterating $j_0+1$ steps, we have

$$\|v_{j_0+1}\|_{L^2} \leq \|u_0\|_{L^2} + cN^{-s}, \tag{6.8}$$

$$\|v_{j_0+1}\|_{\dot H^1} \lesssim N^{1-s} + j_0^{1/2}N^{1-\frac{9s}{8}+}, \tag{6.9}$$

where (6.9) is obtained by (6.5), (6.6) and

$$E(v_{j_0+1}) \leq E(v_0) + cj_0N^{2-\frac{9s}{4}+}. \tag{6.10}$$

The number we have to repeat is, at most, $T/T_0 \sim TN^{8(1-s)+}$. In order to succeed the above process up to $j_0+1 \sim T/T_0$, uniform, by (6.8) and (6.10), we need the relations

$$N^{-s} \ll 1, \quad TN^{8(1-s)+}N^{2-\frac{9s}{4}+} \lesssim N^{2(1-s)}, \tag{6.11}$$

where the restriction on $s$ appears. When $s > \frac{32}{33}$, the relations (6.11) hold for large $N \gg 1$. Then we complete the proof of Theorem 1.1.

# 7 Remarks on local well-posedness in $H^{1/2}$

In this section we will show the best local well-posedness for the Cauchy problem (1.1). We recall that in [25] the local well-posedness in $H^{1/2}$ was proven, where the proof uses the Fourier restriction norm method. Moreover, if $s < \frac{1}{2}$, the estimate (1.3) crucial to proof fails for any $b \in \mathbb{R}$. So it seems difficult to expect the well-posedness below $H^{1/2}$ for the usual use of Fourier restriction norm method. The scaling argument suggests the value of $s = 0$ critical for the local well-posedness. Thereby we see the gap between the suggestions of scaling argument and of the local well-posedness results in $H^{1/2}$.

This section is motivated by the work in [2, section 6], where Bourgain observes the well-posedness for KdV equation by the data-map argument. The well-posedness requires the continuous dependence on data. Indeed, if we use the contraction argument for proving the integral equation associated with the Cauchy problem, the data-map; $u_0 \mapsto u(t)$ acts smoothly from $H^s$ to itself.

Let us consider the following Cauchy problem

$$iu_t + u_{xx} = i(|u|^2 u)_x, \quad u(0) = \delta u_0. \tag{7.1}$$

In this section, we show that the data-map; $S_t : \delta u_0 \mapsto u_\delta(t)$ of (7.1) fails in $H^s$ for $s < \frac{1}{2}$. The result is the following proposition.

**Proposition 7.1** *Let $T > 0$. Assume that the data-map $S_t$ of (7.1) is $C^3$ class in the sense of Fréchet derivative on the time interval $[0, T]$. Moreover we suppose that $S_t(\delta u_0)$ is expressed as*

$$S_t(\delta u_0) = \sum_{k=1}^{\infty} \delta^k v_k(t)$$

*in $H^s$, where $v_k \in L_t^\infty([0,T] : H^s)$. If $s < \frac{1}{2}$, then the data-map fails.*

**Proof.** By assumption, we have

$$\frac{\partial u_\delta(t)}{\partial \delta}\Big|_{\delta=0} = v_1(t) = e^{it\partial_x^2} u_0, \quad \frac{\partial^2 u_\delta(t)}{\partial \delta^2}\Big|_{\delta=0} = cv_2(t) = 0,$$

$$\frac{\partial^3 u_\delta(t)}{\partial \delta^3}\Big|_{\delta=0} = cv_3(t) = c\int_0^t e^{i(t-s)\partial_x^2} \partial_x(|v_1|^2 v_1)(s)\, ds. \tag{7.2}$$

The assumption of $C^3$-differentiability implies

$$\|e^{it\partial_x^2} u_0\|_{H^s} \le c\|u_0\|_{H^s}, \quad \Big\| \int_0^t e^{i(t-s)\partial_x^2} \partial_x(|v_1|^2 v_1)(s)\, ds \Big\|_{H^s} \le c\|u_0\|_{H^s}^3. \tag{7.3}$$

Let $T > 0$ be fixed, and for simplicity we suppose $T < 1$. We set $u_0$ as follows

$$u_0(x) = \frac{1}{N^s} \int_{|\lambda - N| \le \frac{1}{10^{10}}} e^{i\lambda x}\, d\lambda.$$

We easily see that $\|u_0\|_{H^s} \sim 1$, and the right hand side of (7.2) is similar to

$$\begin{aligned}\frac{1}{N^{3s}} \int_{|\lambda_j - N| \le \frac{1}{10^{10}}} & \frac{\lambda_1 + \lambda_2 - \lambda_3}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{i(\lambda_1+\lambda_2-\lambda_3)x} \\ & \times \left( e^{i(\lambda_1^2+\lambda_2^2-\lambda_3^2)t} - e^{i(\lambda_1+\lambda_2-\lambda_3)^2 t} \right) d\lambda_1 d\lambda_2 d\lambda_3.\end{aligned}$$

When $\frac{T}{2} \le t \le T$, we note that for the points in $|\lambda_j - N| \le \frac{1}{10^{10}}$, $j = 1, 2, 3$, we have

$$|\lambda_1 + \lambda_2 - \lambda_3| \sim N, \qquad \left| \frac{e^{i(\lambda_1^2+\lambda_2^2-\lambda_3^2)t - i(\lambda_1+\lambda_2-\lambda_3)^2 t} - 1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \right| \gtrsim 1.$$

Then we have the following

$$\Big\| \int_0^t e^{i(t-s)\partial_x^2} \partial_x(|v_1|^2 v_1)(s)\, ds \Big\|_{H^s} \gtrsim \frac{N^s N}{N^{3s}} \sim N^{1-2s}. \tag{7.4}$$

From (7.3), (7.4), we need the relation $N^{1-2s} \lesssim 1$, which implies the necessity of $s \ge \frac{1}{2}$ for sufficiently large $N > 1$. □

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