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A unique continuation property for linear elliptic systems and nonresonance problems *

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Abstract

The aim of this paper is to study the existence of solutions for a quasilinear elliptic system where the nonlinear term is a Caratheodory function on a bounded domain of $\mathbb{R}^N$, by proving the well known unique continuation property for elliptic system in all dimensions: 1, 2, 3, ... and the strict monotonocity of eigensurfaces. These properties let us to consider the above problem as a nonresonance problem.

1 Introduction

We study the existence of solutions for the quasilinear elliptic system

$$\begin{gathered}-\Delta u_i = \sum_{j=1}^{n} a_{ij}u_j + f_i(x, u_1, \dots, u_n, \nabla u_1, \dots, \nabla u_n) \quad \text{in } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega,\ i = 1, \dots, n,\end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain, and the coefficients a_{ij} ($1 \leq i, j \leq n$) are constants satisfying $a_{ij} = a_{ji}$, for all i, j. The nonlinearity $f_i : \Omega \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}(1 \leq i \leq n)$ is a Carathéodory function. The case where $n = 2$ and f_i ($1 \leq i \leq n$) is independent of ∇u_i ($1 \leq i \leq n$) has been studied by several authors, in particular by Costa and Magalhães in [8].

This paper is organized as follows. First, we study the unique continuation property in dimension $N \geq 3$ (section 2), for systems of differential inequalities of the form

$$|\Delta u_i(x)| \leq K \sum_{j=1}^{n} |u_j(x)| + m(x)|u_i(x)| \quad \text{a.e. } x \in \Omega,\ 1 \leq i \leq n,$$

where $m \in F^{\alpha,p}$, $0 < \alpha < 1$ and $p > 1$. Here $F^{\alpha,p}$ denotes the set of functions of class Fefferman-Phong. In our proof of Theorem 2, we make use of a number results and techniques developed in [24, 9, 22]. Secondly, we study the unique

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continuation property in dimension $N=2$ (section 3), for linear elliptic systems of the form

$$-\Delta u_i = \sum_{j=1}^{n} a_{ij}u_j + m(x)u_i \quad \text{in } \Omega \ \ i=1,\dots,n\,,$$

where m satisfies the $L_{\log}L$ integrability condition. There is extensive literature on unique continuation; we refer the reader to [22, 12, 18, 19, 15]. The purpose of section 4 is to show that strict monotonicity of eigenvalues for the linear elliptic system

$$\begin{gathered}-\Delta u_i = \sum_{j=1}^{n} a_{ij}u_j + \mu m(x)u_i \quad \text{in } \Omega,\\ u_i = 0 \quad \text{on } \partial\Omega,\ i=1,\dots,n\end{gathered}$$

holds if some unique continuation property is satisfied by the corresponding eigenfunctions. Here $a_{ij}=a_{ji}$ for all $i\neq j$, $\mu\in\mathbb{R}$ and $m\in\mathcal{M}=\{m\in L^\infty(\Omega);\mathrm{meas}(x\in\Omega/m(x)>0)\neq 0\}$. This result will be used for the applications in section 6. In section 5, we study the first order spectrum for linear elliptic systems and strict monotonicity of eigensurfaces. This spectrum is defined as the set of couples $(\beta,\alpha)\in\mathbb{R}^N\times\mathbb{R}$ such that

$$\begin{gathered}-\Delta u_i = \sum_{j=1}^{n} a_{ij}u_j + \alpha m(x)u_i + \beta\cdot\nabla u_i \quad \text{in } \Omega,\\ u_i = 0 \quad \text{on } \partial\Omega,\ i=1,\dots,n\end{gathered} \tag{2}$$

has a nontrivial solution $U=(u_1,\dots,u_n)\in(H_0^1(\Omega))^n$. We denote this spectrum by $\sigma_1(-\overrightarrow{\Delta}-A,m)$ where $A=(a_{ij})_{1\leq i,j\leq n}$ and $m\in\mathcal{M}$. This spectrum is made by an infinite sequence of eigensurfaces $\Lambda_1,\Lambda_2,\dots$ (cf. section 5 and [2] in the case $n=2$). Finally, in section 6 we apply our results to obtain the existence of solutions to (1) under the condition of nonresonance with respect to $\sigma_1(-\overrightarrow{\Delta}-A,1)$.

We use the notation

$$U=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix},\quad -\overrightarrow{\Delta}U=\begin{pmatrix}-\Delta u_1\\ \vdots\\ -\Delta u_n\end{pmatrix},\quad \nabla U=\begin{pmatrix}\nabla u_1\\ \vdots\\ \nabla u_n\end{pmatrix},\quad F=\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}.$$

We denote by $\sigma(-\Delta)=\{\lambda_1,\lambda_2,\dots,\lambda_j,\dots\}$ the spectrum of $-\Delta$ on $H_0^1(\Omega)$. For $\beta\in\mathbb{R}^N$, we denote

$$(\beta\xi)=\begin{pmatrix}\beta.\xi_1\\ \vdots\\ \beta.\xi_n\end{pmatrix},\quad |s|^2=\sum_{i=1}^{n}|s_i|^2,\quad |\xi|^2=\sum_{i=1}^{n}|\xi_i|^2.$$

In the space $(H_0^1(\Omega))^n$ we use the induced inner product

$$\langle U,\Phi\rangle=\sum_{i=1}^{n}\langle u_i,\varphi_i\rangle\quad\forall U=(u_1,\dots,u_n),\Phi=(\varphi_1,\dots,\varphi_n)\in(H_0^1(\Omega))^n$$

and corresponding norm

$$\|U\|_{1,2,\beta}^2=\int_\Omega e^{\beta.x}|\nabla U|^2dx=\sum_{i=1}^{n}\|u_i\|_{1,2,\beta}^2,$$

which is equivalent to the original norm.

2 The unique continuation property for linear elliptic systems in dimensions $N\geq 3$

We will say that a family of functions has the unique continuation property, if no function, besides possibly the zero function, vanishes on a set of positive measure.

In this section, we proceed to establish the unique continuation property when $m\in F^{\alpha,p}$, $0<\alpha<1$ and $p>1$ in dimension $N\geq 3$. The proof of the main result is based on the Carleman's inequality with weight.

Theorem 1 (Carleman's inequality with weight) *Let $m\in F^{\alpha,p}$, $0<\alpha\leq\frac{2}{N-1}$ and $p>1$. Then there exists a constant $c=c(N,p)$ such that*

$$\Big(\int_{\mathbb{R}^N}|e^{\tau x_N}f|^s m\Big)^{1/s}\leq c\|m\|_{F^{\alpha,p}}^{2/s}\Big(\int_{\mathbb{R}^N}|e^{\tau x_N}\Delta f|^r m^{1-r}\Big)^{1/r},\tag{3}$$

for all $\tau\in\mathbb{R}\backslash\{0\}$, and all $f\in S(\mathbb{R}^N)$ where $\frac{1}{r}-\frac{1}{s}=\frac{2}{N+1}$ and $\frac{1}{r}+\frac{1}{s}=1$.

For the proof of this theorem see [22].

Theorem 2 *Let X be an open subset in $\mathbb{R}^N$ and $U=(u_1,\dots,u_n)\in(H_{\rm loc}^{2,r}(X))^n$ ($r=\frac{2(N+1)}{N+3}$) be a solution of the following differential inequalities:*

$$|\Delta u_i(x)|\leq K\sum_{j=1}^{n}|u_j(x)|+m(x)|u_i(x)|\quad a.e.\ x\in X\ 1\leq i\leq n,\tag{4}$$

where K is a constant and m is a locally positive function in $F^{\alpha,p}$, with $\alpha=\frac{2}{N-1}$ and $p>1$, i.e.

$$\lim_{r\to 0}\|\chi_{\{x:|x-y|<r\}}m\|_{F^{\alpha,p}}\leq c(N,p)\quad\forall y\in X.$$

Then, if U vanishes on an open $X\subset\Omega$, U is identically null in Ω.

Lemma 1 *Let $U=(u_1,\dots,u_n)\in(H_{\rm loc}^{2,r}(X))^n$ ($r=\frac{2(N+1)}{N+3}$) be a solution of (4) in a neighborhood of a sphere S. If U vanishes in one side of S, then U is identically null in the neighborhood of S.*

Proof. We may assume without loss generality that S is centered at $-1 = (0,\dots,-1)$ and has radius 1. By the reflection principle (see [24]), we can also suppose that $U = 0$ in the exterior neighborhood of S.

Now, let $\varepsilon > 0$ small enough such that $U(x) = 0$ when $|(x+1)| > 1$ and $|x| < \varepsilon$. Set $f_i(x) = \eta(|x|)u_i(x)$ for each $i = 1,\dots,n$ where $\eta \in C_0^\infty([-\varepsilon,\varepsilon])$, $\eta(|x|) = 1$ if $|x| < \varepsilon/2$. For fixed ρ such that $0 < \rho < \frac{\varepsilon}{2}$, let B_ρ the ball of radius ρ centered at zero. By the Carleman inequality, theorem 1 yields

$$\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} \le c\|\chi_{B_\rho} m\|^{2/s}_{F^{\alpha,p}} \Big(\int_{\mathbb{R}^N} |e^{\tau x_N} \Delta f_i|^r m^{1-r}\Big)^{1/r}, \quad \forall \tau > 0 \tag{5}$$

for $i = 1,\dots,n$. Inequality (5) implies

$$\begin{aligned}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} &\le c\|\chi_{B_\rho} m\|^{2/s}_{F^{\alpha,p}} \Big\{\Big(\int_{\mathbb{R}^N\setminus B_\rho} |e^{\tau x_N} \Delta f_i|^r m^{1-r}\Big)^{1/r} \\ &\quad + \Big(\int_{B_\rho} |e^{\tau x_n} \Delta f_i|^r m^{1-r}\Big)^{1/r}\Big\}.\end{aligned} \tag{6}$$

¿From (4), we have

$$\begin{aligned}\Big(\int_{B_\rho} |e^{\tau x_N} \Delta f_i|^r m^{1-r}\Big)^{1/r} &\le c\sum_{j=1}^n \Big(\int_{B_\rho} |e^{\tau x_N} f_j|^r m^{1-r}\Big)^{1/r} \\ &\quad + \Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m\Big)^{1/r}\end{aligned} \tag{7}$$

for each $i = 1,\dots,n$. Using the Hölder's inequality, we obtain

$$\begin{aligned}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m\Big)^{1/r} &= \Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m^{r/s} m^{1-r/s}\Big)^{1/r} \\ &\le \Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} \Big(\int_{B_\rho} m\Big)^{\frac{1}{r}-\frac{1}{s}}.\end{aligned} \tag{8}$$

As $m \in F^{\alpha,p}_{\mathrm{loc}}(X)$, it follows that

$$\int_{B_\rho} m \le c\rho^{N-\alpha} \|\chi_{B_\rho} m\|_{F^{\alpha,p}}. \tag{9}$$

Indeed, if $m \in F^{\alpha,p}_{\mathrm{loc}}(X)$ then

$$\begin{aligned}\int_{B_\rho} m &\le \Big(\int_{B_\rho} m^p\Big)^{1/p} |B_\rho|^{1-\frac{1}{p}} \\ &\le |B_\rho|^{1-\alpha/N} \Big(|B_\rho|^{\alpha/N} \Big(\frac{1}{|B_\rho|}\int_{B_\rho} m^p\Big)^{1/p}\Big) \\ &\le c\rho^{N-\alpha} \|\chi_{B_\rho} m\|_{F^{\alpha,p}}.\end{aligned}$$

It follows from (8) and (9) that

$$\begin{aligned}
\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m\Big)^{1/r} &\leq c\rho^{(N-\alpha)(\frac{1}{r}-\frac{1}{s})}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}^{\frac{1}{r}-\frac{1}{s}}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s}, \\
&\leq c\rho^{\frac{2(N-\alpha)}{N+1}}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}^{\frac{1}{r}-\frac{1}{s}}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s}, \quad (10)
\end{aligned}$$

for each $i = 1, \ldots, n$.
We may assume without loss generality that $m \geq 1$, then

$$\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m^{1-r}\Big)^{1/r} \leq \Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m\Big)^{1/r} \quad \forall 1 \leq i \leq n.$$

¿From (10), we deduce

$$\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^r m^{1-r}\Big)^{1/r} \leq c\rho^{\frac{2(N-\alpha)}{N+1}}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}^{\frac{1}{r}-\frac{1}{s}}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s}. \quad (11)$$

Therefore from (10) and (11), we have

$$\begin{aligned}
\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} &\leq c\|\chi_{B_\rho} m\|_{F^{\alpha,p}}^{2/s}\Big(\int_{\mathbb{R}^N\setminus B_\rho} |e^{\tau x_N}\Delta f_i|^r m^{1-r}\Big)^{1/r} \\
&\quad + c\rho^{\frac{2(N-\alpha)}{(N+1)}}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}\sum_{j=1}^n\Big(\int_{B_\rho} |e^{\tau x_N} f_j|^s m\Big)^{1/s} \\
&\quad + c\rho^{\frac{2(N-\alpha)}{(N+1)}}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s}, \quad (12)
\end{aligned}$$

for each $i = 1, \ldots, n$. Replacing α by $\frac{2}{N-1}$ in (12), we obtain

$$\begin{aligned}
\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} &\leq c\|\chi_{B_\rho} m\|_{F^{\alpha,p}}^{2/s}\Big(\int_{\mathbb{R}^N\setminus B_\rho} |e^{\tau x_N}\Delta f_i|^r m^{1-r}\Big)^{1/r} \\
&\quad + c\rho^{\frac{2(N-2)}{(N-1)}}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}\sum_{j=1}^n\Big(\int_{B_\rho} |e^{\tau x_N} f_j|^s m\Big)^{1/s} \\
&\quad + c\rho^{\frac{2(N-2)}{(N-1)}}\|\chi_{B_\rho} m\|_{F^{\alpha,p}}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s},
\end{aligned}$$

for each $= 1, \ldots, n$. Let us choose ρ small enough, such that

$$\|\chi_{B_\rho} m\|_{F^{\alpha,p}} \leq \frac{1}{2nc}.$$

So for $i = 1, \dots, n$,

$$\begin{aligned}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} \quad \leq \quad & c\Big(\int_{\mathbb{R}^N\setminus B_\rho} |e^{\tau x_N}\Delta f_i|^r m^{1-r}\Big)^{1/r} \\ & + \frac{1}{2n}\sum_{j=1}^{n}\Big(\int_{B_\rho} |e^{\tau x_N} f_j|^s m\Big)^{1/s} \\ & + \frac{1}{2n}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s},\end{aligned}$$

Since $f_i(x) = 0$ for all $1 \leq i \leq n$ when $|(x+1)| > 1$ or $|x| > \varepsilon$, we deduce that

$$\frac{n-1}{2n}\sum_{i=1}^{n}\Big(\int_{B_\rho} |e^{\tau x_N} f_i|^s m\Big)^{1/s} \leq ce^{-\rho\tau}\sum_{i=1}^{n}\Big(\int_{\mathbb{R}^N\setminus B_\rho} |\Delta f_i|^r m^{1-r}\Big)^{1/r}.$$

So

$$\frac{n-1}{2n}\sum_{i=1}^{n}\Big(\int_{B_\rho} |e^{\tau(\rho+x_N)} f_i|^s m\Big)^{1/s} \leq c\sum_{i=1}^{n}\Big(\int_{\mathbb{R}^N} |\Delta f_i|^r m^{1-r}\Big)^{1/r}. \tag{13}$$

Taking $\tau \to +\infty$ in (13), we conclude that $U = 0$ in B_ρ. $\square$

Proof of Theorem 2 We assume that $U \not\equiv 0$ on X. Let Ω be a maximal open set on which U vanishes and $\Omega \neq X$, then there exists a sphere S which its interior is contained in Ω, such that there exists $x \in \partial\Omega \cap S$. As U vanishes in one side of S, it follows that $x \in \Omega$, which is absurd. $\square$

3 The unique continuation property for linear elliptic systems in dimension $N = 2$

In this section we prove the unique continuation property where $m \in L_{\log}L$ in lower dimension by using the zero of infinite order theory.

Definition 1 Let Ω be an open subset in $\mathbb{R}^N$. A function $U = (u_1, \dots, u_n) \in (L^2_{\rm loc}(\Omega))^n$ has a zero of infinite order at $x_0 \in \Omega$, if for each $l \in \mathbb{N}$

$$\lim_{R\to 0} R^{-l}\int_{|x-x_0|<R} |U(x)|^2 dx = 0.$$

Let us denote by ψ the N-function

$$\psi(t) = (1+t)\log(1+t) - t, \quad t \geq 0$$

and by L^ψ the corresponding Orlicz space (see [20]).

Theorem 3 *Let Ω be a bounded open subset in $\mathbb{R}^2$ and $m \in L^{\psi}_{\text{loc}}(\Omega)$. Let $U = (u_1, \dots, u_n) \in (H^1_{\text{loc}}(\Omega))^n$ be a solution of the linear elliptic system*

$$-\Delta u_i = \sum_{j=1}^{n} a_{ij} u_j + m(x) u_i \quad \text{in } \Omega; \quad i = 1, \dots, n \tag{14}$$

where the coefficients $a_{ij}(1 \le i, j \le n)$ are assumed to be constants satisfying $a_{ij} = a_{ji}$ $\forall i, j$. If U vanishes on a set $E \subset \Omega$ of positive measure, then almost every point of E is a zero of infinite order for U.

The proof of this theorem is done in several lemmas.

Lemma 2 *Let ω be a bounded open subset in $\mathbb{R}^2$ and $m \in L^{\psi}(\omega)$. Then for any ε there exists $c_\varepsilon = c_\varepsilon(\omega, m)$ such that*

$$\int_\omega m u^2 \le \varepsilon \int_\omega |\nabla u|^2 + c_\varepsilon \int_\omega u^2 \tag{15}$$

for all $u \in H^1_0(\omega)$.

For a proof of this lemma, see [7].

Lemma 3 *Let U be a solution of system (14), B_r and B_{2r} be two concentric balls contained in Ω. Then*

$$\int_{B_r} |\nabla U|^2 \le \frac{c}{r^2} \int_{B_{2r}} |U|^2, \tag{16}$$

where the constant c does not depend on r.

Proof. Let φ, with $\operatorname{supp}\varphi \subset B_{2r}, \varphi(x) = 1$ for $x \in B_r$ and $|\nabla\varphi| \le \frac{2}{r}$. Using $\varphi^2 U$ as test function in (14), we get

$$\int_\Omega -\overrightarrow{\Delta} U.(\varphi^2 U) = \int_\Omega AU.(\varphi^2 U) + \int_\Omega mU.(\varphi^2 U).$$

So

$$\int_\Omega |\nabla U|^2 \varphi^2 = \int_\Omega (AU.U)\varphi^2 - 2\int_\Omega \langle \varphi \nabla U, \nabla\varphi\, U\rangle + \int_\Omega m\varphi^2 U^2. \tag{17}$$

On the other hand, we have

$$AU(x).U(x) \le \rho(A) U(x).U(x) \quad \text{a.e. } x \in \Omega,$$

where $\rho(A)$ is the largest eigenvalue of the matrix A. Using Schwartz and Young's inequalities, we have

$$2|\langle \varphi\nabla U, \nabla\varphi U\rangle| \le \varepsilon |\varphi \nabla U|^2 + \frac{|\nabla\varphi\, U|^2}{\varepsilon} \quad \text{for } \varepsilon > 0. \tag{18}$$

Thus, by lemma 2, we have for any $\varepsilon > 0$, there exists $c_\varepsilon = c_\varepsilon(\Omega, m)$ such that

$$\int_\Omega m|\varphi U|^2 \leq \varepsilon \int_\Omega |\nabla(\varphi U)|^2 + c_\varepsilon \int_\Omega |\varphi U|^2. \tag{19}$$

It follows from (17), (18) and (19) that

$$\begin{aligned}\int_{B_{2r}} \varphi^2|\nabla U|^2 &\leq \rho(A)\int_{B_{2r}} |\varphi U|^2 + \varepsilon\int_{B_{2r}} |\nabla U|^2\varphi^2 + \frac{1}{\varepsilon}\int_{B_{2r}} |\nabla\varphi|^2 U^2 \\ &\quad +\varepsilon\int_{B_{2r}} |\nabla(\varphi U)|^2 + c_\varepsilon\int_{B_{2r}} |\varphi U|^2,\end{aligned}$$

and therefore

$$(1-(\varepsilon^2+2\varepsilon))\int_{B_{2r}} \varphi^2|\nabla U|^2 \leq (\varepsilon+1+\frac{1}{\varepsilon})\int_{B_{2r}} |(\nabla\varphi\, U)|^2 + (\rho(A)+c_\varepsilon)\int_{B_{2r}} |\varphi U|^2.$$

Using the fact that $|\nabla\varphi| \leq \frac{2}{r}$, $|\varphi| \leq \frac{c}{r}$ and $\varphi = 1$ in B_r, we have immediately (16). $\square$

Remark 1 If U has a zero of infinite order at $x_0 \in \Omega$, then ∇U has also a zero of infinite order at x_0.

Lemma 4 ([21]) *Let $u \in W^{1,1}(B_r)$, where B_r is the ball of radius r in $\mathbb{R}^N$ and let $E = \{x \in B_r : u(x) = 0\}$. Then there exists a constant β depending only on N such that*

$$\int_D |u| \leq \beta\frac{r^N}{|E|}|D|^{1/N}\int_{B_r} |\nabla u|$$

for all B_r, u as above and all measurable sets $D \subset B_r$.

Proof of Theorem 3. Let $U = (u_1, \dots, u_n) \in (H^1_{\rm loc}(\Omega))^n$ be a solution of (14) which vanishes on a set E of positive measure. We know that almost every point of E is a point of density of E. Let x_0 be such a point, i.e.

$$\frac{|E^c \cap B_r|}{|B_r|} \to 0 \quad \text{and} \quad \frac{|E \cap B_r|}{|B_r|} \to 1 \quad \text{as } r \to 0, \tag{20}$$

where B_r is the ball of radius r centered at x_0. So, for a given $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) > 0$ such that for $r \leq r_0$

$$\frac{|E^c \cap B_r|}{|B_r|} < \varepsilon \quad \text{and} \quad \frac{|E \cap B_r|}{|B_r|} > 1 - \varepsilon,$$

where E^c denotes the complement of E in Ω. Taking r_0 smaller if necessary, we may assume that $\overline{B}_{2r_0} \subset \Omega$. By lemma 4 we have

$$\begin{aligned}\int_{B_r} |u_i|^2 = \int_{B_r\cap E^c} |u_i|^2 &\leq \beta\frac{r^2}{|E \cap B_r|}|E^c \cap B_r|^{1/2}\int_{B_r} |\nabla(u_i)^2| \\ &\leq 2\beta\frac{r^2}{|E \cap B_r|}|E^c \cap B_r|^{1/2}\int_{B_r} |u_i||\nabla u_i|\end{aligned}$$

for each $i = 1, \dots, n$. The Hölder and Youngs's inequalities lead to

$$\begin{aligned}\int_{B_r} |u_i|^2 &\leq 2\beta \frac{r^2}{|E \cap B_r|} |E^c \cap B_r|^{1/2} \Big(\int_{B_r} |u_i|^2 \Big)^{1/2} \Big(\int_{B_r} |\nabla u_i|^2 \Big)^{1/2} \\ &\leq \beta \frac{r^2}{|E \cap B_r|} |E^c \cap B_r|^{1/2} \Big(\frac{1}{r} \int_{B_r} u_i^2 + r \int_{B_r} |\nabla u_i|^2 \Big), \qquad (21)\end{aligned}$$

for each $i = 1, \dots, n$. We take the sum for $i = 1$ to n, we obtain

$$\int_{B_r} |U|^2 \leq \beta \frac{r^2}{|E \cap B_r|} |E^c \cap B_r|^{1/2} \Big(\frac{1}{r} \int_{B_r} |U|^2 + r \int_{B_r} |\nabla U|^2 \Big).$$

It follows from (16) and (20) that

$$\begin{aligned}\int_{B_r} |U|^2 &\leq \beta \frac{r^2 |B_r|}{|B_r|^{1/2} |E \cap B_r|} \frac{|E^c \cap B_r|^{1/2}}{|B_r|^{1/2}} \Big(\frac{1}{r} \int_{B_{2r}} |U|^2 + \frac{c}{r} \int_{B_{2r}} |U|^2 \Big) \\ &\leq \beta \frac{cr}{|B_r|^{1/2}} \frac{\varepsilon^{1/2}}{1-\varepsilon} \int_{B_{2r}} |U|^2 \\ &\leq c \frac{\varepsilon^{1/2}}{1-\varepsilon} \int_{B_{2r}} |U|^2, \quad \text{for } r \leq r_0. \qquad (22)\end{aligned}$$

Set $f(r) = \int_{B_r} |U|^2$. Let us fix $n \in \mathbb{N}$, we have $\varepsilon > 0$ such that $\frac{c\varepsilon^{1/2}}{1-\varepsilon} = 2^{-n}$. Observe that now r_0 depend on n. ¿From (22), we deduce that

$$f(r) \leq 2^{-n} f(2r), \quad \text{for } r \leq r_0. \qquad (23)$$

Iterating (23), we get

$$f(\rho) \leq 2^{-kn} f(2^k \rho) \quad \text{if } 2^{k-1} \rho \leq r_0. \qquad (24)$$

Thus, given $0 < r < r_0(n)$ and choosing $k \in \mathbb{N}$ such that

$$2^{-k} r_0 \leq r \leq 2^{-(k-1)} r_0.$$

¿From (24), we conclude that

$$f(r) \leq 2^{-kn} f(2^k r) \leq 2^{-kn} f(2r_0),$$

and since $2^{-k} \leq \frac{r}{r_0}$, we get

$$f(r) \leq \Big(\frac{r}{r_0} \Big)^n f(2r_0).$$

This shows that $f(r) = 0(r^n)$ as $r \to 0$. Consequently x_0 is a zero of infinite order for U. $\square$

Theorem 4 *Let Ω be an open subset in $\mathbb{R}^2$. Assume that $U = (u_1, \dots, u_n) \in (H^2_{\mathrm{loc}}(\Omega))^n$ has a zero of infinite order at $x_0 \in \Omega$ and satisfies*

$$|\Delta u_i(x)| \leq K \sum_{j=1}^{n} |u_j(x)| + m(x)|u_i(x)| \quad \text{a.e. } x \in \Omega,\ 1 \leq i \leq n, \qquad (25)$$

where m is a positive function belong to a class of $L_{\log} L_{\mathrm{loc}}(\Omega)$. Then U is identically null in Ω.

Proof. The technique used here is due to S. Chanillo and E. Sawyer (see [9]), we may assume that $m \geq 1$. Since $m+1$ also satisfy the hypotheses of theorem 4 when $m \in L_{\log}L_{\mathrm{loc}}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^2} |I_1 f|^2 m \leq c\|m\| \int_{\mathbb{R}^2} f^2 \quad \forall f \in C_0^\infty(\mathbb{R}^2), \tag{26}$$

(cf. [13, 25]), where $I_\alpha f$ denotes the Riesz potential of order $0 < \alpha < n$, defined by

$$I_\alpha(x) = \int_{\mathbb{R}^N} |x-y|^{-n+\alpha} f(y) dy,$$

where one posed to simplify $\|m\| = \|m\|_{L_{\log}L}$. The inequality (26) is equivalent to the dual inequality

$$\int_{\mathbb{R}^2} |I_2 f|^2 m \leq c\|m\|^2 \int_{\mathbb{R}^2} |f|^2 m^{-1} \quad \forall f \in C_0^\infty(\mathbb{R}^2). \tag{27}$$

(cf. [14]) where $I_2 f = \phi_2 * f$ denotes the Newton Potential with ϕ_2 is the elementary solution of $-\Delta$. On the other hand, from the result of E.Sawyer (cf. [23]), if

$$\phi_2(x) = \frac{1}{2\pi} \log|x|,$$

then

$$|\phi_2(x-y) - \sum_{j=0}^{l-1} \frac{1}{j!} \left(\frac{\partial}{\partial s}\right)^j \phi_2(sx-y)|_{s=0}| \leq c \left(\frac{|x|}{|y|}\right)^l \phi_2(x-y) \quad \forall l \in \mathbb{N}. \tag{28}$$

The constant c does not depend on l, x and y. Let $U = (u_1, \dots, u_n)$ be a solution of (25) and has a zero of infinite order at $x_0 \in \Omega$. We may suppose without loss generality that $0 \in \Omega$ and $x_0 = 0$. Let also η and ψ be two functions such that $\eta \in C_0^\infty(B_{2r}), \eta = 1$ on B_r and $\psi = 0$ on B_1, $\psi = 1$ outside B_2 and $0 \leq \psi \leq 1$. Set $\psi_k(x) = \psi(kx), k \geq 0$. We also assume that $k \geq 4/r$ and $r < 1/2$. Then by

(27, 28) and [16, Theoremm 4.3], for $l \geq 1$ we have

$$\begin{aligned}
&\int_{B_r} \frac{|\psi_k(x)u_i(x)|}{|x|^{2l}} m(x)dx \qquad (29)\\
&= \int_{B_r} |x|^{-2l}|\int \phi_2(x-y)\Delta(\eta\psi_k u_i)dy|^2 m(x)dx\\
&= \int_{B_r} |x|^{-2l}|\int (\phi_2(x-y) - \sum_{j=0}^{l-1}\frac{1}{j!}(\frac{\partial}{\partial s})^j \phi_2(sx-y)|_{s=0})\Delta(\eta\psi_k u_i)dy|^2 m dx\\
&\leq c\int_{B_r} \Big(\frac{\int \phi_2(x-y)|\Delta(\psi_k\eta u_i)|}{|y|^l}\Big)^2 m(x)dx\\
&\leq c\int_{B_r} |I_2(\frac{\Delta(\psi_k\eta u_i)}{|y|^l})|^2 m(x)dx\\
&\leq c\|\chi_{B_r} m\|^2 \int_{B_{2r}} \frac{|\Delta(\psi_k\eta u_i)|^2}{|x|^{2l}} m^{-1}(x)dx\\
&\leq c\|\chi_{B_r} m\|^2 (\int_{B_{2r}} \frac{|\Delta\psi_k|^2 u_i^2}{|x|^{2l}} m^{-1}(x)dx + \int_{B_{2r}} \frac{|\nabla\psi_k|^2|\nabla u_i|^2}{|x|^{2l}} m^{-1}(x)dx\\
&\quad + \int_{B_r} \frac{|\psi_k|^2|\Delta(\eta u_i)|^2}{|x|^{2l}} m^{-1}dx) + \int_{|x|>r} \frac{|\psi_k|^2|\Delta(\eta u_i)|^2}{|x|^{2l}} m^{-1}dx)\\
&= c\|\chi_{B_r} m\|^2 (I_k^i + II_k^i + III_k^i + IV_k^i), \qquad (30)
\end{aligned}$$

for each $i = 1, \dots, n$. Choosing $c\|\chi_{B_r} m\|^2 < \frac{1}{2n}$ (this is possible since the measure $L_{\log}L$ is absolutely continuous) it follows that

$$\begin{aligned}
III_k^i &\leq \frac{1}{2n}\int_{|x|<r} \frac{|\psi_k|^2|\Delta u_i|^2}{|x|^{2l}} m^{-1}(x)dx\\
&\leq \frac{1}{2n}\Big(\sum_{j=1}^{n}\int_{|x|<r} \frac{|\psi_k|^2|u_j|^2}{|x|^{2l}} m^{-1}(x)dx + \int_{|x|<r} \frac{|\psi_k|^2|u_i|^2}{|x|^{2l}} m(x)dx\Big).
\end{aligned}$$

We have

$$\int_{|x|<r} \frac{|\psi_k|^2|u_j|^2}{|x|^{2l}} m^{-1}(x)dx \leq \int_{|x|<r} \frac{|\psi_k|^2|u_j|^2}{|x|^{2l}} m(x)dx \quad \text{whenever } m \geq 1.$$

So

$$III_k^i \leq \frac{1}{2n}\Big(\sum_{j=1}^{n}\int_{|x|<r} \frac{|\psi_k||u_j|^2}{|x|^{2l}} m(x)dx + \int_{|x|<r} \frac{|\psi_k|^2|u_i|^2}{|x|^{2l}} m(x)dx\Big). \qquad (31)$$

As $U = (u_1, \dots, u_n)$ is a solution of (25), from (29) and (31), we conclude that

$$(1-\frac{1}{2n})\int_{B_r} \frac{|\psi_k|^2|u_i|^2}{|x|^{2l}} m(x)dx - \frac{1}{2n}\sum_{j=1}^{n}\int_{B_r} \frac{|\psi_k|^2|u_j|^2}{|x|^{2l}} m(x)dx \leq I_k^i + II_k^i. \qquad (32)$$

On the other hand, we have

$$I_k^i \le \int_{\frac{1}{k}\le|x|\le\frac{2}{k}} \frac{|\psi_k|^2|u_i|^2}{|x|^{2l}} m^{-1}(x)dx \le ck^{2l+4}\int_{|x|\le\frac{2}{k}} |u_i|^2 dx,$$

for each $i = 1,\dots,n$. Hence $\lim_{k\to+\infty} I_k^i = 0 \quad \forall 1 \le i \le n$, since U has a zero of infinite order at 0 by hypothesis. On the other side

$$II_k^i \le ck^{2l+2}\int_{|x|\le\frac{2}{k}} |\nabla u_i|^2 dx.$$

By Remark 1, it follows that $\lim_{k\to+\infty} II_k^i = 0 \quad \forall 1 \le i \le n$. The sum from $i = 1$ to n in the inequality (32), yields

$$\frac{n-1}{2n}\sum_{i=1}^{n}\int_{B_r} \frac{|\psi_k u_i|^2}{|x|^{2l}} m(x)dx \le \sum_{i=1}^{n}(I_k^i + II_k^i + IV_k^i).$$

So that

$$\int_{|x|<r} |\psi_k U|^2 m \le r^{2l}\int_{B_r} \frac{|\psi_k U|^2}{|x|^{2l}} \le \frac{2n}{n-1} r^{2l}\sum_{i=1}^{n}(I_k^i + II_k^i + IV_k^i). \tag{33}$$

Taking the limit as k and $l \to +\infty$ in (33), we conclude that $U = 0$ on B_r. $\square$

Remark 2 In the following sections we take m in $\mathcal{M}$ which is obviously a subspace of $F^{\alpha,p}$ and $L_{\log}L$. Also for those bounded potential we can use the Carleman inequality of N. Arnsajn [5].

4 Strict monotonicity of eigenvalues for linear elliptic systems

In this section we study the strict monotonicity of eigenvalues for the linear elliptic system

$$\begin{gathered} -\Delta u_i = \sum_{j=1}^{n} a_{ij}u_j + \mu m(x) u_i \quad \text{in } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega,\ i = 1,\dots,n\,. \end{gathered} \tag{34}$$

We will assume that

$$\lambda_1 > \rho(A), \tag{35}$$

where $\rho(A)$ is the largest eigenvalue of the matrix A and λ_1 the smallest eigenvalue of $-\Delta$.

As it is well Known [1, 17, 10, 2], that the eigenvalues in (34) form a sequence of positive eigenvalues, which can be written as

$$\mu_1(m) < \mu_2(m) \le \dots\,.$$

Here we use the symbol $\lneqq$ to indicate inequality a.e. with strict inequality on a set of positive measure.

Proposition 1 *Let m_1 and m_2 be two weights of M with $m_1 \lneqq m_2$ and let $j \in \mathbb{N}$. If the eigenfunctions associated to $\mu_j(m_1)$ enjoy the unique continuation property, then $\mu_j(m_1) > \mu_j(m_2)$.*

Proof. We proceed by the similar arguments which has been developed by D.G. de Figueiredo and J.P. Gossez [12]. $\mu_j(m_1)$ is given by the variational characterization

$$\frac{1}{\mu_j(m_1)} = \sup_{F_j} \inf\{\int_\Omega m_1|U|^2dx; U \in F_j \quad \text{and } \mathcal{L}(U,U) = 1\}, \tag{36}$$

where $\mathcal{L}(U,U) = \int_\Omega |\nabla U|^2 - \int_\Omega AU.Udx$ and F_j varies over all j-dimensional subspace of $(H_0^1(\Omega))^n$ (cf. [1, 17, 10]). Since the extrema in (36) are achieved [11], there exists $F_j \subset (H_0^1(\Omega))^n$ of dimension j such that

$$\frac{1}{\mu_j(m_1)} = \inf\{\int_\Omega m_1|U|^2dx; U \in F_j \quad \text{and} \quad \mathcal{L}(U,U) = 1\}. \tag{37}$$

Pick $U \in F_j$ with $\mathcal{L}(U,U) = 1$. Either U achieves tits infimum in (37) or not. In the first case, U is an eigenfunction associated to $\mu_j(m_1)$ (cf. [11]), and so, by the unique continuation property

$$\frac{1}{\mu_j(m_1)} = \int_\Omega m_1|U|^2 < \int_\Omega m_2|U|^2.$$

In the second case

$$\frac{1}{\mu_j(m_1)} < \int_\Omega m_1|U|^2 \leq \int_\Omega m_2|U|^2.$$

Thus, in any case

$$\frac{1}{\mu_j(m_1)} < \int_\Omega m_2|U|^2.$$

It follows, by a simple compactness argument that

$$\frac{1}{\mu_j(m_1)} < \inf\{\int_\Omega m_2|U|^2; U \in F_j \text{ and } \mathcal{L}(U,U) = 1\}.$$

This yields the desired inequality

$$\frac{1}{\mu_j(m_1)} < \frac{1}{\mu_j(m_2)}.$$

5 Spectrum for linear elliptic systems

First order spectrum

Theorem 5 **a)** *$\Lambda_n(.,A,m) : \mathbb{R}^N \to \mathbb{R}$ is the positive function characterized in a variational form by*

$$\frac{1}{\Lambda_n(\beta,A,m)} = \sup_{F_n \in \mathcal{F}_n((H_0^1(\Omega))^n)} \min\Big\{\int_\Omega e^{\beta.x} m(x)|U|^2dx,\ U \in F_n \cap S_\beta(A)\Big\}$$

for all $\beta \in \mathbb{R}^N$, *with*

$$S_\beta(A) = \left\{ U \in (H_0^1(\Omega))^n : \|U\|_{1,2,\beta}^2 - \int_\Omega e^{\beta.x} AU.U dx = 1 \right\},$$

and $\mathcal{F}_n((H_0^1(\Omega))^n)$ *is the set of* n*-dimensional subspaces of* $(H_0^1(\Omega))^n$.

b) *For all* $U \in (H_0^1(\Omega))^n$,

$$\Lambda_1(\beta, A, m) \int_\Omega e^{\beta.x} m(x)|U|^2 dx \le \|U\|_{1,2,\beta}^2 - \int_\Omega e^{\beta.x} AU.U\, dx\,.$$

c) *For all* $\beta \in \mathbb{R}^N$, $\lim_{n\to+\infty} \Lambda_n(\beta, A, m) = +\infty$.

For the proof of this theorem see [2].

Strict monotonicity of eigensurfaces for linear elliptic systems

By theorem 5 it seems that the following result may be proved by arguments similar to those in proposition 1 (see section 4).

Proposition 2 *Let* $m_1, m_2 \in M$, *if* $m_1 \precneqq m_2$ *then* $\Lambda_j(\beta, A, m_1) > \Lambda_j(\beta, A, m_2)$ *for all* $j \in \mathbb{N}^*$.

6 Nonresonance between consecutives eigensurfaces

In this section, we study the existence of solutions for the quasilinear elliptic system

$$\begin{aligned} -\overrightarrow{\Delta} U = AU + F(x, U, \nabla U) \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{38}$$

Let us consider the situation where the nonlinearity F is asymptotically between two consecutive eigensurfaces in the following sense: we assume that there exists $\alpha_1 < \alpha_2 \in \mathbb{R}$, $\beta \in \mathbb{R}^N$ and for all $\delta > 0$ there exist $a_\delta \in L^2(\Omega)$ such that

$$\begin{aligned} \alpha_1|s|^2 + (\beta\xi).s - \delta(|\xi|^2 + a_\delta(x))|s| &\le s.F(x, s, \xi) \qquad (39) \\ &\le \alpha_2|s|^2 + (\beta\xi).s + \delta(|\xi|^2 + a_\delta(x))|s| \end{aligned}$$

a.e. $\in \Omega$ and for all $(\xi, s) \in \mathbb{R}^{2N} \times \mathbb{R}^2$.

A function U in $(H_0^1(\Omega))^n$ is said to be a solution of (38) if U satisfies (38) in the sense of distributions. With this definition, we state the main result of this section.

Theorem 6 *Let (39) be satisfied with* $\Lambda_k(\beta, A, 1) < \alpha_1 < \alpha_2 < \Lambda_{k+1}(\beta, A, 1)$ *for some* $k \ge 1$, *then (38) admits a solution.*

Remark 3 It is clear that by (39) there exist $b_1 > 0$ such that for all $\delta > 0$ there exists $a_\delta \in L^2(\Omega)$ such that

$$|F(x,s,\xi) - (\beta\xi)| \le b_1|s| + \delta b_1(|\xi| + a_\delta(x)) \tag{40}$$

a.e. $x \in \Omega$ and for all $(\xi, s) \in \mathbb{R}^{2N} \times \mathbb{R}^2$.

Proof of Theorem 6 Let $(T_t)_{t\in[0,1]}$ be a family of operators from $(H_0^1(\Omega))^n$ to $(H^{-1}(\Omega))^n$:

$$T_t(U) = -\overrightarrow{\Delta}^\beta(U) - e^{\beta.x}(t(F(x,U,\nabla U) + (1-t)\alpha U - t(\beta\nabla U))$$

where $\alpha_1 < \alpha < \alpha_2$. Since F verifies (39), the operator T_t is of the type (S_+). Now, we show the a priori estimate:

$$\exists r > 0 \text{ such that } \forall t \in [0,1], \forall U \in (\partial B(0,r))^n, \text{ we have } T_t(U) \ne 0.$$

We proceed by contradiction, if the a priori estimate is not true, then

$$\forall n \in \mathbb{N},\ \exists t_n \in [0,1],\ \exists U_n \in (\partial B(0,n))^n\ (\|U_n\|_{1,2} = n), \text{such that } T_{t_n}(U_n) = 0,$$

so that

$$-\overrightarrow{\Delta}^\beta(U_n) = e^{\beta.x}(t_nF(x,U_n,\nabla U_n) + (1-t_n)\alpha U_n - t_n(\beta\nabla U_n)). \tag{41}$$

Set $V_n = \frac{U_n}{\|U_n\|_{1,2}}$, the sequence (V_n) is bounded in $(H_0^1(\Omega))^n$. Therefore, there exists a subsequence of (V_n) (also noted (V_n)) such that: $V_n \rightharpoonup V$in $(H_0^1(\Omega))^n$, $V_n \to V$ in $(L^q(\Omega))^n$ for all $q \in [1, 2^*[$, with $2^* = \frac{2N}{N-2}$. Then we proceed in several steps.

Step 1: The sequence of functions defined a.e. $x \in \Omega$ by

$$G_n(x) = \frac{F(x,U_n,\nabla U_n)}{\|U_n\|_{1,2}} - (\beta\nabla V_n)$$

is bounded in $(L^2(\Omega))^n$.

To prove this statement we divide (40) by $\|U_n\|_{1,2}$. Then

$$|G_n(x)| \le b_1|V_n| + \delta b_1(|\nabla V_n| + \frac{a_\delta(x)}{n}).$$

and

$$\begin{aligned}\|G_n\| &\le b_1\|V_n\|_2 + \delta b_1(\|V_n\|_{1,2} + \frac{\|a_\delta\|_2}{n})\\ &\le \frac{b_1}{(\lambda_1)^{1/2}} + \delta b_1(1 + \frac{\|a_\delta\|_2}{n}),\end{aligned}$$

which proves step 1.

Since $(L^2(\Omega))^n$ is a reflexive space, there exists a subsequence of (G_n), also denoted by (G_n), and $\tilde{F} \in (L^2(\Omega))^n$ such that

$$G_n \rightharpoonup \tilde{F} \quad \text{in } (L^2(\Omega))^n. \tag{42}$$

Step 2. $\tilde{F}(x) = 0$ a.e. in $\mathcal{A} := \{x \in \Omega : V(x) = 0 \text{ a.e.}\}$

To prove this statement, we define $\phi(x) = \operatorname{sgn}(\tilde{F}(x))\chi_{\mathcal{A}}$. By (40), we have

$$|G_n(x)\phi(x)| \le b_1(|V_n| + \delta(|\nabla V_n| + \frac{a_\delta(x)}{n}))\chi_{\mathcal{A}}(x)$$

and

$$\|G_n\phi\|_2 \le a(\|V_n\chi_A\|_2 + \delta(1 + \frac{\|a_\delta\chi_A\|_2}{n})).$$

Since $V_n \to V$ in $(L^2(\Omega))^n$, we have $V_n\chi_{\mathcal{A}} \to 0$ in $(L^2(\Omega))^n$. Passing to the limit, we obtain

$$\limsup \|G_n\phi\|_2 \le \delta b_1.$$

As δ is arbitrary, it follows that

$$G_n\phi \to 0 \quad \text{in } (L^2(\Omega))^n.$$

On the other hand, (42) implies

$$\int_\Omega G_n.\phi \to \int_\Omega \tilde{F}.\phi = \int_\Omega |\tilde{F}(x)|\chi_{\mathcal{A}}(x).$$

So $\int_{\mathcal{A}} |\tilde{F}(x)| = 0$, which completes the proof of step 2.

Now, we define the function

$$D(x) = \begin{cases} \frac{\tilde{F}(x).V(x)}{|V(x)|^2} & \text{a.e. } x \in \Omega \setminus \mathcal{A}, \\ \alpha & \text{a.e. } x \in \mathcal{A}. \end{cases}$$

Step 3. $\alpha_1 \le D(x) \le \alpha_2$ a.e. $x \in \Omega$.

First, we prove that $\alpha_1 \le \frac{\tilde{F}(x).V(x)}{|V(x)|^2}$ a.e. $x \in \Omega \setminus \mathcal{A}$. then analogously we prove that $\frac{\tilde{F}(x).V(x)}{|V(x)|^2} \le \alpha_2$ a.e. $x \in \Omega \setminus \mathcal{A}$).

Set $B = \{x \in \Omega \setminus \mathcal{A} : \alpha_1|V(x)|^2 > \tilde{F}(x).V(x) \text{ a.e.}\}$. It is sufficient to show that $\operatorname{meas} B = 0$. Indeed, the assumption (39) yields

$$\alpha_1|U_n|^2 - \delta(|\nabla U_n| + a_\delta(x))|U_n| \le U_n.F(x, U_n, \nabla U_n) - (\beta\nabla U_n).U_n, \tag{43}$$

dividing by $\|U_n\|_{1.2}^2$, we obtain

$$\alpha_1|V_n|^2 - \delta(|\nabla V_n| + \frac{a_\delta(x)}{n})|V_n| \le V_n.G_n(x).$$

Multiplying (43) by χ_B and integrating over Ω, we have

$$\begin{aligned} &\alpha_1 \int_\Omega |V_n|^2\chi_B \\ &\le \delta\int_\Omega (|\nabla V_n| + \frac{a_\delta(x)}{n})|V_n|\chi_B + \int_\Omega V_n.G_n(x)\chi_B \\ &\le \int_\Omega V_n.G_n\chi_B + \delta\Big((\int_\Omega |\nabla V_n|^2)^{1/2}(\int_\Omega |V_n|^2)^{1/2} + \frac{\|a_\delta\|_2}{n}\|V_n\|_2\Big) \\ &\le \int_\Omega V_n.G_n(x)\chi_B + \delta\Big(\frac{1}{\lambda_1^{1/2}} + \frac{\|a_\delta\|_2}{n\lambda_1^{1/2}}\Big). \end{aligned}$$

Passing to the limit (Knowing that $G_n \rightharpoonup \tilde{F}$ and $V_n \to V$ in $(L^2(\Omega))^n$), we get

$$\alpha_1 \int_\Omega |V(x)|^2 \chi_B \le \int_\Omega V(x).\tilde{F}(x)\chi_B + \frac{\delta}{\lambda_1^{1/2}},$$

for all $\delta > 0$. Then

$$\int_\Omega \Big(V(x).\tilde{F}(x) - \alpha_1 |V(x)|^2 \Big) \chi_B dx \ge 0.$$

Finally, by the definition of B, we deduce that $\operatorname{meas} B = 0$, and the proof of step 3 concludes.

It is clear that we can suppose that $t_n \to t$. Set $m(x) = tD(x) + (1-t)\alpha$.

Step 4. **1)** The function V is a solution of

$$\begin{gathered} -\overrightarrow{\Delta}^\beta U = e^{\beta.x} AU + e^{\beta.x} m(x) U \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

2) $\alpha_1 \le m(x) \le \alpha_2$ a.e. $x \in \Omega$.

To prove 1), we dividing (41) by $n = \|U_n\|_{1,2}$. Then

$$-\overrightarrow{\Delta}^\beta V_n = e^{\beta.x} AV_n + e^{\beta.x}(t_n G_n(x) + (1-t_n)\alpha V_n). \tag{44}$$

Since $V_n \rightharpoonup V$ in $(H_0^1(\Omega))^n$,

$$\int_\Omega e^{\beta.x} \nabla V_n.\nabla\Phi \to \int_\Omega e^{\beta.x} \nabla V.\nabla\Phi \quad \text{for all } \Phi \in (H_0^1(\Omega))^n.$$

On the other hand, multiplying (44) by $\Phi \in (H_0^1(\Omega))^n$, as $n \to +\infty$ we obtain

$$\begin{aligned} \int_\Omega e^{\beta.x} \nabla V.\nabla\Phi &= \int_\Omega AV.\Phi + \int_\Omega e^{\beta.x}(t\tilde{F}(x) + (1-t)\alpha V(x)).\Phi, \\ &= \int_\Omega e^{\beta.x} AV.\Phi + \int_\Omega e^{\beta.x} \Big(t \frac{\tilde{F}(x).V(x)}{|V(x)|^2} + (1-t)\alpha \Big) V(x).\Phi \\ &= \int_\Omega e^{\beta.x} AV.\Phi + \int_\Omega e^{\beta.x} \left(tD(x) + (1-t)\alpha \right) V(x).\Phi \end{aligned}$$

From the second step and the definition of $D(x)$ it follows that

$$-\overrightarrow{\Delta}^\beta V = e^{\beta.x} AV + e^{\beta.x} m(x) V \quad \text{in } (H^{-1}(\Omega))^n.$$

Then assertion 1) follows.

To prove 2), we combine the result of step 3 and the fact that $\alpha_1 < \alpha < \alpha_2$.

Step 5. $V \not\equiv 0$.

To prove this statement, we multiplying (44) by V_n. Then

$$\int_\Omega e^{\beta.x}(t_n G_n(x).V_n + (1-t_n)\alpha|V_n|^2) + \int_\Omega e^{\beta.x} AV_n.V_n = \int_\Omega e^{\beta.x}|\nabla V_n|^2 \geq M,$$

where $M = \min_{\overline{\Omega}} e^{\beta.x} > 0$. Passing to the limit, we get

$$\int_\Omega e^{\beta.x}(t\tilde{F}(x) + (1-t)\alpha|V(x)|^2) + \int_\Omega e^{\beta.x} AV.V \geq M > 0.$$

This which completes the proof of step 5.

Finally, from the step 4 and step 5, we conclude that $(\beta, 1)$ is a first order eigenvalue of the problem, with

$$\Lambda_k(\beta, A, 1) < \alpha_1 \leq m(x) \leq \alpha_2 < \Lambda_{k+1}(\beta, A, 1).$$

By the strict monotonicity with respect to weight (see proposition 2.), we have

$$\Lambda_k(\beta, A, m) < 1 < \Lambda_{k+1}(\beta, A, m),$$

which is absurd, and present proof is complete. $\square$

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