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Global bifurcation result for the p-biharmonic operator *

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Abstract

We prove that the nonlinear eigenvalue problem for the p-biharmonic operator with $p > 1$, and Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary, has principal positive eigenvalue λ_1 which is simple and isolated. The corresponding eigenfunction is positive in Ω and satisfies $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$, $\Delta u_1 < 0$ in Ω. We also prove that $(\lambda_1, 0)$ is the point of global bifurcation for associated nonhomogeneous problem. In the case $N = 1$ we give a description of all eigenvalues and associated eigenfunctions. Every such an eigenvalue is then the point of global bifurcation.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. For $p \in (1, +\infty)$ consider the nonlinear eigenvalue problem

$$\begin{gathered} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

In this paper we prove that (1.1) has a *principal positive eigenvalue* $\lambda_1 = \lambda_1(p)$ which is *simple* and *isolated*. Moreover, we prove that there exists *strictly positive eigenfunction* $u_1 = u_1(p)$ in Ω associated with $\lambda_1(p)$ and satisfying $\frac{\partial u_1}{\partial n} < 0$ on $\partial\Omega$. We also study the dependence of $\lambda_1(p)$ on p and show that $p \mapsto \lambda_1(p)$ is a *continuous* function in $(1, +\infty)$. Making use of this result we prove that $\lambda_1(p)$ is a *bifurcation point* of

$$\begin{gathered} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u + g(x, \lambda, u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

from which a global continuum of nontrivial solutions emanates.

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In *one dimensional case* ($N = 1, \Omega = (0,1)$) we obtain a *complete characterization* of the *spectrum* of the eigenvalue problem

$$\begin{aligned} (|u''|^{p-2}u'')'' &= \lambda |u|^{p-2}u \quad \text{in } (0,1) \\ u(0) = u''(0) &= u(1) = u''(1) = 0. \end{aligned} \tag{1.3}$$

We prove that the spectrum of (1.3) consists of a sequence of simple eigenvalues $0 < \lambda_1 < \ldots < \lambda_n < \ldots \to +\infty$. The eigenfunction u_n associated with $\lambda_n (n \geq 2)$ has precisely n bumps in $(0,1)$ and it is reproduced from u_1 by using the symmetry of (1.3). As a simple consequence we then obtain that any λ_n is a global bifurcation point of

$$\begin{aligned} (|u''|^{p-2}u'')'' &= \lambda |u|^{p-2}u + g(t,\lambda,u) \quad \text{in } (0,1) \\ u(0) = u''(0) &= u(1) = u''(1) = 0. \end{aligned} \tag{1.4}$$

Our main results are stated in the following theorems.

Theorem 1.1 *The problem (1.1) has the least positive eigenvalue $\lambda_1(p)$ which is simple and isolated in the sense that the set of all solutions of (1.1) with $\lambda = \lambda_1(p)$ forms a one dimensional linear space spanned by a positive eigenfunction $u_1(p)$ associated with $\lambda_1(p)$ such that $\Delta u_1(p) < 0$ in Ω and $\frac{\partial u_1(p)}{\partial n} < 0$ on $\partial\Omega$ and that there exists a positive number δ so that $(\lambda_1(p), \lambda_1(p) + \delta)$ does not contain any eigenvalues of $(\mathrm{E_N})_\mathrm{p}$. Moreover, (1.1) has a positive solution if and only if $\lambda = \lambda_1$ and the function $p \mapsto \lambda_1(p)$ is continuous.*

Theorem 1.2 *Let $p > 1$ be fixed and the function $g = g(x,\lambda,s)$, $g(x,\lambda,0) = 0$, represents higher order terms in (1.2) (see Section 4 for precise assumptions). Then there exists a continuum of nontrivial solutions (λ, u) of (1.2) bifurcating from $(\lambda_1(p), 0)$ which is either unbounded or meets the point $(\lambda_e(p), 0)$, where $\lambda_e(p) > \lambda_1(p)$ is some eigenvalue of (1.1).*

Theorem 1.3 *The set of all eigenvalues of (1.3) is formed by a sequence*

$$0 < \lambda_1(p) < \lambda_2(p) < \ldots < \lambda_n(p) < \ldots \to +\infty.$$

For any $n = 1, 2, \ldots,$ the function $p \mapsto \lambda_n(p)$ is continuous. Every $\lambda_n(p)$ is simple and the corresponding one dimensional space of solutions of (1.3) with $\lambda = \lambda_n(p)$ is spanned by a function having precisely n bumps in $(0,1)$. Each n-bump solution is constructed by the reflection and compression of the eigenfunction $u_1(p)$ associated with $\lambda_1(p)$.

Theorem 1.4 *Let $p > 1$ be fixed and $g = g(t,\lambda,s), g(t,\lambda,0) = 0$, represents higher order terms in (1.4) (see Section 5 for precise assumptions). Then for every $n = 1, 2, \ldots$ there exists a continuum of nontrivial solutions (λ, u) of (1.4) bifurcating from $(\lambda_n(p), 0)$ which is either unbounded or meets the point $(\lambda_k(p), 0)$, with $k \neq n$.*

The paper is organized as follows. In Section 2 we define the notion of the solution, and prepare some auxiliary results. Section 3 contains the proof of Theorem 1.1. The essential part of it relies on the abstract result of Idogawa and Ôtani [7] and the verification of its assumptions. In Section 4 we prove the bifurcation result stated in Theorem 1.2 using the degree argument and the well-known result of Rabinowitz [R]. The last Section 5 is devoted to the one dimensional case and Theorems 1.3, 1.4 are proved there.

2 Auxiliaries

For $p > 1$ we define the function $\psi_p : \mathbb{R} \to \mathbb{R}$ by $\psi_p(s) = |s|^{p-2}s, s \neq 0$ and $\psi_p(0) = 0$. Denoting $p' = \frac{p}{p-1}$ we immediately obtain that $z = \psi_p(s)$ if and only if $s = \psi_{p'}(z)$. The eigenvalue problem (1.1) can be thus written in the form

$$\begin{aligned} \Delta\psi_p(\Delta u) = \lambda\psi_p(u) \quad \text{in } \Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Before we define the weak solution to (2.1) we recall some properties of the Dirichlet problem for Poisson equation:

$$\begin{aligned} -\Delta w = f \quad \text{in } \Omega \\ w = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

It is well known that (2.2) is uniquely solvable in $L^p(\Omega)$ for any $p \in (1, \infty)$ and that the linear solution operator $\Lambda : L^p(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\Lambda f = w$, has the properties stated in the following lemma, (see, e.g., [6]).

Lemma 2.1 *(i) (Continuity) There exists a constant $c_p > 0$ such that*

$$\|\Lambda f\|_{W^{2,p}} \leq c_p\|f\|_{L^p}$$

holds for all $p \in (1, \infty)$ and $f \in L^p(\Omega)$.

(ii) (Continuity) Given $k \geq 1, k \in \mathbb{N}$, there exists a constant $c_{p,k} > 0$ such that

$$\|\Lambda f\|_{W^{k+2,p}} \leq c_{p,k}\|f\|_{W^{k,p}}$$

holds for all $p \in (1, \infty)$ and $f \in W^{k,p}(\Omega)$.

(iii) (Symmetry) The following identity

$$\int_\Omega \Lambda u \cdot v dx = \int_\Omega u \cdot \Lambda v dx$$

holds for all $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $p \in (1, \infty)$.

(iv) (Regularity) Given $f \in L^\infty(\Omega)$, we have $\Lambda f \in C^{1,\alpha}(\bar{\Omega})$ for all $\alpha \in (0, 1)$; moreover, there exist $c_\alpha > 0$ such that

$$\|\Lambda f\|_{C^{1,\alpha}} \leq c_\alpha\|f\|_{L^\infty}.$$

(v) *(Regularity and Hopf-type maximum principle) Let $f \in C(\bar{\Omega})$ and $f \geq 0$, then $w = \Lambda f \in C^{1,\alpha}(\bar{\Omega})$, for all $\alpha \in (0,1)$ and w satisfies: $w > 0$ in Ω, $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$.*

(vi) *(Order preserving property) Given $f, g \in L^p(\Omega), f \leq g$ in Ω, we have $\Lambda f < \Lambda g$ in Ω.*

Let us denote $v := -\Delta u$ in (1.1). Then the problem (1.1) can be restated as an operator equation

$$\psi_p(v) = \lambda \Lambda \psi_p(\Lambda v) \quad \text{in } \Omega \tag{2.3}$$

or as

$$v = \lambda^{\frac{1}{(p-1)}} \psi_{p'}(\Lambda \psi_p(\Lambda v)) \quad \text{in } \Omega. \tag{2.4}$$

Indeed, let us assume that $v \in L^p(\Omega)$ solves (2.3). Then from Lemma 2.1 (i) and the properties of the Nemytskii operator induced by ψ_p we obtain:

$$\begin{aligned} & u = \Lambda v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \Rightarrow \psi_p(\Lambda v) \in L^{p'}(\Omega) \Rightarrow \\ & \Rightarrow \Lambda \psi_p(\Lambda v) \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \Rightarrow \\ & \Rightarrow \psi_p(v) \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \Rightarrow \\ & \Rightarrow -\Delta \psi_p(-\Delta u) = \lambda \psi_p(u) \text{ holds in } L^{p'}(\Omega). \end{aligned}$$

This enables us to give the following definition of the solution of (1.1).

Definition 2.2 The function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is called *a solution* of (1.1) if $v = -\Delta u$ solves (2.3) in $L^{p'}(\Omega)$. The parameter λ_e is called an *eigenvalue* of (1.1) if there is a nonzero solution u_e of (1.1) with $\lambda = \lambda_e$. The function u_e is then called the *eigenfunction* associated with the eigenvalue λ_e.

Lemma 2.3 *(Duality). Let $\lambda_e = \lambda_e(p) \neq 0$ be the eigenvalue of $(E_N)_p$ and $u_e(p)$ be the eigenfunction associated with λ_e. Define $\lambda_e^{(p')}$ and $u_e(p')$ by $\lambda_e^{1/p}(p) = \lambda_e^{1/p'}(p')$ and $u_e(p') = \lambda_e^{-1}(p)\psi_p(\Delta u_e(p))$. Then $\lambda_e(p')$ becomes an eigenvalue of $(E_N)_{p'}$ with $p' = \frac{p}{p-1}$ and $u_e(p')$ gives the eigenfunction associated with $\lambda_e(p')$.*

Proof. We have

$$\begin{gathered} \Delta \psi_p(\Delta u_e(p)) = \lambda_e(p)\psi_p(u_e(p)) \quad \text{in } \Omega \\ u_e(p) = \Delta u_e(p) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.5}$$

Let $w_p := \psi_p(\Delta u_e(p))$, then $w_p \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. It is easy to see that to solve (2.5) is nothing but to find $(u_e(p), w_p)$ satisfying the system

$$\begin{gathered} \Delta w_p = \lambda_e(p)\psi_p(u_e(p)) \\ \Delta u_e(p) = \psi_{p'}(w_p). \end{gathered} \tag{2.6}$$

Since $u_e(p') = \frac{1}{\lambda_e(p)} w_p \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ satisfies $\psi_{p'}(u_e(p')) = \lambda_e(p)^{1-p'}\psi_{p'}(w_p) = \lambda_e(p')^{-1}\psi_{p'}(w_p)$, we easily find that $(u_e(p'), w_{p'})$ with $w_{p'} = u_e(p)$ solves (2.6) with $p = p'$.

Remark 2.4 The duality proved in the previous lemma enables us to deduce several properties of (1.1) for $p > 2$ from those for $p \in (1,2)$ and vice versa.

The following technical lemma will be useful for the verification of certain abstract assumptions in the next section.

Lemma 2.5 *Let A, B, C and p be real numbers satisfying $A \geq 0, B \geq 0, C \geq \max\{B - A, 0\}$ and $p > 1$. Then*

$$|A + C|^p + |B - C|^p \geq A^p + B^p. \tag{2.7}$$

Proof. If $C = 0$ (i.e, $B \leq A$), then (2.7) is trivial. So it suffices to show (2.7) when $B \geq A$. Due to the strict convexity of the function $s \mapsto s^p$, in $(0, +\infty)$ we have

$$\begin{aligned} |A + C|^p &\geq B^p + pB^{p-1}[C - (B - A)], \\ |B - C|^p &\geq A^p - pA^{p-1}[C - (B - A)]. \end{aligned}$$

Adding these inequalities, we derive the assertion. □

3 Eigenvalue problem

Let us define convex functionals $f_p^1, f_p^2 : L^p(\Omega) \to \mathbb{R}$ as follows:

$$f_p^1(v) = \frac{1}{p}\int_\Omega |v|^p dx, \ f_p^2(v) = \frac{1}{p}\int_\Omega |\Lambda v|^p dx.$$

Then it is clear that f_p^1 and f_p^2 are Fréchet differentiable in $L^p(\Omega)$. Since for every Fréchet differentiable convex functional f, its subdifferential ∂f coincides with its Fréchet derivative f', we get that (2.3) is equivalent to

$$\partial f_p^1(v) = \lambda \partial f_p^2(v) \text{ in } L^{p'}(\Omega), \tag{3.1}$$

where ∂f_p^i are the subdifferentials of $f_p^i, (i = 1, 2)$. We are going to verify the hypotheses $(A0), (A0)', (6.1) - (6.10)$ of [7] with $A = \partial f_p^1$, $B = \partial f_p^2$ and $V = L^p(\Omega)$. The assumptions (6.1) (i)–(iii), (6.2) (i)–(iii), (6.3), (6.4) (i) and (6.5) are clearly satisfied. Concerning (6.4) (ii) we should verify that

$$f_p^2(\max\{u, w\}) + f_p^2(\min\{u, w\}) \geq f_p^2(u) + f_p^2(w) \tag{3.2}$$

for any $u, w \in L^p(\Omega)$ satisfying $u \geq 0$ and $w \geq 0$ a.e. in Ω. We have $\max\{u, w\} = u + (w - u)^+$ and $\min\{u, w\} = w - (w - u)^+$. By Lemma 2.1

(vi), the inequality $w-u \le (w-u)^+$ implies $\Lambda(w-u)^+ \ge \Lambda(w-u) = \Lambda w - \Lambda u$. Hence Lemma 2.5 with $A = \Lambda u, B = \Lambda w$ and $C = \Lambda(w-u)^+$ gives

$$\int_\Omega |\Lambda u + \Lambda(w-u)^+|^p dx + \int_\Omega |\Lambda w - \Lambda(w-u)^+|^p dx \ge \int_\Omega |\Lambda u|^p dx + \int_\Omega |\Lambda w|^p dx. \tag{3.3}$$

Then (3.3) implies (3.2). The assumption (6.10) is a consequence of Lemma 2.1 (vi). Hence it remains to verify (A0) and (A0)'.

Lemma 3.1 *Let $v \in L^p(\Omega)$ solve (2.3) in $L^{p'}(\Omega)$. Then $v \in C(\bar\Omega)$.*

Proof. The main part of the proof is to show the following fact:
Suppose, that $v \in L^{p_0}(\Omega)$, then we find that

(i) $v \in L^{p_1}(\Omega)$ with $\frac{1}{p_1} = \frac{1}{p_0} - \frac{p'}{N}$ if $p_0 < \frac{N}{2p'}$

(ii) $v \in C(\bar\Omega)$ if $p_0 > \frac{N}{2p'}$, $p' = \frac{p}{p-1}$.

Let $v \in L^{p_0}(\Omega)$, and $p_0 < \frac{N}{2p}$, then $\Lambda v \in W^{2,p_0}(\Omega)$ by Lemma 2.1(i). Then, by Sobolev's embedding theorem and the property of the Nemytskii operator: $r \mapsto \psi_p(r)$, we get $\Lambda v \in L^{r_0}(\Omega)$ and $\psi_p(\Lambda v) \in L^{\frac{r_0}{p-1}}$ with $r_0 = \frac{Np_0}{N-2p_0}$ Again, by Sobolev's embedding theorem and the property of the Nemytskii operator, we obtain

$$\Lambda\psi_p(\Lambda v) \in W^{2,\frac{ro}{p-1}}(\Omega) \hookrightarrow L^{r_1}(\Omega)$$

and

$$\psi_{p'}(\Lambda\psi_p(\Lambda v)) \in L^{\frac{r_1}{p'-1}}(\Omega) = L^{r_1(p-1)}(\Omega)$$

with $r_1 = \frac{Nr_0}{N(p-1)-2r_0}$. Consequently, (2.4) implies that $v \in L^{p_1}(\Omega)$ with $p_1 = r_1(p-1)$, i.e., $\frac{1}{p_1} = \frac{1}{p_0} - \frac{2p'}{N}$, whence follows assertion (i). If $\frac{N}{2} < p_0$ it is obvious by Sobolev's embedding theorem that $v \in C(\bar\Omega)$. As for the case $\frac{N}{2p'} < p_0 < \frac{N}{2}$ (or $p_0 = \frac{N}{2}$), noting that $W^{2,\frac{r_0}{p-1}}(\Omega) \hookrightarrow C(\bar\Omega)$ (or $W^{2,\frac{r}{p-1}}(\Omega) \hookrightarrow C(\bar\Omega)$ for sufficiently large r) we easily see that $v \in C(\bar\Omega)$. Then assertion (ii) is verified. Now take suitable $p_0 \in (1,p]$ and $k \in \mathbb{N}$ such that

$$p_{k-1} < \frac{N}{2p'} < p_k \text{ with } \frac{1}{p_k} = \frac{1}{p_0} - \frac{2p'}{N}k.$$

Then applying assertion (i) with $p_0 = p_0, p_1 \dots, p_{k-1}$, we deduce $v \in L^{p_k}(\Omega)$. Hence from assertion (ii), $v \in C(\bar\Omega)$ follows. $\square$

Remark 3.2 In particular, it follows from above proof that given bounded sequences $\{p_n\} \subset (1,\infty)$ and $\{\lambda_n\} \subset (0,\infty)$, the sequence of elements v_n solving (2.3) with $\lambda = \lambda_n$ and $p = p_n$ which are normalized by $\|v_n\|_{L^q} = 1, q \in (1,\infty)$, we find a constant $c > 0$ (independent of n) such that

$$\|v_n\|_{L^\infty} \le c.$$

By the same reason, if $\lambda_n \to \lambda_0$ and v_0 solves (2.3) with $\lambda = \lambda_0, \|v_0\|_{L^q} = 1$, the proof of Lemma 3.1 implies that

$$\lim_{n\to\infty} \|v_n - v_0\|_{L^\infty} = 0.$$

Lemma 3.3 *Let $p \geq 2$ and $v \in L^p(\Omega), v \geq 0$ a.e. in Ω, and let v solve (2.3) in $L^{p'}(\Omega)$. Then $v \in C^1(\Omega), v > 0$ everywhere in Ω and $\frac{\partial v}{\partial n} = -\infty$ on $\partial\Omega$.*

Proof. It follows from Lemma 2.1 (v), Lemma 3.1 and (2.3) that $w := \psi_p(v)$ satisfies $w \in C^{1,\alpha}(\bar{\Omega}), \alpha \in (0,1), w > 0$ in Ω and $\frac{\partial w}{\partial n} < 0$ on $\partial\Omega$. This fact assures that $v > 0$ in Ω and $(p-1)|v|^{p-2}\frac{\partial v}{\partial n} < 0$ on $\partial\Omega$. Then $\frac{\partial v}{\partial n} = -\infty$ follows from the fact that $v = 0$ on $\partial\Omega$. $\square$

For $p \geq 2$ the assumption (A0) now follows from Lemma 3.3 while instead of (A0)' we obtain the following property - (A0)": *Every positive solution v of (3.1) satisfies $v \in C^1(\Omega), v = 0$ on $\partial\Omega$ and $\frac{\partial v}{\partial n} = -\infty$ on $\partial\Omega$.*
It is easy to see that the results of [7] remain true even if (A0)' is substituted by (A0)". Applying now the results of [7] we deduce that, for $p \geq 2$,

$$0 < \lambda_1(p) := \left(\sup_{v\in L^p(\Omega)\ v\neq 0} \frac{f_p^2(v)}{f_p^1(v)} \right)^{-1},$$

is the least simple eigenvalue of (3.1) with associated positive eigenfunction $v_1(p), \|v_1(p)\|_{L^p} = 1$ and (3.1) has a positive solution if and only if $\lambda = \lambda_1(p)$. The assertion for $p \in (1,2)$ now follows from Lemma 2.3 and Remark 2.4.

As a consequence of this fact we find that $u_1(p) = \Lambda v_1(p)$ is the corresponding first eigenfunction of $(\mathrm{E_N})_\mathrm{p}$ satisfying $u_1(p) > 0$ in $\Omega, \Delta u_1(p) < 0$ in Ω and $\frac{\partial u_1(p)}{\partial n} < 0$ on $\partial\Omega$ due to Lemma 2.1 (vi). Moreover, if u is another positive solution of $(\mathrm{E_N})_\mathrm{p}$ then $v = -\Delta u > 0$ solves (2.3) in $L^{p'}(\Omega)$. Therefore (2.4) holds with $\Lambda v = u$. Hence according to the above mentioned argument, it holds that $\lambda = \lambda_1(p)$ and $v = v_1(p)$, i.e. $u = u_1(p)$.

Lemma 3.4 *$\lambda_1(p)$ is isolated, i.e. there is $\delta > 0$ such that the interval $(\lambda_1(p), \lambda_1(p) + \delta)$ does not contain any eigenvalue of (3.1).*

Proof. Assume the contrary, i.e., there are sequences $\{\lambda_n\}, \{v_n\}$ such that $\lambda_n \to \lambda_1(p), \|v_n\|_{L^p} = 1$ and that v_n solves (3.1) with $\lambda = \lambda_n$. Then both v_n and Λv_n must change sign in Ω and

$$\lim_{n\to\infty} \|v_n - v_1(p)\|_{L^\infty} = 0$$

according to Remark 3.2. But Lemma 2.1 (iv) implies that $\Lambda v_n \to \Lambda v_1(p)$ in $C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ which leads to a contradiction with the fact that $\Lambda v_1(p) > 0$ in Ω and $\frac{\partial \Lambda v_1(p)}{\partial n} < 0$ on $\partial\Omega$. $\square$

It remains to show the continuity of $p \mapsto \lambda_1(p)$. Let us note first that

$$\lambda_1(p) = \inf \frac{1}{f_p^2(v)},$$

where the infimum is taken over all $v \in L^p(\Omega), \|v\|_{L^p} = p$. It follows from Lemma 2.1 (i) that $\lambda_1(p)$ is bounded uniformly away from zero and infinity for any p belonging to a compact subinterval of $(1,\infty)$. Let $p_n \to p \in (1,\infty)$. Then $\{\lambda_1(p_n)\}$ is a bounded sequence. Denote by $v_1(p_n)$ the positive eigenfunction associated with $\lambda_1(p_n)$ and normalized by

$$\|v_1(p_n)\|_{L^p} = p. \tag{3.4}$$

Extracting a suitable subsequence we can assume that

$$\lambda_1(p_n) \to \lambda_0, v_1(p_n) \rightharpoonup v_0 \text{ in } L^p(\Omega). \tag{3.5}$$

In particular, we derive from (3.5) that $v_0 \geq 0$ a.e. in Ω, and the compactness of Λ (cf. Lemma 2.1 (i)) yields $\Lambda v_1(p_n) \to \Lambda v_0$ in $L^p(\Omega)$. Extracting again to a subsequence we get

$$\Lambda v_1(p_n) \to \Lambda v_0 \text{ a.e in } \Omega. \tag{3.6}$$

It follows from Remark 3.2 and Lemma 2.1 (iv) that there is a constant $c > 0$ independent of n such that

$$|\Lambda v_1(p_n)| \leq c. \tag{3.7}$$

Hence it follows from (3.6), (3.7) and Lemma 2.1 (iv) that

$$\Lambda\psi_{p_n}(\Lambda v_1(p_n)) \to \Lambda\psi_p(\Lambda v_0) \text{ a.e. in } \Omega, \text{ i.e.,}$$

$$\psi_{p_n'}(\Lambda\psi_{p_n}(\Lambda v_1(p_n))) \to \psi_{p'}(\Lambda(\psi_p(\Lambda v_0))) \text{ a.e. in } \Omega. \tag{3.8}$$

Now taking arbitrary $\varphi \in L^{p'}(\Omega)$, it follows from (3.4), (3.5), (3.7), (3.8), Lemma 2.1 (iv) and the Lebesgue dominated convergence theorem that

$$\int_\Omega \lambda_1^{\frac{1}{p_n-1}}(p_n)\psi_{p_n'}(\Lambda\psi_{p_n}(\Lambda v_1(p_n)))\varphi dx \to \int_\Omega \lambda_0^{\frac{1}{p-1}}\psi_{p'}(\Lambda\psi_p(\Lambda v_0))\varphi dx. \tag{3.9}$$

It also follows from (3.5) that

$$\int_\Omega v_1(p_n)\varphi dx \to \int_\Omega v_0\varphi dx. \tag{3.10}$$

So it follows from (2.4), (3.9) and (3.10) that

$$v_0 = \lambda_0^{\frac{1}{p-1}}\psi_{p'}(\Lambda\psi_p(\Lambda v_0)). \tag{3.11}$$

On the other hand (3.6), (3.7) the definition of λ_1 and the Lebesgue dominated convergence theorem imply

$$1 = \lim_{n\to\infty} \lambda_1(p_n)\int_\Omega |\Lambda v_1(p_n)|^{p_n} dx = \lambda_0 \int_\Omega |\Lambda v_0|^p dx,$$

i.e. $v_0 \not\equiv 0$. It follows from here and (3.11) that v_0 is a positive solution of (2.3) with $\lambda = \lambda_0$. According to the first part of Theorem 1.1 (cf.[7]) it must be $\lambda_0 = \lambda_1(p), v_0 = v_1(p)$. Since the above argument does not depend on the choice of subsequences, the continuity of the function

$$p \mapsto \lambda_1(p)$$

is proved. This also completes the proof of Theorem 1.1

4 Global bifurcation result

For $p > 1$ set $X = L^p(\Omega)$. Then $X^* = L^{p'}(\Omega)$ and the Nemytskii operator

$$\Psi_p : v \mapsto \psi_p(v)$$

is one to one mapping between X and X^*.

Lemma 4.1 *Ψ_p satisfies condition (S_+), i.e.*

$$v_n \rightharpoonup v_0 \text{ weakly in } X. \tag{4.1}$$

and

$$\limsup_{n\to\infty} \int_\Omega \psi_p(v_n)(v_n - v_0)dx \le 0 \tag{4.2}$$

imply $v_n \to v_0$ strongly in X.

Proof. The monotonicity of ψ_p, (4.1) and (4.2) imply

$$\begin{aligned}
0 \quad &\ge \quad \limsup_{n\to\infty} \int_\Omega \psi_p(v_n)(v_n - v_0)dx = \\
&= \quad \limsup_{n\to\infty} \int_\Omega (\psi_p(v_n) - \psi_p(v_0))(v_n - v_0)dx \ge \\
&\ge \quad \limsup_{n\to\infty} \left[\left(\int_\Omega |v_n|^p dx \right)^{1/p'} - \left(\int_\Omega |v_0|^p dx \right)^{1/p'} \right] \times \\
&\qquad \times \left[\left(\int_\Omega |v_n|^p dx \right)^{1/p} - \left(\int_\Omega |v_0|^p dx \right)^{1/p} \right] \ge 0
\end{aligned}$$

Hence $\|v_n\|_X \to \|v_0\|_X$, which together with (4.1) yields the desired strong convergence. □

Let the function $g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathèodory function, i.e. $g(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$ and $g(\cdot, \lambda, s)$ is measurable for all $(\lambda, s) \in \mathbb{R}^2$. Moreover, let $g(x, \lambda, 0) = 0$ for any $(x, \lambda) \in \Omega \times \mathbb{R}$ and given any bounded interval $J \subset \mathbb{R}$ we assume that there exists an exponent $q \in (p, p^{**})$ with

$p^{**} = \frac{Np}{N-p}$ (for $N > 2p$); $p^{**} = \infty$ (for $N \le 2p$) such that for any $\varepsilon > 0$, there exists a constant C_ε such that

$$|g(x,\lambda,s)| \le \varepsilon |s|^{p-1} + C_\varepsilon |s|^{q-1} \text{ for a.e. } x \in \Omega \text{ and all } \lambda \in J, s \in \mathbb{R}. \tag{4.3}$$

Note that (1.2) can be written in the equivalent form

$$\psi_p(v) = \lambda \Lambda \psi_p(\Lambda v) + \Lambda g(x,\lambda,\Lambda v). \tag{4.4}$$

Due to (4.3) the right hand side of (4.4) defines an operator

$$T_{\lambda,g} : v \mapsto \lambda \Lambda \psi_p(\Lambda v) + \Lambda g(x,\lambda,\Lambda v)$$

from X into X^* which is compact. Indeed, by Lemma 2.1 (i) we get $\Lambda v \in W^{2,p}(\Omega)$ and $\Lambda \psi_p(\Lambda v) \in W^{2,p'}(\Omega)$. Furthermore by using (4.3) and the fact that $W^{2,p}(\Omega) \subset L^q(\Omega)$, we find that $\Lambda g(x,\lambda,\Lambda v) \in W^{2,q'}(\Omega)$. Thus $T_{\lambda,g}$ maps any bounded set of X onto a bounded set of $W^{2,q'}(\Omega)$, which is compactly embedded in X^*, since $q < p^{**}$. Then this fact and Lemma 4.1 imply that $\Psi_p - T_{\lambda,g}$ satisfies condition (S_+). So, given an open and bounded set $D \subset X$ such that $\Psi_p(v) - T_{\lambda,g}(v) \ne 0$ for any $v \in \partial D$, the generalized degree of Browder and Petryshin

$$\mathrm{Deg}[\Psi_p - T_{\lambda,g}; D, 0]$$

is well defined.

Lemma 4.2 $\|\Lambda g(x,\lambda,\Lambda v)\|_{X^*} = o(\|v\|_X^{p-1})$ *as* $\|v\|_X \to 0$.

Proof. Since Λ is symmetric, we have

$$\|\Lambda g(x,\lambda,\Lambda v)\|_{X^*} = \sup_{\|\varphi\|_X \le 1} \int_\Omega \Lambda g(x,\lambda,\Lambda v)\varphi dx = \sup_{\|\varphi\|_X \le 1} \int_\Omega g(x,\lambda,\Lambda v)\Lambda\varphi dx. \tag{4.5}$$

Then, for any $\varepsilon > 0$, by virtue of (4.3) and Lemma 2.1 (i), we find

$$\begin{aligned} \Big| \int_\Omega g(x,\lambda,\Lambda v)\Lambda\varphi dx \Big| &\le \int_\Omega \varepsilon |\Lambda v|^{p-1} |\Lambda\varphi| dx + \int_\Omega C_\varepsilon |\Lambda v|^{q-1} |\Lambda \varphi| dx \\ &\le \varepsilon \|\Lambda v\|_{L^p}^{p-1} \|\Lambda\varphi\|_{L^p} + C_\varepsilon \|\Lambda v\|_{L^q}^{q-1} \|\Lambda\varphi\|_{L^q} \\ &\le \varepsilon c_p^p \|v\|_X^{p-1} \|\varphi\|_X + C_\varepsilon c^q \|\Lambda v\|_{W^{2,p}}^{q-1} \|\Lambda\varphi\|_{W^{2,p}} \\ &\le \varepsilon c_p^p \|v\|_X^{p-1} + C_\varepsilon c^q c_p^q \|v\|_X^{q-1}, \end{aligned} \tag{4.6}$$

where c_p is the constant appearing in Lemma 2.1 (i) and $c > 0$ is the embedding constant for $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$. Thus the assertion follows from (4.5) and (4.6), since $p < q$. $\square$

Let $\delta > 0$ be as in Lemma 3.4 and consider $\lambda < \lambda_1(p) + \delta, \lambda \ne \lambda_1(p)$. Then Lemma 4.2 and simple homotopy argument yields

$$\mathrm{Deg}[\Psi_p - T_{\lambda,g}; B_r(0), 0] = \mathrm{Deg}[\Psi_p - T_{\lambda,0}; B_\lambda(0), 0] \tag{4.7}$$

if $r > 0$ is chosen sufficiently small (cf. [4], [5], [2], [3] or [R]). Here $B_r(0)$ is the ball centred at the origin and with radius $r > 0$.

Lemma 4.3 $\mathrm{Deg}[\Psi_p - T_{\lambda,0}; B_r(0), 0] = \pm 1$ *for* $\lambda < \lambda_1(p) + \delta, \lambda \neq \lambda_1(p)$ *and* $sgn(\lambda_1(p) - \lambda) = \pm 1$.

Proof. To prove the "jump" of the degree we adopt the method developed in [5] (see also [4]). Thus we just sketch the proof and refer to [DKN, Theorem 3.7] or [D, Theorem 14.18] for the details. Consider the functional

$$F_\lambda(v) = \frac{1}{p}\int_\Omega |v|^p dx - \frac{\lambda}{p}\int_\Omega |\Lambda v|^p dx.$$

It follows from the variational characterization of $\lambda_1(p)$ (see Section 3) that for $\lambda < \lambda_1(p)$ we have

$$\langle F'_\lambda(v), v\rangle_X > 0$$

for $v \in \partial B_r(0)$ and $v = 0$ is the only critical point of F_λ (here $\langle\cdot,\cdot\rangle_X$ denotes the duality between X^* and X) and hence

$$\mathrm{Deg}[\Psi_p - T_{\lambda,0}; B_r(0), 0] = 1 \tag{4.8}$$

by the properties of the degree (cf.[9]). Let now $\lambda \in (\lambda_1(p), \lambda_1(p) + \delta)$. As in (DKN, Theorem 3.7] we define a function $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(t) = \begin{cases} 0, & \text{for } t < K, \\ \frac{2\delta}{\lambda_1(p)}(t - 2K), & \text{for } t \geq 3K, \end{cases}$$

The function $\eta(t)$ is continuously differentiable, positive and strictly convex in $(K, 3K), K > 0$. Let us modify F_λ as follows

$$\tilde{F}_\lambda(v) := F_\lambda(v) + \eta(\frac{1}{p}\int_\Omega |v|^p dx).$$

The properties of $\lambda_1(p)$ stated in Theorem 1.1 now imply the following properties of $\tilde{F}_\lambda$:

- $\tilde{F}_\lambda$ is continuously Fréchet differentiable and its critical point $v_0 \in X$ corresponds to a solution of the equation

$$\psi_p(v_0) - \frac{\lambda}{1 + \eta'(\frac{1}{p}\int_\Omega |v_0|^p dx)}\Lambda\psi_p(\Lambda v_0) = 0.$$

- For $\lambda \in (\lambda_1(p), \lambda_1(p) + \delta)$ the only nontrivial critical points of $\tilde{F}_\lambda$ occur if

$$\eta'\left(\frac{1}{p}\int_\Omega |v_0|^p dx\right) = \frac{\lambda}{\lambda_1(p)} - 1.$$

- Due to the definition of η we then have

$$\frac{1}{p}\int_\Omega |v_0|^p dx \in (K, 3K)$$

  and due to the simplicity of $\lambda_1(p)$, either $v_0 = -tv_1(p)$ or $v_0 = tv_1(p)$, for some $t \in ((pK)^{1/p}, (3pK)^{1/p}), v_1(p)$ as in the Section 3.

- $\tilde{F}_\lambda$ has precisely three isolated critical points $-tv_1(p), 0, tv_1(p)$.
- $\tilde{F}_\lambda$ is weakly lower semicontinuous and even.
- $\tilde{F}_\lambda$ is coercive, i.e.
$$\lim_{\|v\|_X \to \infty} \tilde{F}_\lambda(v) = \infty$$
- $-tv_1(p), tv_1(p)$ are the points of the global minimum of $\tilde{F}_\lambda$; 0 is an isolated critical point of "saddle type".
- $\langle \tilde{F}'_\lambda(v), v\rangle_X > 0$ for $\|v\|_X = R$ if $R > 0$ is large enough.

The properties of the degree now imply that for small $\rho > 0$ and large $R > 0$ we have
$$\mathrm{Deg}[\tilde{F}'_\lambda; B_\rho(\pm tv_1(p)), 0] = \mathrm{Deg}[\tilde{F}'_\lambda; B_R(0), 0] = 1\,.$$
The additivity property of the degree then yields for $0 < r < (pK)^{1/p}$,
$$\mathrm{Deg}[\Psi_p - T_{\lambda,0}; B_r(0), 0] = \mathrm{Deg}[\tilde{F}'_\lambda; B_r(0), 0] = -1\,. \tag{4.9}$$
The assertion of Lemma 4.3 follows now from (4.8) and (4.9). $\square$

If we combine (4.7) with Lemma 4.3 we come to the following conclusion: for $r > 0$ sufficiently small
$$\mathrm{Deg}[\Psi_p - T_{\lambda,g}; B_r(0), 0] = \pm 1$$
for $\mathrm{sgn}(\lambda_1(p) - \lambda) = \pm 1$. Following the proof of [R, Theorem 1.3] we prove that continuum of nontrivial solutions $(\lambda, v) \in \mathbb{R} \times X$ of (4.4) bifurcates from $(\lambda_1(p), 0)$ and it is either unbounded in $\mathbb{R} \times X$ or meets the point $(\lambda_e(p), 0)$, where $\lambda_e(p) > \lambda_1(p)$ is an eigenvalue of (3.1). The assertion of Theorem 1.2 now follows from the fact that (λ, u) solves $(\mathrm{BP_N})_\mathrm{p}$ if and only if $(\lambda, -\Delta u)$ solves (4.4).

5 One-dimensional problem

Let $N = 1$ and $\Omega = (0, 1)$. Then $(\mathrm{E_N})_\mathrm{p}$ reduces to (1.3) and obviously the assertions of Theorems 1.1, 1.2 remain true. We point out that $W^{2,p}(0,1) \hookrightarrow\hookrightarrow C^1([0,1])$ in the case $N = 1$, and so $\psi_p(v) \in C^1([0,1])$, $v(0) = v(1) = 0$ for any solution v of (2.3). Hence we do not need Lemmas 3.1 and 3.3 in this case. For the sake of brevity we shall write $\lambda_1 := \lambda_1(p), u_1 := u_1(p)$. It follows from the symmetry of (1.3) and Theorem 1.1 (simplicity of λ_1) that $u_1(t) = u_1(1-t)$ for $t \in [0,1]$, i.e. u_1 is even with respect to $\frac{1}{2}$. Making use of this observation, we give a precise description of all eigenvalues and eigenfunctions of $(\mathrm{E_1})_\mathrm{p}$. Indeed, set

$$u_n(t) = u_1(nt); t \in [0, \frac{1}{n}],$$
$$u_n(t) = -u_1(nt-1), t \in [\frac{1}{n}, \frac{2}{n}],$$
$$\dots$$
$$u_n(t) = (-1)^n u_1(nt-n+1), t \in [\frac{n-1}{n}, 1].$$

Then $u_n = u_n(t), t \in [0,1]$, is an eigenfunction of (1.3) associated with the eigenvalue $\lambda_n = n^{2p}\lambda_1$. On the other hand, let $u = u(t)$ be an eigenfunction of $(\mathrm{E}_1)_\mathrm{p}$ associated with some eigenvalue λ_e. According to Theorem 1.1 it must be $\lambda_e > \lambda_1$ and u changes sign in $(0,1)$. By Lemma A.4 the number of nodes of u in (0,1) is finite. Assume first that $\lambda_e = \lambda_n$, for some $n > 1$. Let us normalize u as follows: $u'(0) = u_n'(0) > 0$. Note that since u and u_n are oscillatory, we must have, according to Lemma A.3, that

$$(\psi_p(u''(t)))'|_{t=0} < 0 \quad \text{and} \quad (\psi_p(u_n''(t)))'|_{t=0} < 0,$$

respectively. Let $(\psi_p(u''(t)))'|_{t=0} = (\psi_p(u_n''(t)))'|_{t=0}$. Then Lemma A.1 implies that $u(t) = u_n(t), t \in [0,1]$. Let $(\psi_p(u''(t)))'|_{t=0} \neq (\psi_p(u_n''(t)))'|_{t=0}$. Then Lemma A.2 implies that $u(1) \neq 0$, a contradiction. Let $\lambda_e \neq \lambda_k$ for any $k \geq 2$. Define

$$\tilde{u}(t) = u_1\left(\left(\frac{\lambda_e}{\lambda_1}\right)^{1/(2p)} t\right), t \in \left[0, \left(\frac{\lambda_1}{\lambda_e}\right)^{1/(2p)}\right],$$
$$\tilde{u}(t) = -u_1\left(\left(\frac{\lambda_e}{\lambda_1}\right)^{1/(2p)} t - 1\right), t \in \left[\left(\frac{\lambda_1}{\lambda_e}\right)^{1/(2p)}, 2\left(\frac{\lambda_1}{\lambda_e}\right)^{1/(2p)}\right], \text{ etc.}$$

Then $\tilde{u}(1)\tilde{u}''(1) < 0$. Let us normalize u as $u'(0) = \tilde{u}'(0) > 0$. Then it follows from Lemma A.2 that $u(1) = u''(1) = 0$ cannot hold at the same time. Thus Theorem 1.3 is proved.

Let $X = C([0,1])$. Let $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying $g(t, \lambda, 0) = 0$ for any $(t, \lambda) \in (0,1) \times \mathbb{R}$ and given any bounded interval $J \subset \mathbb{R}$ we assume that

$$|g(t, \lambda, s)| = o(|s|^{p-1}) \tag{5.1}$$

holds near $s = 0$ uniformly for all $(t, \lambda) \in [0,1] \times J$. Note that $(\mathrm{BP}_1)_\mathrm{p}$ can be written in the equivalent form

$$v = \psi_{p'}(\lambda\Lambda\psi_p(\Lambda v) + \Lambda g(t, \lambda, \Lambda v)). \tag{5.2}$$

Due to Lemma 2.1 (i), the right hand side of (5.2) defines an operator

$$R_{p,\lambda,g} : (p, \lambda, v) \mapsto \psi_{p'}(\lambda\Lambda\psi_p(\Lambda v) + \Lambda g(t, \lambda, \lambda v))$$

which is compact from $(1, \infty) \times \mathbb{R} \times X$ into X. If $I : X \to X$ denotes the identity mapping, the Leray-Schauder degree

$$\deg[I - R_{p,\lambda,g}; D, 0]$$

is well defined for any open bounded set D such that $v - R_{p,\lambda,g}(v) \neq 0$ for $v \in \partial D$.

Lemma 5.1 *Let $\lambda \neq \lambda_n$. Then there is $r > 0$ (sufficiently small) such that*

$$\deg[I - R_{p,\lambda,g}; B_r(0), 0] = \deg[I - R_{p,\lambda,0}; B_r(0), 0]. \tag{5.3}$$

Proof. Standard argument based on (5.1) yields that the homotopy

$$H(\tau, v) = v - \psi_{p'}(\lambda\Lambda\psi_p(\Lambda v) + \tau\Lambda g(t, \lambda, \Lambda v))$$

satisfies $H(\tau, v) \neq 0$ for all $\tau \in [0,1]$ and $v \in \partial B_r(0)$ if $r > 0$ is small enough. So (5.3) follows from the homotopy invariance property of the Leray-Schauder degree. $\square$

Let $\lambda \in (\lambda_n(p), \lambda_{n+1}(p)), n = 0, 1, 2, \dots$, where we set $\lambda_0(p) = -\infty$ and $\lambda_1(p), \lambda_2(p), \dots$ are as above, then we have.

Lemma 5.2 $\deg[I - R_{p,\lambda,0}; B_r(0), 0] = (-1)^n$.

Proof. We follow the idea in [2]. Note that it follows from Theorems 1.1, 1.3 that

$$\lambda_n : p \mapsto \lambda_n(p), n = 1, 2, \dots,$$

are continuous functions on $(1, \infty)$. Assume that $p < 2$. Define $\lambda(q), q \in [p, 2]$, by the following way

$$\begin{aligned}\lambda(q) :=& \frac{\lambda - \lambda_n(p)}{\lambda_{n+1}(p) - \lambda_n(p)} \cdot (\lambda_{n+1}(q) - \lambda_n(q)) + \lambda_n(q), \quad n \geq 1,\\ \lambda(q) :=& \lambda_1(q) - (\lambda_1(p) - \lambda), n = 0.\end{aligned}$$

Then

$$H(q, v) := v - R_{q,\lambda(q),0}(v) = v - \psi_{q'}(\lambda(q)\Lambda\psi_q(\Lambda v))$$

satisfies $H(q, v) \neq 0$ for all $q \in [p, 2]$ and $v \in \partial B_r(0)$. It follows from the homotopy invariance property of the Leray-Schauder degree that

$$\deg[I - R_{p,\lambda,0}; B_r(0), 0] = \deg[I - R_{2,\lambda(2),0}; B_r(0), 0]. \tag{5.4}$$

The same approach but in the interval $[2, p]$ yields to the same conclusion also for $p > 2$. Since $\lambda_n(2) < \lambda(2) < \lambda_{n+1}(2)$, the classical Leray-Schauder index formula implies that

$$\deg[I - R_{2,\lambda(2),0}; B_r(0), 0] = (-1)^n. \tag{5.5}$$

The assertion of lemma follows now from (5.4) and (5.5). $\square$

With Lemmas 5.1 and 5.2 in hand we can follow the proof of [R, Theorem 1.3] to prove that continua of nontrivial solutions $(\lambda, v) \in \mathbb{R} \times X$ of (5.2) bifurcate from $(\lambda_n(p), 0), n = 1, 2 \dots$, and they are either unbounded in $\mathbb{R} \times X$ or meet the point $(\lambda_m(p), 0)$ with $m \neq n$. The assertion of Theorem 1.4 follows from the fact that (λ, u) solves (1.4) if and only if $(\lambda, -\Delta u)$ solves (5.2).

6 Appendix

To justify some statements in Section 5 we present here a brief study of the initial value problem associated with the equation in $(E_1)_p$ with $\lambda > 0$:

$$\begin{aligned} u'' &= \psi_{p'}(w), \quad u(t_0) = \alpha, \quad u'(t_0) = \beta, \\ w'' &= \lambda\psi_p(u), \quad w(t_0) = \gamma, \quad w'(t_0) = \delta\,. \end{aligned} \tag{6.1}$$

By a *solution* of (6.1) we understand a couple of functions (u, w) which are of class C^2 and fulfil the equations and initial conditions in (6.1).

Lemma 6.1 *The solution to (6.1) is locally unique.*

Proof. Without loss of generality we can restrict ourselves to $t_0 = 0$ and $p \in (1, 2)$ (the case $p > 2$ is treated similarly). Local existence is a consequence of the Schauder fixed point theorem. For its uniqueness we have to distinguish among several cases:

(I) $\alpha \neq 0$ implies that both functions $\psi_p(u(t))$ and $\psi_{p'}(w(t))$ are of class C^1 in the neighbourhood of $t = 0$ and so the assertion follows from the classical theory.

(II) $\alpha = 0$, in this case $\psi_p(u(t))$ is not C^1 in $t = 0$.

(II)(i) $\alpha = 0, \beta \neq 0$. Let $(u, w_1), (v, w_2)$ be two solutions of (6.1) in $(0, \varepsilon)$ with some $\varepsilon > 0$. Then

$$\psi_p(u''(t)) - \psi_p(v''(t)) = \lambda \int_0^t (t-\tau)(\psi_p(u(\tau)) - \psi_p(v(\tau)))d\tau. \tag{6.2}$$

By the assumption, $\frac{u(\tau)}{\tau}, \frac{v(\tau)}{\tau}$ lie in the neighbourhood of $\beta \neq 0$ for $\tau \in (0, \varepsilon)$ wiht ε small enough. We thus have $K_1 > 0$ such that

$$\left|\psi_p\left(\frac{u(\tau)}{\tau}\right) - \psi_p\left(\frac{v(\tau)}{\tau}\right)\right| \le K_1 \left|\frac{u(\tau)}{\tau} - \frac{v(\tau)}{\tau}\right|, \tag{6.3}$$

$\tau \in (0, \varepsilon), K_1$ independent of $\varepsilon << 1$. On the other hand there is $K_2 > 0$ such that

$$|\psi_p(u''(t)) - \psi_p(v''(t))| \ge K_2|u''(t) - v''(t)|, \tag{6.4}$$

$t \in (0, \varepsilon)$. Now, it follows from (6.2)–(6.4)

$$K_2|u''(t) - v''(t)| \le \lambda \int_0^t (t-\tau)\tau^{p-1}K_1 \left|\frac{u(\tau)}{\tau} - \frac{v(\tau)}{\tau}\right| d\tau.$$

Taking into account

$$u(\tau) - v(\tau) = \int_0^\tau (\tau - \sigma)(u''(\sigma) - v''(\sigma))d\sigma,$$

we arrive at
$$\|u'' - v''\|_\varepsilon \le \lambda \frac{K_1}{K_2} \varepsilon^{p+2} \|u'' - v''\|_\varepsilon, \tag{6.5}$$
where $\|\cdot\|_\varepsilon$ is the sup norm on $[0,\varepsilon]$. This implies $u = v$ (and thus $w_1 = w_2$) for ε small enough.

(II) (ii) $\alpha = \beta = 0, \gamma \ne 0$ and (iii) $\alpha = \beta = \gamma = 0, \delta \ne 0$. Instead of (6.2) we use the following fact
$$\psi_{p'}(w_1''(t)) - \psi_{p'}(w_2''(t)) = \psi_{p'}(\lambda) \int_0^t (t-\tau)(\psi_{p'}(w_1(\tau)) - \psi_{p'}(w_2(\tau)))d\tau. \tag{6.6}$$
Since $p' > 2$, we have
$$|\psi_{p'}(w_1(\tau)) - \psi_{p'}(w_2(\tau))| \le K_1 |w_1(\tau) - w_2(\tau)|,$$
$\tau \in (0,\varepsilon)$. Hence
$$\Big| \int_0^t (t-\tau)(\psi_{p'}(w_1(\tau)) - \psi_{p'}(w_2(\tau)))d\tau \Big| \le K_1 \varepsilon^2 \|w_1 - w_2\|_\varepsilon. \tag{6.7}$$
It follows from the initial conditions that $\frac{w_i''(t)}{t^{2(p-1)}}$, $i = 1,2$, lie near $\lambda\gamma\psi_p(\frac{1}{2}) \ne 0$ in the case (ii) and $\frac{w_i(t)}{t^{2p-1}}, i = 1,2$, lie near $\lambda\delta\psi_p\left(\frac{1}{p'(p'+1)}\right) \ne 0$ in the case (iii). Hence there exists $K_2 > 0$ such that
$$\left|\psi_{p'}\left(\frac{w_1''(t)}{t^{2(p-1)}}\right) - \psi_{p'}\left(\frac{w_2''(t)}{t^{2(p-1)}}\right)\right| \ge K_2 \left|\frac{w_1''(t)}{t^{2(p-1)}} - \frac{w_2''(t)}{t^{2(p-1)}}\right| \tag{6.8}$$
in the case (ii) and
$$\left|\psi_{p'}\left(\frac{w_1''(t)}{t^{2p-1}}\right) - \psi_{p'}\left(\frac{w_2''(t)}{t^{2p-1}}\right)\right| \ge K_2 \left|\frac{w_1''(t)}{t^{2p-1}} - \frac{w_2''(t)}{t^{2p-1}}\right| \tag{6.9}$$
in the case (iii). Taking into account
$$w_1(t) - w_2(t) = \int_0^t (t-\tau)(w_1''(\tau) - w_2''(\tau))d\tau$$
we derive from (6.6), (6.7), (6.8) and (6.9) that
$$\|w_1 - w_2\|_\varepsilon \le \frac{K_1}{K_2} \psi_{p'}(\lambda) \varepsilon^{2p+2} \|w_1 - w_2\|_\varepsilon$$
in the case (ii) and
$$\|w_1 - w_2\|_\varepsilon \le \frac{K_1}{K_2} \psi_{p'}(\lambda) \varepsilon^{2p+3} \|w_1 - w_2\|_\varepsilon$$
in the case (iii).

(II)(iv) $\alpha = \beta = \gamma = \delta = 0$. In this case (6.1) has always the trivial solution $u_0 = w_0 = 0$. Let (u, w) be a nontrivial solution. Then

$$|\psi_p(u''(t))| \le \lambda \int_0^t (t-\tau)\psi_p(|u(\tau)|)d\tau \le \lambda\varepsilon^2 \|u\|_\varepsilon^{p-1}, t \in (0,\varepsilon),$$

which yields

$$\|u''\|_\varepsilon^{p-1} \le \lambda\varepsilon^{2p}\|u''\|_\varepsilon^{p-1},$$

i.e. $u = w = 0$. This completes the proof. □

Lemma 6.2 *Let (u, w) and $(\tilde{u}, \tilde{w})$ be solutions of (6.1) defined on [0,1], respectively, $u(0) = w(0) = \tilde{u}(0) = \tilde{w}(0) = 0$, $u'(0) = \tilde{u}'(0) > 0$, $w'(0) < \tilde{w}'(0)$. Then $u(t) < \tilde{u}(t)$ and $w(t) < \tilde{w}(t)$ for any $t \in (0, 1]$.*

Proof. Assume that the assertion is not true . Then it follows from Lemma A.1 that there is $t_1 > 0$ such that $u(t_1) = \tilde{u}(t_1)$ and $u(t) < \tilde{u}(t), t \in (0, t_1)$. Simultaneously, the fact that both u and $\tilde{u}$ solve $(E_1)_p$ imply that

$$\begin{aligned} &\int_0^{t_1} (t_1-\tau)\psi_{p'}\Big(\lambda\int_0^\tau (\tau-\sigma)\psi_p(u(\sigma))d\sigma + w'(0)\tau\Big)d\tau \\ &= \int_0^{t_1} (t_1-\tau)\psi_{p'}\Big(\lambda\int_0^\tau (\tau-\sigma)\psi_p(\tilde{u}(\sigma))d\sigma + \tilde{w}'(0)\tau\Big)d\tau \end{aligned}$$

which contradicts the monotone character of the functions ψ_p and $\psi_{p'}$. The same argument applies for w and $\tilde{w}$. □

Lemma 6.3 *Let (u, w) be a nonzero solution of (6.1) defined on $[0, 1]$ and satisfying $u(0) = w(0) = u(1) = w(1) = 0$. Then $u'(0)w'(0) < 0$.*

Proof. Multiply the first (second) equation in (6.1) by $w'(u')$ and add to get

$$u'(x)w'(x) = \frac{|w(x)|^{p'}}{p'} + \lambda\frac{|u(x)|^p}{p} - C \quad \text{for all} \quad x \in [0, 1]. \tag{6.10}$$

Let $x_0 \in (0, 1)$ be the point satisfying

$$|u(x_0)| = \max_{x\in[0,1]} |u(x)| > 0\,.$$

Then (6.10) implies

$$0 = \frac{|w(x_0)|^{p'}}{p'} + \lambda\frac{|u(x_0)|^p}{p} - C,$$

i.e. $C > 0$. Hence $u'(0)w'(0) < 0$ by (6.10). □

Lemma 6.4 *Let us assume the same as in the previous lemma. Then u (and also w) changes sign in $(0, 1)$ at most finitely many times.*

Proof. Let u have an infinite number of bumps in $(0,1)$. Then there exist sequences x_n, y_n such that $u(x_n) = u'(y_n) = 0, x_n \to x_0, y_n \to x_0, x_n, y_n, x_0 \in [0,1]$. Then $u(x_0) = u'(x_0) = 0$, hence (6.10) gives

$$0 = \frac{|w(x_0)|^{p'}}{p'} - C.$$

Since $C > 0$, we have

$$w(x_0) > 0 \quad \text{or} \quad w(x_0) < 0.$$

Due to

$$u'(x) = \int_{x_0}^{x} \psi_{p'}(w(y))dy + \psi_{p'}(w(x_0)),$$

the function $u'(x)$ should be of definite sign in a neighbourhood of $x = x_0$, which contradicts the observation that $u'(y_n) = 0, y_n \to x_0$. $\square$

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