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On periodic solutions of superquadratic Hamiltonian systems *

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Abstract

We study the existence of periodic solutions for some Hamiltonian systems $\dot{z} = JH_z(t,z)$ under new superquadratic conditions which cover the case $H(t,z) = |z|^2(\ln(1+|z|^p))^q$ with $p, q > 1$. By using the linking theorem, we obtain some new results.

1 Introduction

We consider the superquadratic Hamiltonian system

$$\dot{z} = JH_z(t,z) \tag{1.1}$$

where $H \in C^1([0,1] \times \mathbb{R}^{2N}, \mathbb{R})$ is a 1-periodic function in t, $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard $2N \times 2N$ symplectic matrix, and

$$\frac{H(t,z)}{|z|^2} \to +\infty \text{ as } |z| \to +\infty \text{ uniformly in } t. \tag{1.2}$$

We assume H satisfies the following conditions.

(H1) $H(t,z) \geq 0$, for all $(t,z) \in [0,1] \times \mathbb{R}^{2N}$.

(H2) $H(t,z) = o(|z|^2)$ as $|z| \to 0$ uniformly in t.

In [12], Rabinowitz established the existence of periodic solutions for (1.1) under the following superquadratic condition: there exist $\mu > 0$ and $r_1 > 0$ such that for all $|z| \geq r_1$ and $t \in [0,1]$

$$0 < \mu H(t,z) \leq z \cdot H_z(t,z). \tag{1.3}$$

Since then, the condition (1.3) has been used extensively in the literature; see [1-14] and the references therein.

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It is easy to see that (1.3) does not include some superquadratic nonlinearity like

$$H(t,z) = |z|^2(\ln(1+|z|^p))^q, \quad p,q>1. \tag{1.4}$$

In this paper, we shall study the periodic solutions of (1.1) under some superquadratic conditons which cover the cases like (1.4). We assume H satisfies the following condition.

(H3) There exist constants $\beta > 1$, $1 < \lambda < 1 + \frac{\beta-1}{\beta}$, $c_1, c_2 > 0$ and $L > 0$ such that

$$\begin{gathered} z \cdot H_z(t,z) - 2H(t,z) \geq c_1|z|^\beta, \quad \forall |z| \geq L, \ \forall t \in [0,1]; \\ |H_z(t,z)| \leq c_2|z|^\lambda, \quad \forall |z| \geq L, \ \forall t \in [0,1]. \end{gathered}$$

Theorem 1.1 *Suppose $H \in C^1([0,1]\times \mathbb{R}^{2N}, \mathbb{R})$ is 1-periodic in t and satisfies (1.2), (H1)–(H3). Then (1.1) possesses a nonconstant 1-periodic solution.*

A straightforward computation shows that if H satisfies (1.4), for any $T > 0$, the system (1.1) has a nonconstant T-periodic solution with minimal period T. One can see Remark 2.2 and Corollary 2.3 for more examples.

For the second order Hamiltonian system

$$\begin{gathered} \ddot{u}(t) + V'(t,u(t)) = 0, \\ u(0) - u(1) = \dot{u}(0) - \dot{u}(1) = 0 \end{gathered} \tag{1.5}$$

we have a similar result.

Theorem 1.2 *Suppose $V \in C^1([0,1]\times \mathbb{R}^N, \mathbb{R})$ is 1-periodic in t and satisfies*

(V1) $V(t,x) \geq 0$, for all $(t,x) \in [0,1]\times\mathbb{R}^N$

(V2) $V(t,x) = o(|x|^2)$ as $|x| \to 0$ uniformly in t

(V3) $V(t,x)/|x|^2 \to +\infty$ as $|x| \to +\infty$ uniformly in t

(V4) There exist constants $1 < \lambda \leq \beta$, $d_1, d_2 > 0$ and $L > 0$ such that

$$x \cdot V'(t,x) - 2V(t,x) \geq d_1|x|^\beta, \quad \forall |x| \geq L, \ \forall t \in [0,1];$$

$$|V'(t,x)| \leq d_2|x|^\lambda, \quad \forall |x| \geq L, \ \forall t \in [0,1]. \tag{1.6}$$

$$(\text{ or } V(t,x) \leq d_2|x|^{\lambda+1}, \quad \forall |x| \geq L, \ \forall t \in [0,1]). \tag{1.7}$$

Then (1.5) possesses a nonconstant 1-periodic solution.

We shall use the linking theorem [13, Theorem 5.29] to prove our results. The idea comes from [11, 12, 13]. Theorem 1.1 is proved in Section 2 while the proof of Theorem 1.2 is carried out in Section 3.

2 First order Hamiltonian system

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $E = W^{1/2,2}(S^1, \mathbb{R}^{2N})$. Then E is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. We define

$$\langle Ax, y\rangle = \int_0^1 (-J\dot{x}, y)\, dt, \quad \forall x, y \in E; \tag{2.1}$$

$$f(z) = \frac{1}{2}\langle Az, z\rangle - \int_0^1 H(t,z)\, dt, \quad \forall z \in E. \tag{2.2}$$

Then A is a bounded selfadjoint operator and $\ker A = \mathbb{R}^{2N}$. (H1)–(H3) imply that

$$|H(t,z)| \le a_1 + a_2|z|^{\lambda+1}, \quad \forall z \in \mathbb{R}^{2N}.$$

This implies that $f \in C^1(E,\mathbb{R})$ and looking for the solutions of (1.1) is equivalent to looking for the critical points of f [12, 13]. Let $E^0 = \ker(A)$, $E^+ =$ positive definite subspace of A, and $E^- =$ negative definite subspace of A. Then $E = E^0 \oplus E^- \oplus E^+$.

Lemma 2.1 *Under the conditions of Theorem 1.1, f satisfies the (PS) condition.*

Proof. Let $\{z_m\}$ be a (PS)-sequence, i.e.,

$$|f(z_m)| \le M; \quad f'(z_m) \to 0 \quad \text{as } m \to \infty.$$

We want to show that $\{z_m\}$ is bounded. Then by a standard argument, $\{z_m\}$ has a convergent subsequence [13]. Suppose $\{z_m\}$ is not bounded, then passing to a subsequence if necessary, $\|z_m\| \to +\infty$ as $m \to +\infty$. By (H3), there exists $C_3 > 0$ such that for all $z \in \mathbb{R}^{2N}$, $t \in [0,1]$

$$z \cdot H_z(t,z) - 2H(t,z) \ge C_1|z|^\beta - C_3.$$

Therefore, we have

$$\begin{aligned} 2f(z_m) - \langle f'(z_m), z_m\rangle &= \int_0^1 [z_m \cdot H_z(t,z_m) - 2H(t,z_m)]dt \\ &\ge \int_0^1 [C_1|z_m|^\beta - C_3]dt = C_1\int_0^1 |z_m|^\beta dt - C_3. \end{aligned}$$

This implies

$$\frac{\int_0^1 |z_m|^\beta dt}{\| z_m \|} \to 0 \quad \text{as } m \to \infty. \tag{2.3}$$

Note that from (H3), $1 < \lambda < 1 + \frac{\beta-1}{\beta}$. Let $\alpha = \frac{\beta-1}{\beta(\lambda-1)}$. Then

$$\alpha > 1, \quad \alpha\lambda - 1 = \alpha - \frac{1}{\beta}. \tag{2.4}$$

By (H3), there exists $C_4 > 0$ such that

$$|H_z(t,z)|^\alpha \le C_2^\alpha |z|^{\lambda\alpha} + C_4, \quad \forall (t,z) \in [0,1] \times \mathbb{R}^{2N}. \tag{2.5}$$

Denote $z_m = z_m^+ + z_m^- + z_m^0 \in E^+ \oplus E^- \oplus E^0$. We have

$$\begin{aligned}
\langle f'(z_m), z_m^+ \rangle &= \langle Az_m^+, z_m^+ \rangle - \int_0^1 [H_z(t,z_m) \cdot z_m^+] dt \\
&\ge \langle Az_m^+, z_m^+ \rangle - \int_0^1 |H_z(t,z_m)||z_m^+| dt \\
&\ge \langle Az_m^+, z_m^+ \rangle - (\int_0^1 |H_z(t,z_m)|^\alpha)^{\frac{1}{\alpha}} \cdot C_\alpha \|z_m^+\|,
\end{aligned} \tag{2.6}$$

where $C_\alpha > 0$ is a constant independent of m. By (2.5),

$$\begin{aligned}
\int_0^1 |H_z(t,z_m)|^\alpha dt &\le \int_0^1 (C_2^\alpha |z_m|^{\lambda\alpha} + C_4) dt \\
&\le C_5 (\int_0^1 |z_m|^\beta dt)^{1/\beta} (\int_0^1 |z_m|^{(\alpha\lambda - 1)\cdot\frac{\beta}{\beta-1}} dt)^{1-\frac{1}{\beta}} + C_4 \\
&\le C_6 (\int_0^1 |z_m|^\beta)^{1/\beta} \|z_m\|^{(\alpha\lambda-1)} + C_4.
\end{aligned}$$

Combining this inequality with (2.3) and (2.4) yields that

$$\frac{(\int_0^1 |H_z(t,z_m)|^\alpha dt)^{\frac{1}{\alpha}}}{\|z_m\|} \le [\frac{C_6 (\int_0^1 |z_m|^\beta dt)^{1/\beta}}{\|z_m\|^{1/\beta}} \cdot \frac{\|z_m\|^{(\alpha\lambda-1)}}{\|z_m\|^{\alpha - \frac{1}{\beta}}} + \frac{C_4}{\|z_m\|^\alpha}]^{\frac{1}{\alpha}} \to 0$$

as $m \to \infty$. By (2.6) we have

$$\frac{\langle Az_m^+, z_m^+ \rangle}{\|z_m\| \ \|z_m^+\|} \le \frac{\|f'(z_m)\| \ \|z_m^+\|}{\|z_m\| \ \|z_m^+\|} + \frac{(\int_0^1 |H_z(t,z_m)|^\alpha dt)^{\frac{1}{\alpha}}}{\|z_m\|} \cdot \frac{C_\alpha \|z_m^+\|}{\|z_m^+\|} \to 0$$

as $m \to \infty$. This implies

$$\frac{\|z_m^+\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \tag{2.7}$$

Similary, we have

$$\frac{\|z_m^-\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \tag{2.8}$$

By (H3) there exist $C_7, C_8 > 0$ such that

$$z \cdot H_z(t,z_m) - 2H(t,z) \ge C_7|z| - C_8, \quad \forall (t,z) \in [0,1] \times \mathbb{R}^{2N}.$$

This implies

$$\begin{aligned}2f(z_m)-\langle f'(z_m),z_m\rangle &= \int_0^1 [z_m\cdot H_z(t,z_m)-2H(t,z_m)]dt \geq \int_0^1 [C_7|z_m|-C_8]dt\\ &\geq \int_0^1 [C_7|z_m^0|-C_7|z_m^+|-C_7|z_m^-|-C_8]dt\\ &\geq C_9\|z_m^0\|-C_{10}(\|z_m^+\|+\|z_m^-\|+1).\end{aligned}$$

Therefore, by (2.7) and (2.8)

$$\frac{\|z_m^0\|}{\|z_m\|}\to 0 \quad \text{as } m\to\infty.$$

Combine this with (2.7) and (2.8), we get

$$1=\frac{\|z_m\|}{\|z_m\|}\leq \frac{\|z_m^+\|+\|z_m^-\|+\|z_m^0\|}{\|z_m\|}\to 0 \quad \text{as } m\to\infty,$$

a contradiction. Therefore, $\{z_m\}$ must be bounded. □

Proof of Theorem 1.1 We prove that f satisfies the conditions of Theorem 5.29 in [13].

Step 1: By (H1)–(H3), we have

$$H(t,z)\leq a_1+a_2|z|^{\lambda+1},\quad \forall (t,z)\in[0,1]\times\mathbb{R}^{2N}.$$

By (H2), for any $\varepsilon>0$, there exists $\delta>0$ such that

$$H(t,z)\leq \varepsilon|z|^2,\quad \forall t\in[0,1],\ |z|\leq\delta.$$

Therefore, there exists $M=M(\varepsilon)>0$ such that

$$H(t,z)\leq\varepsilon|z|^2+M|z|^{\lambda+1},\quad \forall (t,z)\in[0,1]\times\mathbb{R}^{2N}.$$

Note that $\lambda+1>2$. By the same arguements as in [13, Lemma 6.16], there exist $\rho>0$ and $\tilde a>0$, such that for $z\in E_1=E^+$

$$f(z)\geq\tilde a \quad \text{if } \|z\|=\rho,$$

i.e., f satisfies $(I_7)(i)$ in [13, Theorem 5.29] with $S=\partial B_\rho\cap E_1$.

Step 2: Let $e\in E^+$ with $\|e\|=1$ and $\tilde E=E^-\oplus E^0\oplus span\{e\}$. We denote

$$K=\{z\in\tilde E:\|z\|=1\},\quad \lambda^-=\inf_{z\in E^-,\|z^-\|=1}|\langle Az^-,z^-\rangle|,\quad \gamma=(\frac{\|A\|}{\lambda^-})^{1/2}.$$

For $z\in K$, we write $z=z^-+z^0+z^+\in\tilde E$.

i) If $\|z^-\|>\gamma\|z^++z^0\|$, by (H1) we have, for any $r>0$,

$$\begin{aligned}f(rz)&=\frac12<Arz^-,rz^-\rangle+\frac12\langle Arz^+,rz^+\rangle-\int_0^1 H(t,z)dt\\ &\leq -\frac12\lambda^-r^2\|z^-\|^2+\frac12\|A\|r^2\|z^+\|^2\leq 0.\end{aligned}$$

ii) If $\|z^-\| \leq \gamma\|z^+ + z^0\|$, we have

$$1 = \|z\|^2 = \|z^-\|^2 + \|z^+ + z^0\|^2 \leq (1+\gamma^2)\|z^+ + z^0\|^2,$$

i.e.,

$$\|z^+ + z^0\|^2 \geq \frac{1}{1+\gamma^2} > 0. \tag{2.9}$$

Denote $\tilde{K} = \{z \in K : \|z^-\| \leq \gamma\|z^+ + z^0\|\}$.
Claim: There exists $\varepsilon_1 > 0$ such that, $\forall u \in \tilde{K}$,

$$\operatorname{meas}\{t \in [0,1] : |u(t)| \geq \varepsilon_1\} \geq \varepsilon_1. \tag{2.10}$$

For otherwise, $\forall k > 0$, $\exists u_k \in \tilde{K}$ such that

$$\operatorname{meas}\{t \in [0,1] : |u_k(t)| \geq \frac{1}{k}\} < \frac{1}{k}. \tag{2.11}$$

Write $u_k = u_k^- + u_k^0 + u_k^+ \in \tilde{E}$. Notice that $\dim(E^0 \oplus span\{e\}) < +\infty$ and $\|u_k^0 + u_k^+\| \leq 1$. In the sense of subsequence, we have

$$u_k^0 + u_k^+ \to u_0^0 + u_0^+ \in E^0 \oplus span\{e\} \quad \text{as } k \to +\infty.$$

Then (2.9) implies that

$$\|u_0^0 + u_0^+\|^2 \geq \frac{1}{\gamma^2 + 1} > 0. \tag{2.12}$$

Note that $\|u_k^-\| \leq 1$, in the sense of subsequence $u_k^- \rightharpoonup u_0^- \in E^-$ as $k \to +\infty$. Thus in the sense of subsequences,

$$u_k \rightharpoonup u_0 = u_0^- + u_0^0 + u_0^+ \quad \text{as } k \to +\infty.$$

This means that $u_k \to u_0$ in L^2, i.e.,

$$\int_0^1 |u_k - u_0|^2 dt \to 0 \quad \text{as } k \to +\infty. \tag{2.13}$$

By (2.12) we know that $\|u_0\| > 0$. Therefore, $\int_0^1 |u_0|^2 dt > 0$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\operatorname{meas}\{t \in [0,1] : |u_0(t)| \geq \delta_1\} \geq \delta_2. \tag{2.14}$$

Otherwise, for all $n > 0$, we must have

$$\operatorname{meas}\{t \in [0,1] : |u_0(t)| \geq \frac{1}{n}\} = 0, \quad \text{i.e., } \operatorname{meas}\{t \in [0,1] : |u_0(t)| < \frac{1}{n}\} = 1;$$
$$0 < \int_0^1 |u_0|^2 dt < \frac{1}{n^2} \cdot 1 \to 0 \quad \text{as } n \to +\infty.$$

We get a contradiction. Thus (2.14) holds. Let $\Omega_0 = \{t \in [0,1] : |u_0(t)| \geq \delta_1\}$, $\Omega_k = \{t \in [0,1] : |u_k(t)| < 1/k\}$, and $\Omega_k^{\perp} = [0,1]\backslash\Omega_k$. By (2.11), we have

$$\operatorname{meas}(\Omega_k \cap \Omega_0) = \operatorname{meas}(\Omega_0 - \Omega_0 \cap \Omega_k^{\perp}) \geq \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_0 \cap \Omega_k^{\perp}) \geq \delta_2 - \frac{1}{k}. \tag{2.15}$$

Let k be large enough such that $\delta_2 - \frac{1}{k} \geq \frac{1}{2}\delta_2$ and $\delta_1 - \frac{1}{k} \geq \frac{1}{2}\delta_1$. Then we have

$$|u_k(t) - u_0(t)|^2 \geq (\delta_1 - \frac{1}{k})^2 \geq (\frac{1}{2}\delta_1)^2, \quad \forall t \in \Omega_k \cap \Omega_0.$$

This implies that

$$\begin{aligned}\int_0^1 |u_k - u_0|^2 dt &\geq \int_{\Omega_k \cap \Omega_0} |u_u - u_0|^2 dt \geq (\frac{1}{2}\delta_1)^2 \cdot \operatorname{meas}(\Omega_k \cap \Omega_0) \\ &\geq (\frac{1}{2}\delta_1)^2 \cdot (\delta_2 - \frac{1}{k}) \geq (\frac{1}{2}\delta_1)^2(\frac{1}{2}\delta_2) > 0.\end{aligned}$$

This is a contradiction to (2.13). Therefore the claim is true and (2.10) holds.

For $z = z^- + z^0 + z^+ \in \tilde{K}$, let $\Omega_z = \{t \in [0,1] : |z(t)| \geq \varepsilon_1\}$. By (1.2), for $M = \frac{\|A\|}{\varepsilon_1^3} > 0$, there exists $L_1 > 0$ such that

$$H(t,x) \geq M|x|^2, \quad \forall |x| \geq L_1, \text{ uniformly in } t.$$

Choose $r_1 \geq L_1/\varepsilon_1$. For $r \geq r_1$,

$$H(t, rz(t)) \geq M|rz(t)|^2 \geq Mr^2\varepsilon_1^2, \quad \forall t \in \Omega_z.$$

By (H1), for $r \geq r_1$

$$\begin{aligned}f(rz) &= \frac{1}{2}r^2\langle Az^+, z^+\rangle + \frac{1}{2}r^2\langle Az^-, z^-\rangle - \int_0^1 H(t, rz)dt \\ &\leq \frac{1}{2}\|A\|r^2 - \int_{\Omega_z} H(t, rz)dt \leq \frac{1}{2}\|A\|r^2 - Mr^2\varepsilon_1^2 \cdot \operatorname{meas}(\Omega_z) \\ &\leq \frac{1}{2}\|A\|r^2 - M\varepsilon_1^3 r^2 = -\frac{1}{2}\|A\|r^2 < 0.\end{aligned}$$

Therefore, we have proved that

$$f(rz) \leq 0, \quad \text{for any } z \in K \text{ and } r \geq r_1. \tag{2.16}$$

Let $E_2 = E^- \oplus E^0$, $Q = \{re : 0 \leq r \leq 2r_1\} \oplus \{z \in E_2 : \|z\| \leq 2r_1\}$. By (H1) and (2.16) we have $f|_{\partial Q} \leq 0$, i.e., f satisfies $(I_7)(ii)$ in [13, Theorem 5.29].

Step 3: By Lemma 2.1, f satisfies the (PS) condition. Similar to the proof of [13, Theorem 6.10], by the linking theorem [13, Theorem 5.29], there exists a critical point $z^* \in E$ of f such that $f(z^*) \geq \tilde{a} > 0$. Moreover, z^* is a classical solution of (1.1) and z^* is nonconstant by (H1). $\square$

Remark 2.2 i) Suppose $H(t,z) = \frac{1}{2}\langle B(t)z,z\rangle + \tilde{H}(t,z)$ with $B(t)$ being a $2N\times 2N$ matrix, continuous and 1-periodic in t and $\tilde{H}(t,z)$ satisfies (1.2) and (H1)-(H3). We have the same conclusion as Theorem 1.1. The proof is similar and we omit it.

ii) Suppose $H(t,z) = H(z)$ is independent on t, i.e., (1.1) is an autonomous Hamiltonian system. Then under similar conditions as (1.2) and (H1)-(H3), for any $T>0$, the system (1.1) has a nonconstant T-periodic solution. Moreover, if $H(z)\in C^2(\mathbb{R}^{2N},\mathbb{R})$ and satisfies some strictly convex conditions such as $H''(x)$ is positive defininte for $x\neq 0$, then for any $T>0$, (1.1) has a nonconstant T-periodic solution with minimal period T. We omit the proof which is similar to the one in [4, 5].

iii) Suppose (1.4) holds, i.e.,

$$H(t,z) = H(z) = |z|^2(\ln(1+|z|^p))^q, \quad \forall (t,z)\in[0,1]\times\mathbb{R}^{2N},$$

where $p>1$ and $q>1$. Obviously, (1.2), (H1) and (H2) hold. Note that

$$z\cdot H_z(z) - 2H(z) = |z|^2 q(\ln(1+|z|^q))^{q-1}\frac{p|z|^p}{1+|z|^p} \geq |z|^2\frac{pq(\ln 2)^{q-1}}{2}, \quad \forall |z|\geq 1.$$

$$|H_z(z)| \leq 2(\ln(1+|z|^p))^q|z| + \frac{p|z|^p}{1+|z|^p}q(\ln(1+|z|^p))^{q-1}|z| \leq 2|z|^{\frac{5}{4}},\ \forall |z|\geq L,$$

for L being large enough. This implies (H3). By directly computation, $H''(z)$ is positive definite for $z\neq 0$. Therefore, for any $T>0$, (1.1) possesses a T-periodic solution with minimal period T.

iv) There are many examples which satisfy (H1)-(H3) and (1.2) but do not satisfy (1.3). For example

$$H(t,z) = |z|^2\ln(1+|z|^2)\ln(1+2|z|^3).$$

Corollary 2.3 *Suppose $H(t,z) = |z|^2 h(t,z)$ with $h\in C^1([0,1]\times\mathbb{R}^{2N},\mathbb{R})$ being 1-periodic in t and satisfies*

(H1′) $h(t,z)\geq 0$, for all $(t,z)\in[0,1]\times\mathbb{R}^{2N}$.

(H2′) $h(t,z)\to 0$ as $|z|\to 0$; $h(t,z)\to+\infty$ as $|z|\to+\infty$.

(H3′) There exist $0\leq\delta<1$, $L>0$, $\varepsilon_0>0$ and $M>0$ such that

$$|z|^\delta h_z(t,z)\cdot z\geq\varepsilon_0, \quad |z||h_z(t,z)|\leq Mh, \quad \forall |z|\geq L;$$
$$\frac{h(t,z)}{|z|^\gamma}\to 0 \quad \text{as } |z|\to\infty \text{ for any } \gamma>0.$$

Then system (1.1) possesses a nonconstant 1-periodic solution.

Proof Obviously, $(H1')-(H3')$ imply (1.2), (H1) and $(H2)$.

$$z\cdot H_z(t,z) - 2H(t,z) = |z|^2|h_z(t,z)\cdot z\geq\varepsilon_0|z|^{2-\delta}, \quad \forall |z|\geq L;$$
$$\begin{aligned}|H_z(t,z)| &\leq |2h(t,z)||z| + |z|^2|h_z(t,z)| \\ &\leq (2+M)|z|h(t,z)\leq (2+M)|z|^{1+\gamma}, \ \forall |z|\geq L'.\end{aligned}$$

Let $\beta = 2-\delta$ and $\lambda = 1+\gamma$ with $0 < \gamma < (1-\delta)/(2-\delta)$. Then (H3) holds. By Theorem 1.1 we get the conclusion. $\square$

3 Second order Hamiltonian System

Let $E = W^{1,2}(S^1, \mathbb{R}^N)$ with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Then $E \subset C(S^1, \mathbb{R}^N)$ and $\|u\|^2 = \int_0^1 (|\dot{u}|^2 + |u|^2)dt$. Define

$$\langle Kx, y\rangle = \int_0^1 x\cdot y dt, \quad \forall x, y \in E;$$
$$f(z) = \frac{1}{2}\langle (id-K)z, z\rangle - \int_0^1 V(t,z)dt, \quad \forall z \in E.$$

Then K is compact, $\ker(id-K) = \mathbb{R}^N$, and the negative definite subspace of $\mathrm{id}-K$, $M^-(\mathrm{id}-K) = \{0\}$, i.e., $E = E^0 \oplus E^+$ where $E^0 = \ker(id-K)$ and E^+ is the positive definite subspace of $id-K$. Note that (V1)–(V4) imply

$$V(t,x) \le d_2|x|^{\lambda+1} + d_3. \tag{3.1}$$

This implies that $f \in C^1(E, \mathbb{R})$ and critical points of f are 1-periodic solutions of (1.5) [11].

Lemma 3.1 *Suppose (V1)–(V4) hold. Then f satisfies the (PS) condition.*

Proof Let $\{z_m\}$ be a (PS) sequence. Suppose $\{z_m\}$ is not bounded. Passing to a subsequence if necessary, $\|z_m\| \to +\infty$ as $m \to \infty$. Then by (V4)

$$2f(z_m) - \langle f'(z_m), z_m\rangle = \int_0^1 [z_m\cdot V'(t,z_m) - 2V(t,z_m)]dt \ge d_1\int_0^1 |z_m|^\beta dt - d_4.$$

This implies

$$\frac{\int_0^1 |z_m|^\beta dt}{\|z_m\|} \to 0 \quad \text{as } m \to +\infty.$$

If (1.6) holds, we have

$$\begin{aligned}\langle f'(z_m), z_m^+\rangle &= \langle (id-K)z_m^+, z_m^+\rangle - \int_0^1 V'(t,z_m)\cdot z_m^+ dt\\ &\ge \langle (id-K)z_m^+, z_m^+\rangle - \|z_m^+\|_\infty \int_0^1 |V'(t,z_m)|dt\\ &\ge \langle (id-K)z_m^+, z_m^+\rangle - d_5\|z_m^+\|(\int_0^1 |z_m|^\lambda dt + d_6).\end{aligned}$$

Since $\lambda \le \beta$, we have

$$\frac{\|z_m^+\|}{\|z_m\|} \to 0 \quad \text{as} \quad m \to +\infty. \tag{3.2}$$

If (1.7) holds, we have

$$\begin{aligned} f(z_m) &= \frac{1}{2}\langle (id-K)z_m^+, z_m^+\rangle - \int_0^1 V(t,z_m)dt \\ &\geq \frac{1}{2}\langle (id-K)z_m^+, z_m^+\rangle - d_5\int_0^1 |z_m|^{1+\lambda}dt - d_7 \\ &\geq \langle (id-K)z_m^+, z_m^+\rangle - d_8\|z_m\|\int_0^1 |z_m|^{\lambda}dt - d_7. \end{aligned}$$

Since $\lambda \leq \beta$, we obtain (3.2). On the other hand, (V1)–(V4) imply

$$x\cdot V'(t,x) - 2V(t,x) \geq d_9|x| - d_{10}, \quad \forall t \in S^1 \times \mathbb{R}^N.$$

$$\begin{aligned} 2f(z_m) - \langle f'(z_m), z_m\rangle &= \int_0^1 [z_m\cdot V'(t,z_m) - 2V(t,z_m)]dt \\ &\geq d_9\int_0^1 |z_m|dt - d_{10} \\ &\geq d_9\int_0^1 |z_m^0|dt - d_9\int_0^1 |z_m^+|dt - d_{10} \\ &\geq d_9\|z_m^0\| - d_{11}\|z_m^+\| - d_{10}. \end{aligned}$$

This implies

$$\frac{\|z_m^0\|}{\|z_m\|} \to 0 \quad \text{as } m \to +\infty. \tag{3.3}$$

By (3.2) and (3.3), we get a contradiction. Therefore $\{z_m\}$ is bounded. By a standard argument, $\{z_m\}$ has a convergent subsequence [11]. □

Proof of Theorem 1.2 As in Step 1 of the proof of Theorem 1.1, by (V2) and (3.1), there exist $\tilde{a} > 0$, $\rho > 0$ such that

$$f(z) \geq \tilde{a}, \quad \forall z \in E^+ \quad \text{with } \|z\| = \rho.$$

Choose $e \in E^+$ with $\|e\| = 1$. Let $\tilde{E} = \text{span}\{e\} \oplus E^0$ and $K = \{u \in \tilde{E} : \|u\| = 1\}$. Note that $\dim \tilde{E} < +\infty$. By using similar arguments as in the proof of (2.10), there exists $\varepsilon_1 > 0$ such that

$$\text{meas}\{t \in [0,1] : |u(t)| \geq \varepsilon_1\} \geq \varepsilon_1, \quad \forall u \in K. \tag{3.4}$$

By (V1), (V3) and similar arguments as in the proof of Theorem 1.1, there exists $r_1 > 0$ such that

$$f|_{\partial Q} \leq 0, \quad \text{where} \quad Q = \{re : 0 \leq r \leq 2r_1\} \oplus \{z \in E^0 : \|z\| \leq 2r_1\}.$$

Now by Lemma 3.1, [13, Theorem 5.29], and (V1), f has a nonconstant critical point z^* such that $f(z^*) \geq \tilde{a} > 0$. z^* is 1-periodic solution of (1.5). □

Remark 3.2 (i) Suppose $V(t,x) = V(x)$ is independent on t and $V(x)$ satisfies (V1)–(V4). Then for any $T > 0$, (1.5) possesses a nonconstant T-periodic solution.

(ii) There are many examples which satisfy (V1)–(V4) but do not satisfy a condition similar to (1.3). For example,

$$\begin{aligned} &V(t,x) = [1 + (\sin 2\pi t)^2] \cdot |x|^2 \ln(1 + 2|x|^2); \quad \text{or} \\ &V(t,x) = |x|^2 \ln(1 + |x|^2) \ln(1 + 2|x|^4). \end{aligned}$$

By using similar arguments as in the proof of Theorem 1.2, we can prove the following corollary. Details are omited.

Corollary 3.3 *Suppose $V(t,x) = |x|^2 h(t,x)$ with $h \in C^1(S^1 \times \mathbb{R}^N, \mathbb{R})$ satisfies*

(V1′) $h(t,x) \geq 0, \quad \forall (t,x) \in S^1 \times \mathbb{R}^N$.

(V2′) $h(t,x) \to 0$ *as* $|x| \to 0$; $\quad h(t,x) \to +\infty$ *as* $|x| \to +\infty$.

(V3′) There exist $L > 0$, $\lambda > 0$, $C_1, C_2 > 0$ such that for $t \in S^1$

$$C_1 |x| (h'(t,x) \cdot x) \geq h(t,x), \quad h(t,x) \leq C_2 |x|^\lambda, \quad \forall |x| \geq L.$$

Then (1.5) possesses a nonconstant 1-periodic solution.

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