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Two functionals for which C_0^1 minimizers are also $W_0^{1,p}$ minimizers *

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Abstract

Brezis and Niremberg [1] showed that for a certain functional the C_0^1 minimizer is also the H_0^1 minimizer. In this paper, we present two functionals for which a local minimizer in the C_0^1 topology is also a local minimizer in the $W_0^{1,p}$ topology. As an application, we show some existence results involving the sub and super solution method for elliptic equations.

1 Introduction

It is well known that for a domain Ω with smooth boundary in $\mathbb{R}^n$, the $W_0^{1,p}$ topology is much weaker than the C_0^1 topology. Therefore, a $W_0^{1,p}(\Omega)$ neighborhood of function u contains much more elements than the corresponding $C_0^1(\Omega)$ neighborhood. As an example, let $B_1(0)$ be the unit ball in $\mathbb{R}^n$,

$$f(x) = \frac{1}{|x|^\alpha} - 1\,,$$

and α, p satisfy $0 < \alpha < n-1$, $(\alpha+1)p < n$. Then $f(x) \notin C_0^1(B_1(0))$, while $f(x) \in W_0^{1,p}(B_1(0))$. Hence, the C_0^1 minimizer of a functional Φ, if exist, is not necessarily the $W_0^{1,p}$ minimizer. In the following example, $\Omega = (0,1)$ in $\mathbb{R}^1$. For a function u in $C_0^1((0,1))$ or in $W_0^{1,p}((0,1))$, we write du as the weak differential quotient of u. Since weak differential quotients different only on a set of measure zero, we denote by u' weak differential quotient with the least number of discontinuous points. Let $\Gamma = \{x \in \operatorname{supp} u : |u'(x)| \le \alpha < 1\}$ and

$$\Phi(u) = \begin{cases} \limsup_{x\in\Gamma}\{u'(x)\}, & \text{if } \Gamma \neq \emptyset,\\ 0, & \text{if } \Gamma = \emptyset. \end{cases}$$

If $u \in C_0^1((0,1))$ then $u' \in C_0((0,1))$. It is easy to see that $u_0 \equiv 0$ is a local minimizer of Φ and $\Phi(u_0) = 0$. However, u_0 is not a local minimizer in the $W_0^{1,p}$

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topology. Let

$$u_n = \begin{cases} (x-a), & x \in [a, \delta_n^1], \\ -\frac{1}{n}(x - \frac{a+b}{2}), & x \in [\delta_n^1, \delta_n^2], \\ (x-b), & x \in [\delta_n^2, b], \\ 0, & x \in (0,1)\backslash[a,b]. \end{cases}$$

where $[a,b]$ is a closed subinterval of $(0,1)$, $\delta_n^1 = (a + \frac{a+b}{2n})/(1+\frac{1}{n})$ and $\delta_n^2 = (b + \frac{a+b}{2n})/(1+\frac{1}{n})$. Thus $u_n \in W_0^{1,p}((0,1))$, and $\|u_n - u_0\|_{W_0^{1,p}} \le \epsilon$, for any given $\epsilon > 0$ as $n \to \infty$. Moreover,

$$u_n' = \begin{cases} 1, & x \in [a, \delta_n^1) \cup (\delta_n^2, b], \\ -\frac{1}{n}, & [\delta_n^1, \delta_n^2], \\ 0, & (0,1)\backslash[a,b]\,. \end{cases}$$

Then $\Phi(u_n) = \limsup_{x\in\Gamma}\{u'(x)\} = -\frac{1}{n}$ which is less than $\Phi(u_0) = 0$.

In some special cases, the minimizer in the C_0^1 topology of is also the minimizer in the $W_0^{1,p}$ topology. Brezis and Nirenberg [1] showed that for the functional

$$\Psi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \int_\Omega F(x,u),$$

the C_0^1 minimizer is also a minimizer in H_0^1, under certain conditions on $F(x,u)$. In this paper, we present another two of these kinds of functionals: The first functional is

$$\Phi(u) = \frac{1}{2}\int_\Omega (\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j} + cu^2) - \int_\Omega F(x,u),$$

whose C_0^1 minimizer is also a minimizer in H_0^1 for coefficient functions $a^{i,j}, c$ $(i,j = 1,\cdots,n)$ satisfying an ellipticity condition. The second functional is

$$\widetilde{\Phi}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u).$$

Its C_0^1 minimizer is also a minimizer in $W_0^{1,p}$, $p \le n$, under certain conditions on $F(x,u)$.

We will give examples of functionals for which the $W_0^{1,p}$ (or the H_0^1) minimizer is not easy to find, but we can find C_0^1 minimizers instead. Then we show that it is also a minimizer in the $W_0^{1,p}$ (or the H_0^1) topology. In some cases, it is easier to find minimizers of functional truncated by a constant, and then show that the minimizer of the truncated functional is also the local minimizer of the original functional in the C_0^1 topology.

Next, we introduce some Lemmas to be used later.

Lemma 1.1 *Let Ω be a domain in $\mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function such that for almost every $x \in \Omega$,*

$$|g(x,u)| \le a(x)(1+|u|)$$

with $a \in L^{n/2}_{\rm loc}(\Omega)$. Let $u \in W^{1,2}_{\rm loc}(\Omega)$ be a weak solution of $-\Delta u = g(\cdot,u)$ in Ω, then $u \in L^q_{\rm loc}(\Omega)$, for any $q < \infty$. If $u \in W^{1,2}_0(\Omega)$, and $a \in L^{n/2}(\Omega)$, then $u \in L^q(\Omega)$, for any $q < \infty$.

The proof of this lemma is given in the appendix; see also [2, p. 244]. The conclusion can also be obtained for divergence elliptic equations, as stated in the lemma below.

Lemma 1.2 *Suppose $u \in H^1_0(\Omega)$ is a weak solution of $Lu = g(\cdot,u)$ where*

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j}$$

with bounded coefficients $a^{ij} = a^{ji}$ satisfying the uniformly ellipticity condition

$$\sum_{i,j=1}^n a^{ij}\xi_i\xi_j \ge \theta|\xi|^2, \quad \theta > 0, \text{ for any } \xi \in \mathbb{R}^n,$$

and $|g(x,u)| \le \widetilde{a}(x)(1+|u|)$ with $\widetilde{a}(x) \in L^{n/2}(\Omega)$. Then $u \in L^q(\Omega)$, for any $q < \infty$.

Proof. Let $u \in H^1_0$ be a weak solution of $Lu = g(\cdot,u)$, in the sense that

$$\int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}\cdot\varphi_{x_j} = \int_\Omega g\cdot\varphi, \quad \text{for any } \varphi \in H^1_0(\Omega). \tag{1.1}$$

We choose $s \ge 0$, $M \ge 0$. Let $\varphi = u\min\{|u|^{2s}, M^2\} \in H^1_0(\Omega)$, then

$$\varphi_{x_j} = \begin{cases} u_{x_j}\ \min\{|u|^{2s}, M^2\} + 2s|u|^{2s}u_{x_j}, & |u(x)|^s \le M, \\ u_{x_j}\ \min\{|u|^{2s}, M^2\}, & |u(x)|^s > M. \end{cases}$$

Multiplying (1.1) by a test function φ, we have

$$\begin{aligned} \int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}\cdot\varphi_{x_j} &= \int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j}\ \min\{|u|^{2s}, M^2\} \\ &\quad + 2s\int_{\{x\in\Omega;\ |u(x)|^s \le M\}} |u|^{2s}(\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j}) \\ &\ge \theta\int_\Omega |\nabla u|^2\ \min\{|u|^{2s}, M^2\} \\ &\quad + 2s\theta\int_{\{x\in\Omega;\ |u(x)|^s \le M\}} |u|^{2s}|\nabla u|^2. \end{aligned} \tag{1.2}$$

From the proof of Lemma 1.1, and (1.2), we have

$$\begin{aligned}\int_\Omega |\nabla(u\ \min\{|u|^s, M\})|^2 &\leq C\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, M^2\}\\ &\leq C\int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j}\ \min\{|u|^{2s}, M^2\}\\ &\quad + 2Cs\int_{\{x\in\Omega;\,|u(x)|^s\leq M\}} |u|^{2s}(\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j})\\ &= C\int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}\cdot\varphi_{x_j} \leq C\int_\Omega |g|\cdot|\varphi|\\ &\leq C\int_\Omega |\tilde{a}||u|^2\ \min\{|u|^{2s}, M^2\}.\end{aligned}$$

Here and hereafter we denote all the constants with the same symbol C. Then as in the proof of Lemma 1.1, see [2, p.244], we obtain $u \in L^q(\Omega)$, for any $q < \infty$. □

Remark 1.1 If $Lu = -\sum_{i,j=1}^n (a^{ij}(x)u_{x_i})_{x_j} + cu$, with bounded coefficient functions a^{ij} and c sufficiently smooth, and $a^{ij} = a^{ji}$ satisfying the uniformly ellipticity condition, then with the condition in Lemma 1.2, the conclusion is also true.

For the operator Δ_p, $\Delta_p u = -\nabla\cdot(|\nabla u|^{p-2}\nabla u)$, we have the following statement.

Lemma 1.3 *Let $1 \leq p \leq n$, $f \in L^s(\Omega)$ for some $s > n/p$, and $u \in W_0^{1,p}(\Omega)$ be a weak solution of*

$$\begin{aligned}\Delta_p u &= f, \quad \text{in } \Omega,\\ u &= 0, \quad \text{on } \partial\Omega.\end{aligned}$$

Then $u \in L^\infty(\Omega)$ and there exists $c = c(n, p, |\Omega|)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq c\,\|f\|_{L^s(\Omega)}^{1/(p-1)}.$$

The proof of this lemma is a straightforward application of Moser's iterative scheme(cf. [3, 6]).

2 Divergence elliptic differential operator

In this section, we are concerned with the relation between the H_0^1 and C_0^1 minimizers of the functional

$$\Phi(u) = \int_\Omega [\frac{1}{2}(\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j} + cu^2) - F(x, u)]dx,$$

where
$$F(x,u)=\int_0^u f(x,s)ds\,.$$
Here $f(x,u)$ is a Carathéodory function, satisfying the natural growth condition
$$|f(x,u)|\le K(1+|u|^p) \tag{2.1}$$
where K is a constant and $p\le\frac{(n+2)}{(n-2)}$ for $n>2$.

We call $u\in H_0^1(\Omega)$ a local minimizer of Φ, if u is a weak solution of
$$\begin{aligned} Lu&=f,\quad \text{in } \Omega,\\ u&=0,\quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$
where $Lu=-\sum_{i,j=1}^n(a^{ij}(x)u_{x_i})_{x_j}+c(x)u$ satisfies the hypotheses in Remark 1.1.

Our main theorem is as follows.

Theorem 2.1 *Assume $u_0\in H_0^1(\Omega)$ is a local minimizer of Φ in the C_0^1 topology: this means that there exits some $r>0$, such that*
$$\Phi(u_0)\le\Phi(u_0+v),\qquad \forall\, v\in C_0^1(\Omega)\quad \textit{with}\quad \|v\|_{C_0^1}\le r. \tag{2.3}$$
Then u_0 is a local minimizer of Φ in the H_0^1 topology, i.e. there exists $\epsilon_0>0$ such that
$$\Phi(u_0)\le\Phi(u_0+v),\quad \forall\, v\in H_0^1(\Omega)\quad \textit{with}\quad \|v\|_{H_0^1}\le\epsilon_0. \tag{2.4}$$

Proof. **1.** We claim that $u_0\in C_0^{1,\alpha}(\Omega)$, for any $0<\alpha<1$. In the case $p<(n+2)/(n-2)$ we can prove the regularity of u_0 by a bootstrap argument [4]. For $p=(n+2)/(n-2)$, the standard bootstrap procedure does not work. We now define
$$\widetilde{a}(x)=\frac{f(x,u_0)}{1+|u_0|}\,,$$
then by (2.1),
$$|\widetilde{a}(x)|\le C\,|u_0(x)|^{p-1}\le C\,|u_0(x)|^{\frac{4}{n-2}}.$$
Note that $u_0(x)\in H_0^1(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, so we have $\widetilde{a}(x)\in L^{n/2}(\Omega)$.

Then we can deduce from Lemma 1.2 that $u_0\in L^q(\Omega)$, for any $q<\infty$, furthermore, since $|f(x,u_0)|\le K(1+|u_0|^p)$, then $f(x,u_0)\in L^q(\Omega)$, for any $q<\infty$. From (2.2) we deduce that $u_0\in W_0^{2,q}(\Omega)$, for any $q<\infty$. By a Sobolev embedding with q large enough, $W_0^{2,q}\hookrightarrow C_0^{1,\alpha}(\Omega)$; therefore $u_0\in C_0^{1,\alpha}(\Omega)$, for any $0<\alpha<1$. Without loss of generality we may now assume that $u_0=0$.

2. Now we prove Theorem 2.1 in the subcritical case $p<(n+2)/(n-2)$. Suppose the conclusion (2.4) does not hold. Then
$$\forall\,\epsilon>0,\ \exists\, v_\epsilon\in B_\epsilon\quad \text{such that}\quad \Phi(v_\epsilon)<\Phi(0), \tag{2.5}$$

where $B_\epsilon = \{u \in H_0^1(\Omega) : \|u\|_{H_0^1} \le \epsilon\}$, For each j consider the truncation map

$$T_j(r) = \begin{cases} -j, & \text{if } r \le -j, \\ r, & \text{if } -j \le r \le j, \\ j, & \text{if } r \ge j. \end{cases}$$

Set

$$f_j(x,s) = f(x, T_j(s)), \quad F_j(x,u) = \int_0^u f_j(x,s)ds,$$
$$\Phi_j(u) = \int_\Omega [\frac{1}{2}(\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j} + c\,u^2) - F_j(x,u)]dx.$$

Then, for each $u \in H_0^1(\Omega)$, $\Phi_j(u) \to \Phi(u)$ as $j \to \infty$. Hence, from (2.5) we know for each $\epsilon > 0$ there is some $j = j(\epsilon)$ s.t. $\Phi_j(v_\epsilon) < \Phi(0)$. Now we point out Φ_j is coercive and weakly lower semi-continuous. Φ_j is coercive, because

$$\begin{aligned} \Phi_j(u) &\ge \theta \int_\Omega (|\nabla u|^2 + |c||u|^2)dx - \int_\Omega \int_0^u (1 + |T_j(s)|^p)dsdx \\ &\ge \theta \int_\Omega (|\nabla u|^2 + c|u|^2)dx - \int_\Omega (1 + j^p)|u|dx) \\ &\ge \theta \int_\Omega (|\nabla u|^2 + c|u|^2)dx - C\epsilon \int_\Omega |u|^2dx - C(\epsilon) \\ &\ge \theta \int_\Omega (|\nabla u|^2 + C'|u|^2)dx - C(\epsilon) \\ &\ge C\,\|u\|_{H_0^1}^2 - C(\epsilon). \end{aligned}$$

Note that $\int_\Omega (\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j} + cu^2)$ is equivalent to the norm $\|u\|_{H_0^1}$ and is weakly lower semi-continuous. Using Lemma 2.2 below, we can deduce that

$$E_j(u) \doteq \int_\Omega F_j(x,u)dx = \int_\Omega (F_j(x,u) + 0 \cdot \nabla u)dx$$

is weakly lower semi-continuous, so Φ_j is weakly lower semi-continuous.

Lemma 2.2 *Assume that $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory function satisfying the conditions*

1. *$F(x,u,p) \ge \phi(x)$ for almost every x, u, p, where $\phi \in L^1(\Omega)$.*
2. *$F(x,u,\cdot)$ is convex in p for almost every x, u.*

Then, if u_m, $u \in W_{\rm loc}^{1,1}(\Omega)$ and $u_m \to u$ in $L^1(\Omega')$, $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^1(\Omega')$ for all bounded $\Omega' \subset\subset \Omega$, it follows that

$$E(u) \le \liminf_{m\to\infty} E(u_m)$$

where $E(u) = \int_\Omega F(x,u,\nabla u)dx$.

The proof of this lemma can be found in [2, p. 9].

Then Φ_j is bounded from below and attains its infimum in B_ϵ (which is closed and convex, so it is a weakly closed subset), suppose

$$\Phi_{j_\epsilon}(w_\epsilon) = \min_{u\in B_\epsilon} \Phi_{j_\epsilon}(u),$$

we have

$$\Phi_{j_\epsilon}(w_\epsilon) \le \Phi_{j_\epsilon}(v_\epsilon) \le \Phi(0). \tag{2.6}$$

The corresponding Euler equation for w_ϵ involves a Lagrange multiplier $\mu_\epsilon \le 0$ (cf. Generalized Kuhn-Tucker Theory in [5]), namely, w_ϵ satisfies

$$\langle\, \Phi_j'(w_\epsilon),\, \zeta\,\rangle_{H^{-1},H_0^1} = \mu_\epsilon(w_\epsilon\cdot\zeta)_{H_0^1}, \quad \forall\,\zeta\in H_0^1(\Omega),$$

i.e.

$$\int_\Omega [\sum_{k,l=1}^n a^{kl}(w_\epsilon)_{x_k}\zeta_{x_l} + cw_\epsilon\zeta - f_j(x,w_\epsilon)\zeta] = \mu_\epsilon\int_\Omega \nabla w_\epsilon\cdot\nabla\zeta, \quad \forall\zeta\in H_0^1(\Omega).$$

This implies $(Lw_\epsilon - \mu_\epsilon\Delta w_\epsilon) = f_j(x,w_\epsilon)$. Let $L'w_\epsilon = (Lw_\epsilon - \mu_\epsilon\Delta w_\epsilon)$, then

$$L'w_\epsilon = \sum_{k,l=1}^n [\hat a^{\,kl}\cdot(w_\epsilon)_{x_k}]_{x_l} + cw_\epsilon, \quad \text{where} \quad \hat a^{kl} = \begin{cases} a^{kl}-\mu_\epsilon, & k=l,\\ a^{kl}, & k\ne l.\end{cases}$$

Note that $\mu_\epsilon \le 0$. It is easy to check that $\hat a^{kl}$ still satisfy the uniformly ellipticity condition.

Since $L'w_\epsilon = f_j(x,w_\epsilon)$ and $p<(n-2)/(n+2)$, using a bootstrap procedure, we can derive from $\|w_\epsilon\|_{H_0^1}\le C$, that $\|w_\epsilon\|_{C_0^{1,\alpha}}\le C$, where C is a constant independent of ϵ. Then

$$\sup_{x,y\in\Omega}\frac{|w_\epsilon(x)-w_\epsilon(y)|}{|x-y|^\alpha} < C < \infty, \quad \forall\epsilon>0.$$

This implies that the w_ϵ's are equicontinuous and $|w_\epsilon| < 2C\,\mathrm{diam}(\Omega)$, for all $x\in\Omega$, which means w_ϵ are uniformly bounded. Then by Ascoli Theorem, $\{w_\epsilon\}$ has a subsequence converging in $C_0^1(\Omega)$, still denoted by $\{w_\epsilon\}$. Then since $\|w_\epsilon\|_{H_0^1}\to 0$ as $\epsilon\to 0$, we can derive $\|w_\epsilon\|_{C_0^1}\to 0$ as $\epsilon\to 0$, i.e. $w_\epsilon\to 0$ in $C_0^1(\Omega)$, as $\epsilon\to 0$. Then for ϵ small enough, we have

$$\Phi(w_\epsilon) = \Phi_{j(\epsilon)}(w_\epsilon) < \Phi(0).$$

This contradicts (2.3); therefore, (2.5) can not hold. Thus Theorem 2.1 is proved for the subcritical case.

3. In the critical case $p=(n+2)/(n-2)$, since $H_0^1(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ is not compact, a bootstrap argument does not work. But we still have

$$L'w_\epsilon = (Lw_\epsilon - \mu_\epsilon\Delta w_\epsilon) = f_j(x,w_\epsilon)$$

and

$$f_j(x,w_\epsilon)=f(x,T_j(w_\epsilon))\le K\,(1+|T_j(w_\epsilon)|^p)\le K\,(1+|w_\epsilon|^p). \tag{2.7}$$

Let

$$\widetilde{a}(x)=\frac{f_j(x,w_\epsilon)}{1+|w_\epsilon|},$$

then

$$|\widetilde{a}(x)|\le C\,|w_\epsilon|^{p-1}\le C\,|w_\epsilon|^{\frac{4}{n-2}}\in L^{n/2}(\Omega).$$

By Remark 1.1, $w_\epsilon\in L^q(\Omega)$, for any $q<\infty$. From (2.7) $f_j(x,w_\epsilon)\in L^q(\Omega)$, for any $q<\infty$. Then $w_\epsilon\in W_0^{2,q}(\Omega)$, for any $q<\infty$, then $w_\epsilon\in C_0^{1,\alpha}(\Omega)$, for any $q<\infty$. Consequently, $w_\epsilon\to 0$ *in* C_0^1 since $w_\epsilon\to 0$ *in* H_0^1. Thus Theorem 2.1 is proved. □

As an application of Theeorem 2.1, we will obtain an existence result of divergence elliptic equation involving the sub and super solution method. First, we give a lemma:

Lemma 2.3 *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $u\in L^1_{\rm loc}(\Omega)$ and assume that, for some constant $k\ge 0$, u satisfies*

$$\begin{gathered} Lu+ku\ge 0,\quad \textit{in } \Omega,\\ u\ge 0\quad \textit{in } \Omega. \end{gathered}$$

Then either $u\equiv 0$, or there exists $\epsilon>0$ such that

$$u(x)\ge \epsilon\,\mathrm{dist}(x,\partial\Omega),\quad \textit{in } \Omega.$$

Proof. Let $\mu=Lu+ku$, then $\mu\ge 0$, we may assume $u\not\equiv 0$.
Case 1: $\mu\equiv 0$. In this case $u\in C^\infty(\Omega)$,

$$Lu+ku=0,\quad u\ge 0,\quad \text{in } \Omega.$$

Since $u\not\equiv 0$, $u\ge\delta\ge 0$ in some closed ball B in Ω. Suppose h solves

$$\begin{gathered} (L+k)h=0\quad \text{in } \Omega\backslash B,\\ h=\delta\quad \text{on } \partial B,\\ h=0\quad \text{on } \partial\Omega. \end{gathered}$$

Using the maximum principle, we have $u-h\ge 0$ in $\Omega\backslash B$. Using the Hopf Lemma, we have

$$\frac{|h(x)-0|}{\mathrm{dist}(x,\partial\Omega)}\ge\epsilon>0,\quad \text{in } \Omega\backslash B$$

for some $\epsilon>0$; that is $h(x)\ge\epsilon\,\mathrm{dist}(x,\partial\Omega)$ in $\Omega\backslash B$. Then

$$u(x)\ge\epsilon\,\mathrm{dist}(x,\partial\Omega),\quad \text{in } \Omega\backslash B.$$

Since $\overline{\Omega}$ is compact, there can be found ϵ, such that $u(x) \geq \epsilon \operatorname{dist}(x, \partial\Omega)$ in Ω.
Case 2: $\mu \not\equiv 0$. Let $\zeta \in C_0^\infty(\Omega)$ be a cutoff function, $0 \leq \zeta \leq 1$, such that $\zeta u \not\equiv 0$. Let v be the solution of

$$\begin{gathered}(L+k)v = \zeta u \quad \text{in } \Omega,\\ v = 0, \quad \text{on } \partial\Omega.\end{gathered}$$

Since $\zeta u \geq 0$, using Hopf Lemma, we have $v(x) \geq \epsilon \operatorname{dist}(x, \partial\Omega)$ in Ω.

Now we claim $u \geq v$ in Ω. Given any $\alpha > 0$, we will prove that $\overline{u} = u + \alpha \geq v$ in Ω. Let $w = \overline{u} - v$, then we have

$$\begin{aligned}(L+k)w &= (L+k)(u+\alpha-v) = (L+k)u - \zeta\mu + (L+k)\alpha\\ &= (L+k)u - \zeta\mu + (c(x)+k)\alpha\\ &= (1-\zeta)\mu + (c(x)+k)\alpha.\end{aligned}$$

Let k be large enough, so that $c(x) + k \geq 0$ for all $x \in \Omega$, then w satisfies

$$(L+k)w = (1-\zeta)\mu + (c(x)+k)\alpha \geq 0, \qquad \text{in } \Omega \tag{2.8}$$

and

$$w \geq 0, \quad \text{in } N_\eta\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < \eta\}, \tag{2.9}$$

provided η is sufficiently small (depend on α). The last property (2.9) follows from the fact v is smooth near $\partial\Omega$ and $v = 0$ on $\partial\Omega$. Let $\{\rho_j\}$ be a sequence of mollifiers with $\operatorname{supp}\rho_j \subset \Omega/N_{1/j}$. Set $w_j(x) = \int_\Omega \rho_j(x-y)w(y)$.
Clearly w_j is smooth,and by (2.8) we have

$$(L+k)w_j \geq 0, \quad \text{in } \Omega/\overline{N}_{1/j}.$$

On the other hand, from (2.9) we deduce $w_j \geq 0$ in $N_{(\eta-1/j)}$. Provided that j is large enough, $2/j < \eta$, then $w_j \geq 0$ *in* $N_{(\frac{1}{j}+\epsilon)}$, thus $w_j \geq 0$ on $\partial(\Omega/\overline{N}_{1/j})$, using the maximum principle, we have

$$w_j \geq 0, \quad \text{in } \Omega/\overline{N}_{1/j}.$$

Passing to the limit as $j \to \infty$ we see that $w \geq 0$ in Ω, which is the desired conclusion. □

Theorem 2.4 *Assume that $\underline{u}$ and $\overline{u}$ are sub and super solutions in $C^1(\overline{\Omega})$ in the weak sense:*

$$\begin{gathered}L\underline{u} - f(x,\underline{u}) \leq 0 \leq L\overline{u} - f(x,\overline{u}) \quad \text{in } \Omega,\\ \underline{u} \leq 0 \leq \overline{u} \quad \text{on } \partial\Omega.\end{gathered}$$

Moreover, assume that neither $\underline{u}$ nor $\overline{u}$ is a solution of (2.2). Then there is a solution u_0 of (2.2), $\underline{u} \leq u_0 \leq \overline{u}$, such that, in addition, u_0 is a local minimum of Φ in $H_0^1(\Omega)$.

Proof. 1. We introduce an auxiliary function

$$\widetilde{f}(x,s) = \begin{cases} f(x,\underline{u}(x)), & \text{if } s < \underline{u}(x), \\ f(x,s), & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ f(x,\overline{u}(x)), & \text{if } s > \overline{u}(x). \end{cases}$$

which is continuous in s. Also set

$$\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,s)ds$$

$$\widetilde{\Phi}(u) = \int_\Omega [\frac{1}{2}(\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j} + c(x)u^2) - \widetilde{F}(x,u)]dx.$$

Let u_0 be a minimizer of $\widetilde{\Phi}$ on $H_0^1(\Omega)$; as before, we can say that the minimizer is achieved and satisfies

$$Lu_0 = \widetilde{f}(x,u_0) \quad \text{in } \Omega.$$

Thus $u_0 \in W_0^{2.p}(\Omega)$, $\forall p < \infty$.

2. We claim that $\underline{u} \le u_0 \le \overline{u}$; we will just prove the first inequality. Indeed we have

$$L(\underline{u} - u_0) \le f(x,\underline{u}) - \widetilde{f}(x,u_0), \tag{2.10}$$

and in particular

$$L(\underline{u} - u_0) \le 0, \quad \text{in } A = \{x \in \Omega; u_0(x) < \underline{u}(x)\}.$$

Since $\underline{u} - u_0 \le 0$ on ∂A, it follows from the maximum principle that $\underline{u} - u_0 \le 0$ in A. Therefore $A = \emptyset$ and the claim is proved.

3. Returning to (2.10), we have

$$L(\underline{u} - u_0) + K(\underline{u} - u_0) \le (f(x,\underline{u}) + k\underline{u}) - (f(x,u_0) + ku_0) \le 0,$$

here we let k be large enough, so that $f(x,u) + ku$ is nondecreasing in u, for a.e. $x \in \Omega$. Since $\underline{u}$ is not a solution, it follows from Lemma 2.3 that there is some $\epsilon > 0$ such that

$$\underline{u}(x) - u_0(x) \le -\epsilon \operatorname{dist}(x,\partial\Omega), \quad \forall x \in \Omega.$$

Similar inequality is obtained for for $\overline{u}$. therefore,

$$\underline{u}(x) + \epsilon \operatorname{dist}(x,\partial\Omega) \le u_0(x) \le \overline{u}(x) - \epsilon \operatorname{dist}(x,\partial\Omega), \quad \forall x \in \Omega.$$

It follows that if $u \in C_0^1(\overline{\Omega})$ and $\|u - u_0\|_{C_0^1} \le \epsilon$ then

$$\underline{u} \le u \le \overline{u} \quad \text{in } \Omega.$$

Next, we use the fact that $\widetilde{F}(x,u) - F(x,u)$ is a function of x alone for $u \in [\underline{u}(x), \overline{u}(x)]$. In particular, $\Phi(u) - \widetilde{\Phi}(u)$ is constant for $\|u - u_0\|_{C_0^1} \le \epsilon$. Hence, u_0 is a local minimum of Φ in C_0^1 topology (since it is a global minimum for $\widehat{\Phi}$). Now, from Theorem 2.1, we claim that u_0 is also a local minimum of Φ in H_0^1 topology.

3 The Δ_p operator

Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of

$$\begin{aligned} \Delta_p u = -\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

in the sense that u satisfy the equation

$$\int_\Omega (\nabla u|\nabla u|^{p-2}\nabla\varphi - f\varphi)dx = 0, \quad \forall\, \varphi \in W_0^{1,p}(\Omega).$$

We consider the functional

$$\Phi(u) = \frac{1}{p}\int_\Omega [|\nabla u|^p - F(x,u)]dx, \tag{3.2}$$

where $F(x,u) = \int_0^u f(x,s)ds$, with $f(x,r)$ continuous in $\overline{\Omega} \times \mathbb{R}^n$, and

$$|f(x,r)| \le K(1+|r|^\gamma) \quad \text{with} \quad \gamma \le p^* - 1.$$

Here $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent corresponding to the noncompact embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*(\Omega)}$. Note that u is a weak solution of (3.1) if u is a minimizer of (3.2).

For the operator Δ_p, the conclusion of Theorem 2.1 is still true. In fact, we have the following statement.

Theorem 3.1 *Assume Φ is as in (3.2), $u_0 \in W_0^{1,p}(\Omega)$ is a local minimizer of Φ in the C_0^1 topology. Then u_0 is a local minimizer of Φ in the $W_0^{1,p}$ topology.*

To prove this theorem, we need the following Lemma, whose proof relies partially on Lemma 1.3. The rest of the proof is almost the same as that of Theorem 2.1.

Lemma 3.2 *Assume $1 < p \le n$ and $f(x,u)$ is continuous in $\overline{\Omega} \times \mathbb{R}^n$ satisfies*

$$|f(x,r)| \le C|r|^\gamma + D, \quad \forall\,(x,r) \in \overline{\Omega} \times \mathbb{R}^n,$$

where C and D are real constants and $\gamma \le p^* - 1$ if $p < n$, or any positive real number if $p = n$. If $u_0 \in W_0^{1,p}(\Omega)$ satisfies (3.1), then $u \in C_0^{1,\alpha}(\overline{\Omega})$.

The proof of this lemma can be found in [6].

As a counterpart of Theorem 2.3, we present the following theorem with a new proof.

Theorem 3.3 *Assume (3.1) has sub and super solutions in $C^1(\overline{\Omega})$, and suppose $\overline{u}$ and $\underline{u}$ satisfy*

$$\begin{aligned} \Delta_p\underline{u} - f(x,\underline{u}) < 0 < \Delta_p\overline{u} - f(x,\overline{u}), \quad \text{in } \Omega, \\ \underline{u} \le 0 \le \overline{u}, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Suppose $|f_z'(\cdot,z)| < C$ for some constant C, then there is also a solution u_0 of (3.1), $\underline{u} \le u_0 \le \overline{u}$, such that u_0 is a local minimum of Φ in $W_0^{1,p}(\Omega)$.

Proof: 1. Set

$$\widetilde{f}(x,s)=\begin{cases} f(x,\underline{u}(x)) & \text{if } s<\underline{u}(x),\\ f(x,s) & \text{if } \underline{u}(x)\le s\le \overline{u}(x)\\ f(x,\overline{u}(x)) & \text{if } s>\overline{u}(x);\end{cases}$$

$$\widetilde{F}(x,u)=\int_0^u \widetilde{f}(x,s)ds,\quad \widetilde{\Phi}(u)=\frac{1}{p}\int_\Omega[|\nabla u|^p-\widetilde{F}(x,u)]dx.$$

Since $|f'_u|<C$, we can fix a number $\lambda>0$ large enough so that the mapping

$$z\mapsto \widetilde{f}(\cdot,z)+\lambda z \text{ is nondecreasing.} \tag{3.4}$$

Now write $u_0=\underline{u}$, and define u_k $(k=0,1,2,\cdots)$ inductively: $u_{k+1}\in W_0^{1,p}(\Omega)$ is a nonzero weak solution of

$$\begin{gathered} -\nabla\cdot(|\nabla u_{k+1}|^{p-2}\nabla u_{k+1})+\lambda u_{k+1}=\widetilde{f}(x,u_k)+\lambda u_k \quad \text{in } \Omega,\\ u_{k+1}=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.5}$$

The nonzero weak solution exists. In fact, u_{k+1} is a weak solution of (3.5), in the sense that u_{k+1} is a local minimum of the functional

$$\hat{\Phi}(u)=\int_\Omega[\frac{1}{p}|\nabla u|^p+\frac{\lambda}{2}u^2-(\widetilde{f}(x,u_k)+\lambda u_k)u]dx.$$

It is easy to check that $\hat{\Phi}(u)$ is coercive and weakly lower semi-continuous, hence it attains a local minimum in $W_0^{1,p}(\Omega)$.
2. Now we claim

$$\underline{u}=u_0\le u_1\le\cdots\le u_k\le\cdots \quad a.e \quad \text{in } \Omega. \tag{3.6}$$

Note from (3.5) for k=0, we have

$$-\Delta_p u_1+\lambda u_1=\widetilde{f}(x,u_0)+\lambda u_0,$$

in the sense that u_1 satisfies

$$\int_\Omega(\nabla u_1|\nabla u_1|^{p-2}\nabla\varphi+\lambda u_1\varphi)dx=\int_\Omega(\widetilde{f}(x,u_0)+\lambda u_0)\varphi dx. \tag{3.7}$$

for each $\varphi\in H_0^1(\Omega)$. From its definition, $\underline{u}$ satisfies

$$\int_\Omega \nabla\underline{u}|\nabla\underline{u}|^{p-2}\nabla\varphi dx\le\int_\Omega \widetilde{f}(x,\underline{u})\varphi dx,\quad \forall\,\varphi\in W_0^{1,p}(\Omega). \tag{3.8}$$

Compare (3.8) with (3.7), note that $u_0=\underline{u},$. For any $\varphi\in W_0^{1,p}(\Omega)$ we get

$$\begin{gathered} \int_\Omega[(\nabla u_0|\nabla u_0|^{p-2}-\nabla u_1|\nabla u_1|^{p-2})\nabla\varphi+\lambda(u_0-u_1)\varphi]dx\le 0,\\ \varphi=(u_0-u_1)^+=\begin{cases} u_0-u_1, & u_0>u_1,\\ 0, & u_0\le u_1.\end{cases} \end{gathered} \tag{3.9}$$

Then $\varphi \in W_0^{1,p}$, and

$$D\varphi = D(u_0 - u_1)^+ = \begin{cases} D(u_0 - u_1), & \text{a.e. on}\{u_0 > u_1\}, \\ 0, & \text{a.e. on}\{u_0 \le u_1\}. \end{cases}$$

Multiplying (3.9) with φ, we have

$$\int_{\{u_0>u_1\}} [(\nabla u_0|\nabla u_0|^{p-2} - \nabla u_1|\nabla u_1|^{p-2})(\nabla u_0 - \nabla u_1) + \lambda(u_0 - u_1)^2]dx \le 0,$$

so that $\mathcal{L}\{u_0 > u_1\} = 0$ with $\mathcal{L}$ for Lebesgue measurement. If $\nabla u_0 = \nabla u_1$ a.e. in Ω, then testing (3.9) with $\varphi = (u_0 - u_1)^+$, we have

$$\int_{\{u_0>u_1\}} \lambda(u_0 - u_1)^2 dx \le 0,$$

there still has $\mathcal{L}\{u_0 > u_1\} = 0$, that is, $u_0 \le u_1$ a.e. in Ω.

Now assume inductively $u_{k-1} \le u_k$ a.e. in Ω, from (3.5), for any $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega (\nabla u_{k+1}|\nabla u_{k+1}|^{p-2}\nabla\varphi + \lambda u_{k+1}\varphi)dx = \int_\Omega (\widetilde{f}(x, u_k) + \lambda u_k)\varphi dx, \tag{3.10}$$

$$\int_\Omega (\nabla u_k|\nabla u_k|^{p-2}\nabla\varphi + \lambda u_k\varphi)dx = \int_\Omega (\widetilde{f}(x, u_{k-1}) + \lambda u_{k-1})\varphi dx, \tag{3.11}$$

Subtract (3.10) from (3.11) and set $\varphi = (u_k - u_{k+1})^+$, noting $\widetilde{f}(\cdot, z) + \lambda z$ is nondecreasing to z, we deduce

$$\begin{aligned} \int_{\{u_k>u_{k+1}\}} & \Big[(\nabla u_k|\nabla u_k|^{p-2} - \nabla u_{k+1}|\nabla u_{k+1}|^{p-2})(\nabla u_k - \nabla u_{k+1}) \\ & \qquad + \lambda(u_k - u_{k+1})^2\Big]dx \\ & = \int_\Omega [(\widetilde{f}(u_{k-1}) + \lambda u_{k-1}) - (\widetilde{f}(u_k) + \lambda u_k)](u_k - u_{k+1})^+ dx \le 0, \end{aligned}$$

while $\mathcal{L}\{\nabla u_k \ne \nabla u_{k+1}\} \ne 0$; and we have

$$\begin{aligned} \int_{\{u_k>u_{k+1}\}} & \lambda(u_k - u_{k+1})^2 dx \\ & = \int_\Omega [(\widetilde{f}(u_{k-1}) + \lambda u_{k-1}) - (\widetilde{f}(u_k) + \lambda u_k)](u_k - u_{k+1})^+ dx \le 0, \end{aligned}$$

while $\nabla u_k = \nabla u_{k+1}$ a.e. in Ω. Hence $\mathcal{L}\{u_k > u_{k+1}\} = 0$, i.e. $u_k \le u_{k+1}$ a.e. in Ω, as asserted.

3. Next we show that

$$u_k \le \overline{u}, \quad \text{a.e in}\Omega\ (k = 0, 1, \cdots),$$

while $k=0$, there hold $u_0=\underline{u}\le\overline{u}$. Assume inductively $u_k\le\overline{u}$, a.e in Ω. Then from the definition of $\overline{u}$, there holds

$$\int_\Omega \nabla\overline{u}|\nabla\overline{u}|^{p-2}\nabla\varphi dx \ge \int_\Omega \widetilde{f}(x,\overline{u})\varphi dx, \quad \forall\,\varphi\in W_0^{1,p}(\Omega),$$

compare with (3.10) and setting $\varphi=(u_{k+1}-\overline{u})^+$, we find

$$\begin{aligned}\int_{\{u_{k+1}>\overline{u}\}} &[(\nabla u_{k+1}|\nabla u_{k+1}|^{p-2}-\nabla\overline{u}|\nabla\overline{u}|^{p-2})(\nabla u_{k+1}-\nabla\overline{u})+\lambda(u_{k+1}-\overline{u})^2]dx\\ &= \int_\Omega [(\widetilde{f}(u_k)+\lambda u_k)-(\widetilde{f}(\overline{u})+\lambda\overline{u})](u_{k+1}-\overline{u})^+dx\le 0.\end{aligned}$$

As before, we conclude $u_{k+1}\le\overline{u}$ a.e. in Ω.
4. From steps 2 and 3, we have

$$\underline{u}\le\cdots\le u_k\le u_{k+1}\le\cdots\le\overline{u}, \quad \text{a.e} \quad \text{in } \Omega,$$

as u_k $(k=1,2\cdots)$ are solutions of (3.5), using Moser's iterative scheme which we cited in the proof of Lemma 1.3, we have $u_k\in L^q$ for all $q<\infty$, $(k=1,2\cdots)$, since $|f'_z(\cdot,z)|<C$, then $\widetilde{f}(\cdot,z)\le c(1+|z|)$, so that $f(\cdot,u_k)\in L^q$, then from (3.5), we deduce $u_k\in W_0^{2,q}$ for all $q<\infty$, $(k=1,2\cdots)$. We can also deduce from (3.5) that

$$\begin{aligned}\|u_k\|^q_{W_0^{2,q}} &\le C(\|\widetilde{f}(u_{k-1})\|^q_{L^q}+\|u_{k-1}\|^q_{L^q})\\ &\le C(1+\|u_{k-1}\|^q_{L^q})\\ &\le C(1+\max\{\|\underline{u}\|^q_{L^q},\,\|\overline{u}\|^q_{L^q}\}).\end{aligned}$$

So that u_k is unified bounded in $W_0^{2,q}(\Omega)$, hence there exists a subsequence converging in $W_0^{2,q}(\Omega)$, still denote u_k, i.e.

$$u_k\rightharpoonup u_0, \quad \text{in } W_0^{2,q}(\Omega), \tag{3.12}$$

set q be large enough, then $W_0^{2,q}\hookrightarrow\hookrightarrow C_0^{1,\alpha}$, and

$$u_k\to u_0, \quad \text{in } C_0^{1,\alpha}(\Omega).$$

From (3.12), let $k\to\infty$ in (3.10) and cancelling the same item on both sides, we get

$$\int_\Omega \nabla u_0|\nabla u_0|^{p-2}\nabla\varphi dx = \int_\Omega \widetilde{f}(x,u_0)\varphi dx, \quad \forall\,\varphi\in W_0^{1,p}(\Omega).$$

This means $u_0\in C_0^{1,\alpha}$ is a weak solution of

$$\begin{gathered}\Delta_p u=\widetilde{f} \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega.\end{gathered}$$

Hence we have a local minimizer of $\widetilde{\Phi}$.

Then given $\overline{u}$ and $\underline{u}$ in $C^1(\overline{\Omega})$ satisfying the assumption (3.3), we can deduce

$$\underline{u} < u_0 < \overline{u}, \quad \text{a.e.} \quad \text{in } \Omega,$$

Let $\Lambda = \{x \in \Omega : \underline{u}(x) = u_0(x)\} \cup \{x \in \Omega : \overline{u}(x) = u_0(x)\}$, then $\mathcal{L}(\overline{\Lambda}) = 0$ (if not, $\mathcal{L}(\overline{\Lambda}) > 0$, since u_0, $\underline{u}$, and $\overline{u}$ are all continuous, then there must be $\underline{u}(x) = u_0(x)$ or $\overline{u}(x) = u_0(x)$ on $\overline{\Lambda}$, so $\Delta_p\underline{u} = \Delta_p u_0$ or $\Delta_p\overline{u} = \Delta_p u_0$ on $\overline{\Lambda}$, and this contradicts (3.3).). Thus $\Omega' := \Omega\backslash\overline{\Lambda} \subset\subset \Omega$ is still a domain in $\mathbb{R}^n$. and

$$\underline{u}(x) < u_0(x) < \overline{u}(x), \quad \text{for any x in } \Omega',$$

so when set ϵ be small enough, and $\|u - u_0\|_{C_0^1} \le \epsilon$ there has

$$\underline{u} \le u \le \overline{u}, \quad \text{in } \Omega'.$$

Denote $\hat{\Phi}(u) = \frac{1}{p}\int_{\Omega'}[|\nabla u|^p - F(x,u)]dx$, and $\hat{\widetilde{\Phi}}(u) = \frac{1}{p}\int_{\Omega'}[|\nabla u|^p - \widetilde{F}(x,u)]dx$, then u_0 is a $C^1(\overline{\Omega'})$ local minimizer of $\hat{\Phi}$, then as we do in Lemma 2.4, noted the fact that $\hat{\Phi}(u) - \hat{\widetilde{\Phi}}(u)$ is constant for $\|u-u_0\|_{C_0^1} \le \epsilon$, we deduce u_0 is a local minimizer of $\hat{\Phi}$ in C^1 topology, and since $\mathcal{L}\{\Omega\backslash\Omega'\} = 0$, so the integral functional Φ and $\hat{\Phi}$ share the same minimizers, thus we have u_0 is a local minimizer of Φ in C_0^1 topology.

Finally, using Theorem 3.1, we deduce that u_0 is a local minimizer of (3.2) in $W_0^{1,p}(\Omega)$ topology, $\underline{u} \le u_0 \le \overline{u}$. Thus complete the proof of Theorem 3.3. $\square$

4 Appendix

Proof of lemma 1.1 1. Note that u is a weak solution of equation $-\Delta u = g(\cdot, u)$ in Ω, in the sense that u satisfies

$$\int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega g\cdot\varphi, \quad \forall\,\varphi \in H_0^1(\Omega) \tag{4.1}$$

Then we choose $\eta \in \mathbf{C}_0^\infty$ and for $s \ge 0, M \ge 0$, let $\varphi = u\min\{|u|^{2s}, M^2\}\eta^2 \in W_0^{1,2}(\Omega)$, with $supp\,\varphi \subset\subset \Omega$. then we have

$$\nabla\varphi = \begin{cases} \nabla u\min\{|u|^{2s}, M^2\}\eta^2 + 2s|u|^{2s}\nabla u\eta^2 & \\ \quad +2u\min\{|u|^{2s}, M^2\}\eta\nabla\eta, & \text{if } \min\{|u|^{2s}, M^2\} = |u|^{2s}, \\ \nabla u\min\{|u|^{2s}, M^2\}\eta^2 & \\ \quad +2u\ \min\{|u|^{2s}, M^2\}\eta\nabla\eta, & \text{otherwise.} \end{cases} \tag{4.2}$$

Multiplying (4.1) with φ, we obtain

$$\begin{aligned}\int_\Omega \nabla u\nabla\varphi &= \int_\Omega |\nabla u|^2 \min\{|u|^{2s},M^2\}\eta^2+2s\int_{\{|u|^s<M\}}|u|^{2s}|\nabla u|^2\eta^2\\ &\quad +2\int_\Omega \nabla u\, u\,\min\{|u|^{2s},M^2\}\eta\nabla\eta\\ &= \int_\Omega g\varphi\le\int_\Omega |g||\varphi|\\ &\le \int_\Omega |a|(|1+|u|)|u|\min\{|u|^{2s},M^2\}\eta^2(*) \qquad (4.3)\end{aligned}$$

2. Suppose $u\in L^{2s+2}_{\rm loc}(\Omega)$. As in (4.2), we have

$$\begin{aligned}&\nabla(u\min\{|u|^s,M\}\eta)\\ &=\begin{cases}\nabla u\min\{|u|^s,M\}\eta+s|u|^s\nabla u\eta & \\ \quad +u\min\{|u|^s,M\}\nabla\eta, & \text{if } \min\{|u|^s,\ M\}=|u|^s,\\ \nabla u\min\{|u|^s,M\}\eta+u\min\{|u|^s,M\}\nabla\eta, & \text{otherwise.}\end{cases}\end{aligned}$$

We write this expression as

$$\begin{aligned}&\nabla(u\min\{|u|^s,M\}\eta)\\ &\qquad=\nabla u\min\{|u|^s,M\}\eta+\langle s|u|^s\nabla u\eta\rangle_\Theta+u\min\{|u|^s,M\}\nabla\eta\end{aligned}$$

Here Θ denotes the set $\{x\in\Omega:\ |u(x)|^s\le M\}$, then there holds

$$\begin{aligned}&|\nabla(u\,\min\{|u|^s,M\}\eta)|^2\\ &= |(\nabla u\min\{|u|^s,M\}\eta+\langle s|u|^s\nabla u\eta\rangle_\Theta+u\min\{|u|^s,M\}\nabla\eta)|^2\\ &= |\nabla u|^2\min\{|u|^{2s},M^2\}\eta^2+\langle s|u|^{2s}|\nabla u|^2\eta^2\rangle_\Theta\\ &\quad +u^2\min\{|u|^{2s},M^2\}|\nabla\eta|^2+2(\langle s|\nabla u|^2|u|^s\min\{|u|^s,M\}\eta^2\rangle_\Theta\\ &\quad +\langle s|u|^s\nabla uu\min\{|u|^s,M\}\nabla\eta\eta\rangle_\Theta+u\nabla u\min\{|u|^{2s},M^2\}\nabla\eta\eta)\\ &= |\nabla u|^2\min\{|u|^{2s},M^2\}\eta^2+\langle s|\nabla u|^2\min\{|u|^{2s},M^2\}\eta^2\rangle_\Theta\\ &\quad +u^2\min\{|u|^{2s},M^2\}|\nabla\eta|^2+2<s|\nabla u|^2\min\{|u|^{2s},M^2\}\eta^2\rangle_\Theta\\ &\quad +2<s\nabla u\,u\min\{|u|^{2s},M^2\}\nabla\eta\eta\rangle_\Theta+2\nabla u\,u\min\{|u|^{2s},M^2\}\nabla\eta\eta\\ &\le C|\nabla u|^2\min\{|u|^{2s},M^2\}\eta^2+C\nabla u\,u\min\{|u|^{2s},M^2\}\nabla\eta\eta\\ &\quad +u^2\min\{|u|^{2s},M^2\}|\nabla\eta|^2\end{aligned}$$

Note that $\eta \in C_0^\infty$, so $\int_\Omega u^2 \min\{|u|^{2s}, M^2\}|\nabla\eta|^2 \leq \infty$. Then, from (4.3),

$$\begin{aligned}
&\int_\Omega |\nabla(u\min\{|u|^s, M\}\eta)|^2 \\
&\leq C + C\int_\Omega (\nabla u\, u\min\{|u|^{2s}, M^2\}\nabla\eta\eta + |\nabla u|^2\min\{|u|^{2s}, M^2\}\eta^2 \\
&\leq C + C\int_\Omega |a|(|1+|u|)|u|\ \min\{|u|^{2s}, M^2\}\eta^2 \\
&\leq C + C\int_\Omega |a|\min\{|u|^{2s}, M^2\}\eta^2 + \int_\Omega |a||u|^2\min\{|u|^{2s}, M^2\}\eta^2 \\
&\leq C + C\int_\Omega |a||u|^2\min\{|u|^{2s}, M^2\}\eta^2 \\
&\leq C + CB\int_\Omega |u|^2\min\{|u|^{2s}, M^2\}\eta^2 \\
&\quad + \int_{\{x\in\Omega;\ |a(x)|\geq B\}} |a||u|^2\min\{|u|^{2s}, M^2\}\eta^2 \\
&\leq C(1+B) + (C\int_{\{x\in\Omega;\ |a(x)|\geq B\}} |a|^{n/2})^{\frac{2}{n}} \cdot \\
&\quad (\int_\Omega |u\min\{|u|^s, M\}\eta|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \\
&\leq C(1+B) + \varepsilon(B)\cdot\int_\Omega |\nabla(u\min\{|u|^s, M\}\eta)|^2
\end{aligned}$$

The last step come from $H_0^1(\Omega) \hookrightarrow M^{\frac{2n}{n-2}}$, hence $\|\cdot\|^2_{M^{\frac{2n}{n-2}}} \leq c\|\cdot\|^2_{H_0^1}$. And where

$$\varepsilon(K) = (\int_{\{x\in\Omega;\ |a(x)|\geq B\}} |a|^{n/2})^{2/n} \to 0, \quad (B \to \infty).$$

Fix B such that $\varepsilon(B) = \frac{1}{2}$ and observe that for this choice of B, and as above, we now may conclude that

$$\int_{\{x\in\Omega;\ |u(x)|^s\leq M\}} |\nabla(|u|^{s+1}\eta)|^2 \leq\ C\int_\Omega |\nabla(u\min\{|u|^s, M\}\eta)|^2 \leq\ C(1+B)$$

remains uniformly bounded in M. Hence let $M \to \infty$ we derive that

$$|u|^{s+1}\eta \in W_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega);$$

that is, u $\in L_{\rm loc}^{\frac{(2s+2)n}{n-2}}(\Omega)$. Now iterate, to obtain the conclusion of the lemma. If u $\in W_0^{1,2}(\Omega)$, we may let $\eta = 1$ to obtain that u $\in L^q(\Omega)$ for all $q < \infty$.

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