Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 103?, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

On the unique constructive solvability of Hammerstein equations *

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Abstract

We study the unique constructive solvability of Hammerstein operator equation in Banach spaces using iterative and projection like methods. Some error estimates are also given. The linear part is assumed to be either selfadjoint or non-selfadjoint. Applications to Hammerstein integral equations are given.

1 Introduction

In this paper, we shall study the unique constructive solvability of operator equation

$$x - KFx = f \tag{1.1}$$

where K is linear and F is a nonlinear map. We first study (1.1) in the operator form using an iterative process, the A-proper mapping approach and the Brouwer degree theory. We shall consider (1.1) in a general setting between two Banach spaces. To that end, we shall use two approaches. One is based on applying the Brouwer degree theory directly to the finite dimensional approximations of the map $I-KF$, and the other one is based on splitting first the map K as a product of two suitable maps and then use the Brouwer degree. Then we apply the obtained results to Hammerstein integral equations. This work is a continuation of our study of these equations in [11]. There is an extensive literature on Hammerstein equations and we refer to [5, 6, 16].

2 Some preliminaries on A-proper maps

Let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of Banach spaces X and Y respectively such that $\dim X_n = \dim Y_n$ for each n and $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. Let $P_n : X \to Y_n$ and $Q_n : Y \to Y_n$ be linear projections onto X_n and Y_n respectively such that $P_n x \to x$ for each $x \in X$ and $\delta = \max \|Q_n\| < \infty$. Then $\Gamma = \{X_n, P_n; Y_n, Q_n\}$ is a projection scheme for (X, Y).

* *Mathematics Subject Classifications:* 47H14, 35L70, 35L75, 35J40.
Key words: Unique approximation, solvability, Hammerstein equations, A-proper maps, surjectivity, error estimates.

Submitted July 13, 2002. Published December 10, 2002.

Definition A map $T : D \subset X \to Y$ is said to be *approximation-proper* (A-proper for short) with respect to Γ if
(i) $Q_nT : D \cap X_n \to Y_n$ is continuous for each n and
(ii) whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $\|Q_{n_k}Tx_{n_k} - Q_{n_k}f\| \to 0$ for some $f \in Y$, then a subsequence $x_{n_{k(i)}} \to x$ and $Tx = f$. T is said to be *pseudo A-proper* with respect to Γ if in (ii) above we do not require that a subsequence of $\{x_{n_k}\}$ converges to x for which $Tx = f$. If f is given in advance, we say that T is (pseudo) A-proper at f.

For the developments of the (pseudo) A-proper mapping theory and applications to differential equations, we refer to [9, 10, 12, 14, 15]. To demonstrate the generality and the unifying nature of the (pseudo) A-proper mapping theory, we state now a number of examples of A-proper and pseudo A-proper maps.

To look at ϕ-condensing maps, we recall that the *set measure of non-compactness* of a bounded set $D \subset X$ is defined as $\gamma(D) = \inf\{d > 0 : D$ has a finite covering by sets of diameter less than $d\}$. The *ball-measure of non-compactness* of D is defined as $\chi(D) = \inf\{r > 0 | D \subset \cup_{i=1}^{n} B(x_i, r),\ x_i \in X,\ n \in N\}$. Let ϕ denote either the set or the ball-measure of non-compactness. Then a map $N : D \subset X \to X$ is said to be *$k - \phi$ contractive* (*ϕ-condensing*) if $\phi(N(Q)) \leq k\phi(Q)$ (respectively $\phi(N(Q)) < \phi(Q)$) whenever $Q \subset D$ (with $\phi(Q) \neq 0$).

Recall that $N : X \to Y$ is K-monotone for some $K : X \to Y^*$ if $(Nx - Ny, K(x - y)) \geq 0$ for all $x,\ y \in X$. It is said to be generalized pseudo-K-monotone (of type (KM)) if whenever $x_n \rightharpoonup x$ and $\limsup(Nx_n, K(x_n - x)) \leq 0$ then $(Nx_n, K(x_n - x)) \to 0$ and $Nx_n \rightharpoonup Nx$ (then $Nx_n \rightharpoonup Nx$). Recall that N is said to be of type (KS_+) if $x_n \rightharpoonup x$ and $\limsup(Nx_n, K(x_n - x)) \leq 0$ imply that $x_n \to x$. If $x_n \rightharpoonup x$ implies that $\limsup(Nx_n\ , K(x_n - x)) \geq 0$, N is said to be of type (KP). If $Y = X^*$ and K is the identity map, then these maps are called monotone, generalized pseudo monotone, of type (M) and (S_+) respectively. If $Y = X$ and $K = J$ the duality map, then J-monotone maps are called accretive. It is known that bounded monotone maps are of type (M). We say that N is demicontinuous if $x_n \to\ x$ in X implies that $Nx_n \rightharpoonup Nx$. It is well known that $I - N$ is A-proper if N is ball-condensing and that K-monotone like maps are pseudo A-proper under some conditions on N and K. Moreover, their perturbations by Fredholm or hyperbolic like maps are A-proper or pseudo A-proper [10, 11, 12, 14].

The following result states that ball-condensing perturbations of stable A-proper maps are also A-proper.

Theorem 2.1 *([8]) Let $D \subset X$ be closed, $T : X \to Y$ be continuous and A-proper with respect to a projectional scheme Γ and a-stable, i.e.,for some $c > 0$ and n_0*

$$\|Q_nTx - Q_nTy\| \geq c\|x - y\| \quad \textit{for } x, y \in X_n,\ n \geq n_0$$

and $F : D \to Y$ be continuous. Then $T + F : D \to Y$ is A-proper with respect to Γ if F is k-ball contractive with $k\delta < c$, or it is ball-condensing if $\delta = c = 1$.

Remark The A-properness of T in Theorem 2.1 is equivalent to T being surjective. In particular, as T we can take a c-strongly K- monotone map for a suitable $K : X \to Y^*$, i.e., $(Tx - Ty, K(x-y)) \geq c\|x-y\|^2$ for all $x, y \in X$. In particular, since c-strongly accretive maps are surjective, we have the following important special case [8].

Corollary 2.2 *Let X be a π_1 space, $D \subset X$ be closed, $T : X \to X$ be continuous and c-strongly accretive and $F : D \to X$ be continuous and either k-ball contractive with $k < c$, or it is ball-condensing if $c = 1$. Then $T + F : D \to X$ is A-proper with respect to Γ.*

To study error estimates of approximate solutions for non-differentiable maps, we need a notion of a multivalued derivative. Let $U \subset X$ be an open set and $T : \bar{U} \to Y$. A positively homogeneous map $A : X \to 2^Y$, with Ax closed and convex for each $x \in X$, is said to be a *multivalued derivative* of T at $x_0 \in U$ if there is a map $R = R(x_0) : \bar{U} - x_0 \to 2^Y$ such that $\|y\|/\|x - x_0\| \to 0$ as $x \to x_0$ for each $y \in R(x - x_0)$ and

$$Tx - Tx_0 \in A(x - x_0) + R(x - x_0) \quad \text{for } x \text{ near } x_0.$$

A map $A : X \to 2^Y$ is m-bounded if there is $m > 0$ such that $\|y\| \leq m\|x\|$ for each $y \in Ax$, $x \in X$. It is c-coercive if $\|y\| \geq c\|x\|$ for each $y \in Ax$, $x \in X$.

The following result from [9] will be needed below.

Theorem 2.3 *Let $T : \overline{U} \subset X \to Y$ be A-proper with respect to Γ and x_0 be a solution of $Tx = f$. Suppose that A is an odd multivalued derivative of T at x_0 and there exist constants $c_0 > 0$ and $n_0 \geq 1$ such that*

$$\|Q_n u\| \geq c_0\|x\| \quad \textit{for } x \in X_n, \quad u \in Ax, \quad n \geq n_0. \tag{2.1}$$

(a) If x_0 is an isolated solution, then the equation $Tx = f$ is strongly approximation solvable in $B_r(x_0)$ for some $r > 0$.
(b) If, in addition, A is c_1-coercive for some $c_1 > 0$, then x_0 is an isolated solution, the conclusion of (a) holds and, for $\epsilon \in (0, c_0)$, approximate solutions x_n satisfy

$$\|x_n - x_0\| \leq (c_0 - \epsilon)^{-1}\|Tx_n - f\| \quad \textit{for } n \geq n_1 \geq n_0. \tag{2.2}$$

(c) If x_0 is an isolated solution in $B_r(x_0)$, A is c_2-bounded for some c_2 and

$$Tx - Ty \in A(x-y) + R(x-y) \quad \textit{whenever } x - y \in B_r \tag{2.3}$$

and $z/\|x - y\| \to 0$ as $x \to x_0$ and $y \to x_0$ for each $z \in R(x - y)$, then the equation $Tx = f$ is uniquely approximation solvable in $B_r(x_0)$ and the unique solutions $x_n \in B_r(x_0) \cap X_n$ of $Q_n Tx = Q_n f$ satisfy

$$\|x_n - x_0\| \leq k\|P_n x_0 - x_0\| \leq c \operatorname{dist}(x_0, X_n), \tag{2.4}$$

where k depends on c_0, c_2, ϵ and δ and $c = 2k\delta_1$, $\delta_1 = \sup \|P_n\|$.

3 Unique constructive solvability of Hammerstein operator equations

In this section, we shall prove a number of constructive solvability results for Eq. (1.1) using an iterative process and finite dimensional approximations and give the error estimates in the second case. We shall consider (1.1) in a general setting between two Banach spaces. To that end, we shall use two approaches. One is based on using the contraction mapping principal or applying the Brouwer degree theory directly to the finite dimensional approximations of the map $I-KF$, and the other one is based on splitting first the map K as a product of two suitable maps and then use the Brouwer degree. We study (1.1) with A selfadjoint as well as non selfadjoint. Our first result is an extension of a theorem of Dolph [3].

Theorem 3.1 *Let $K:X\to X$ be a continuous linear map, $\lambda^{-1}\notin\sigma(K)$, $d=\|(I-\lambda K)^{-1}K\|^{-1}$ and $F:X\to X$ be nonlinear and continuous.*
a) If for some $k\in(0,d)$,

$$\|Fx-\lambda x-(Fy-\lambda y)\|\le k\|x-y\| \quad \text{for all } x,y\in H. \tag{3.1}$$

then (1.1) is uniquely solvable for each $f\in X$ and the solution is the limit of the iteration process

$$x_n-\lambda Kx_n=KFx_{n-1}-\lambda Kx_{n-1}+f. \tag{3.2}$$

b) If, in addition, either K is compact or $\delta=\max\|P_n\|=1$ and $k\|(I-\lambda K)^{-1}\|\ \|K\|<1$ and $P_nKx=Kx$ for $x\in X_n$, then (1.1) is approximation solvable with respect to Γ for each $f\in X$ and the approximate solutions $\{x_n\in X_n\}$ of $x-P_nKFx=P_nf$ satisfy

$$\|x_n-x\|\le c\|x_n-KFx_n-f\| \quad \text{for some } c \text{ and all large } n. \tag{3.3}$$

and

$$\|x_n-x\|\le c\|P_nx-x\|\le c_1\operatorname{dist}(x,X_n). \tag{3.4}$$

c) If condition (3.1) holds with $k=d$, X is a uniformly convex space with $\delta=1$ and

$$\|Fx-\lambda x\|\le a\|x\|+b \quad \text{for some } a<k, b>0, x\in X. \tag{3.5}$$

then (1.1) is solvable for each $f\in X$.

Proof. Equation (1.1) is equivalent to $Ax-Nx=f$ with $A=I-\lambda K$ and $N=K(F-\lambda I)$. Hence, it is easy to show that $A^{-1}N$ is $k_1=k\|A^{-1}K\|$-contractive with $k_1<1$. Thus, part a) follows from the contractive fixed point principle and c) follows from Theorem 3.3 in [11]. Regarding part b), we need only to show that condition (2.1) of Theorem 2.3 holds. Assume first that K is compact. Then $I-KF$ is A-proper with respect to Γ. Set $B_1x=\{K(y-\lambda x):\|y-\lambda x\|\le k\|x\|\}$ and $Bx=Ax-B_1x$ for $x\in X$. Then B is homogeneous with Bx convex for each $x \in X$ and $A(x-y) - (Nx - Ny) \in B(x-y)$ for each $x, y \in X$. Moreover, if $0 \in Bx$, then $Ax = K(y - \lambda x)$ for some y and

$$\|x\| \le \|A^{-1}K\| \, \|y - \lambda x\| < \|x\|.$$

Hence, $x = 0$. Since B_1 is upper semicontinuous and compact, $B = A - B_1$ is a multivalued A-proper map with respect to Γ (cf. [14]). This implies that condition (2.1) holds. Since also $Nx - Ny \in B_1(x-y)$, the conclusions follow from Theorem 2.3.

Next, let $\delta = 1$ and $l = k\|(I - \lambda K)^{-1}\| \, \|K\| < 1$. Then $I - KF$ is A-proper with respect to $\Gamma = \{X_n, P_n\}$. Indeed, let $\{x_n \in X_n\}$ be bounded and $x_n - P_nKFx_n \to f$. Set $y_n = (I - \lambda K)x_n$. Then $y_n - P_nK(F - \lambda I)(I - \lambda K)^{-1}y_n \to f$ and the map $F_1 = K(F - \lambda I)(I - \lambda K)^{-1}$ is an l-contraction with $l < 1$. Hence, $I - F_1$ is A-proper with respect to Γ and therefore, a subsequence $y_{n_k} \to y$ and $y - F_1y = f$. Hence, $x - KFx = f$ with $x = (I - \lambda K)^{-1}y$, proving that $I - KF$ is A-proper.

Now, let $y \in P_n(Ax - B_1x)$ for some $x \in X_n$. Then $y = P_n(Ax - Kv) = Ax - P_nKv$ for some v with $\|v\| \le k\|x\|$ and $x = A^{-1}P_n(y + Kv)$. Hence,

$$\|x\| \le \delta\|A^{-1}\|\big(\|y\| + k\|K\| \, \|x\|\big)$$

and

$$\big(1 - k\|A^{-1}\| \, \|K\|\big)\|x\| \le \|A^{-1}\| \, \|y\|.$$

Hence, (2.1) holds and similarly we show that B is c-coercive. Thus, Theorem 2.3 applies. $\square$

Let $\Sigma(K)$ be the set of characteristic values of K, i.e., $\Sigma(K) = \{\mu \mid 1/\mu \in \sigma(K)\}$, and $\mu^* = inf\{\mu \mid \mu \in \Sigma(K) \cap (0, \infty)\}$. For $c \in \Sigma(K) \cap (-\infty, \mu^*]$, define $d_c^- = \operatorname{dist}(c, \Sigma(K) \cap (-\infty, c))$. We have (cf. also [11])

Theorem 3.2 *Let $K : H \to H$ be a selfadjoint map, $F : H \to H$ be nonlinear and continuous. Assume that*
(i) $(Fx - Fy, x - y) \ge \alpha\|x - y\|^2$ for all $x, y \in H$,
(ii) $\|Fx - Fy\| \le \beta\|x - y\|$ for all $x, y \in H$.
(a) If (i)-(ii) hold and $\beta^2 < \alpha d_c^- + c(d_c^- - c - 2\alpha)$ for some $c \le \mu^*$, then (1.1) is uniquely solvable for each $f \in X$ and the solution is the limit of the iteration process (3.2). Moreover, if also $P_nKx = Kx$ on X_n and either K is compact or $\beta^2 + \lambda^2$ is sufficiently small, then (1.1) is uniquely approximation solvable for each $f \in H$ and (3.3)-(3.4) hold.
(b) If $\beta^2 \le \alpha d_c^- + c(d_c^- - c - 2\alpha)$ and, for some $a < \lambda = c - d_c^-/2$ and $b > 0$,

$$\|Fx - \lambda x\| \le a\|x\| + b \quad \text{for all } x \in H$$

then (1.1) is solvable for each $f \in H$.

Proof. Let $\lambda = c - d_c^-/2$. Then $\lambda \notin \Sigma(K)$ and $d = \operatorname{dist}(\lambda, \Sigma(K)) > 0$. Since $(I - \lambda K)^{-1}K = -1/\lambda + 1/\lambda(I - \lambda K)^{-1}$, we have that ([4]) $\|(I - \lambda K)^{-1}K\| =$

$\sup_{\mu\in\sigma(K)} |-1/\lambda + 1/\lambda(1-\lambda\mu)^{-1}| = \sup_{\mu\in\Sigma(K)} |(\mu-\lambda)^{-1}| = d^{-1}$. Using conditions (i)-(ii), we get

$$\|Fx + \lambda x - (Fy + \lambda y)\| \le (\beta^2 + \lambda^2 + 2\alpha\lambda)^{1/2}\|x - y\|.$$

By our choice of λ and the condition on β, we get

$$\beta^2 + \lambda^2 + 2\alpha\lambda = \beta^2 + \alpha d_c^- + c(d_c^- - c - 2\alpha) + (d_c^-/2)^2 < (d_c^-/2)^2 = d^2.$$

Hence, the conclusions follow from Theorem 3.1. $\square$

Theorem 3.3 *Let $K : H \to H$ be selfadjoint, $F : H \to H$ be a gradient map and $B^\pm : H \to H$ be selfadjoint maps such that*
(i)$(B^-(x-y), x-y) \le (Fx - Fy, x-y) \le (B^+(x-y), x-y)$ for all $x, y \in H$.
(ii) $\delta\|B^\pm - \lambda I\| \le d = \min\{|\mu| : \mu \in \sigma(I - \lambda K)^{-1}K\}$ for some λ.
(a) If the inequality is strict in (ii), then (1.1) is uniquely solvable for each $f \in X$ and the solution is the limit of the iteration process (3.2). Moreover, if also $P_nKx = Kx$ on X_n and $\|B^\pm - \lambda I\|$ is sufficiently small , then (1.1) is uniquely approximation solvable with respect to Γ for H for each $f \in H$ and the approximate solutions satisfy (3.3)-(3.4).
(b) If, in addition, there are $0 < a < d$ and $b \ge 0$ such that

$$\|Fx - \lambda x\| \le a\|x\| + b \quad \textit{for all } x \in H$$

then (1.1) is solvable for each $f \in H$.

Proof. Since λI is a gradient of the functional $x \to \lambda(x,x)/2$, $N - \lambda I$ is a gradient map and

$$-\|B^- - \lambda I\|\ \|x-y\|^2 \le ((B^- - \lambda I)(x-y), x-y),$$
$$((B^+ - \lambda I)(x-y), x-y) \le \|B^+ - \lambda I\|\ \|x-y\|^2.$$

Hence, by Lemma 1 in [7],

$$\|Fx - \lambda x - (Fy - \lambda y)\| \le k\|x-y\| \quad \text{for all } x, y \in H$$

where $k = \max(\|B^- - \lambda I\|, \|B^+ - \lambda I\|)$. Since $d = \|(I - \lambda K)^{-1}K\|^{-1}$ ([4]), the conclusions follow from Theorem 3.1. $\square$

For $c \in \Sigma(K) \cap (\mu^*, \infty)$, define $d_c^+ = \mathrm{dist}(c, \Sigma(K) \cap (c, \infty))$. We have the following sharper version of Theorem 3.2 (cf. also [11]).

Theorem 3.4 *Let $K : H \to H$ be selfadjoint, $F : H \to H$ be a gradient map and $\alpha, \beta \in R$ be such that*

$$\alpha\|x-y\|^2 \le (Fx - Fy, x-y) \le \beta\|x-y\|^2 \quad \textit{for } x, y \in H.$$

(a) If either $c \in \Sigma(K) \cap (-\infty, \mu^*]$ and $-c < \alpha \le \beta < -c + d_c^-$, or $c \in \Sigma(K) \cap (\mu^*, \infty)$ and $-c - d_c^+ < \alpha \le \beta < -c$, then (1.1) is uniquely solvable for

each $f \in X$ and the solution is the limit of the iterative process (3.2). Moreover, if also $P_nKx = Kx$ on X_n and $\alpha+\lambda$ and $\beta+\lambda$ are sufficiently small, then (1.1) is uniquely approximation solvable for each $f \in H$ and (3.3)-(3.4) hold.
(b) If the conditions in (a) hold with each "$<$" sign replaced by "$\leq$" and, for some $a < \lambda$ with $\lambda = c - d_c^-/2$ if $c \leq \mu^*$ and $\lambda = c + d_c^+/2$ if $c > \mu^*$, and $b > 0$,

$$\|Fx - \lambda x\| \leq a\|x\| + b \ \text{ for all}\, x \in H$$

then (1.1) is solvable for each $f \in H$.

Proof. As above, we have that

$$\|Fx + \lambda x - Fy - \lambda y\| \leq \max(|\alpha + \lambda|, |\beta + \lambda|)\|x - y\|.$$

By our choice of λ as given in b), we conclude that $|\alpha+\lambda| \leq d = \operatorname{dist}(\lambda, \Sigma(K)) = d_c^{\pm}/2$ and $|\beta + \lambda| \leq d$ with the inequalities being strict in part a). Hence, Theorem 3.1 is applicable. $\square$

Remark The unique solvability results of the type in Theorems 3.2-3.4 for semilinear equations have been obtained in [2, 9, 13].

Using a suitable splitting of K, we can still prove the unique approximation solvability of Eq. (1.1) without assuming condition (ii) in Theorem 3.2. Let us first describe the setting for this approach (cf. [16]). Recall that a map K acting in a Hilbert space H is called positive in the sense of Krasnoselskii if there exists a number $\mu > 0$ for which

$$(Kx, Kx) \leq \mu(Kx, x), \quad x \in H.$$

The infimum of all such numbers μ is called the positivity constant of K and is denoted by $\mu(K)$. The simplest example of a positive map is provided by any bounded selfadjoint map K on H. Then $\mu(K) = \|K\|$ for such maps. A compact normal map K in a Hilbert space is positive on H if and only if (cf. [5]) the number

$$[\inf_{\lambda\in\sigma(K),\lambda\neq 0} Re(\lambda^{-1})]^{-1}$$

is well defined and positive. It that case, it is equal to $\mu(K)$.

Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive definite bounded selfadjoint map in H, and $C = K_H^{1/2} : H \to X$, where K_H is the restriction of K to H. We know that the positive square root can be extended to a bounded linear map $T : X^* \to H$ such that $K = T^*T$, where the adjoint of T is $T^* = K_H^{1/2} = C : H \to X$ and $C^* = T$ (cf. [16]). Hence, we can write $K = CT$. Set $\mu(K) = \|C\|^2$. We have the following extension of Theorem 3.2 as well as of a result of Vainberg [16].

Theorem 3.5 *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive definite bounded selfadjoint map in H, and $C =$*

$K_H^{1/2} : H \to X$, where K_H is the restriction of K to H, and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Suppose that $F : X \to X^*$ and c is the smallest number such that

$$(Fx - Fy, x - y) \leq c\|x - y\|^2 \quad \text{for all } x, y \in X$$

a) If $c\mu(K) < 1$, then (1.1) is uniquely approximation solvable in X for each $f \in C(H) \subset X$.
b) If $c\mu(K) = 1$ and $I - TFC$ satisfies condition (+) in H, then Eq. (1.1) is solvable in X for each $f \in C(H) \subset X$.

Proof. a) We have that $K = T^*T$, where the adjoint of T is $T^* = K_H^{1/2} = C : H \to X$ and $C^* = T$. Hence, we can write $K = CT$. Let $f \in C(H)$ so that $f = Ch$ for some $h \in H$. We note that (1.1) is equivalent to the following equation (cf. [16])

$$y - TFCy = h. \tag{3.6}$$

The map $I - TFC : H \to H$ is $1 - c\mu(K)$-strongly monotone. Indeed, for each $x, y \in H$ we have

$$\begin{aligned}
(x - TFCx - y + TFCy, x - y) &= \|x - y)\|^2 - (TFCx - TFCy, x - y) \\
&= \|x - y\|^2 - (FCx - FCy, Cx - Cy) \\
&\geq \|x - y\|^2 - c\|Cx - Cy\|^2 \\
&\geq (1 - c\mu(K))\|x - y\|^2.
\end{aligned}$$

Hence, $I - TFC$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ for H and coercive. It follows that Eq. (3.6) is uniquely approximation solvable for each $h \in H$. Therefore, (1.1) is uniquely approximation solvable for each $f \in C(H)$.
b) If $c\mu(K) = 1$, then as in part a) we get that $I - TFC$ is monotone and satisfies condition (+). Hence, (3.6) is solvable for each $h \in H$ and therefore (1.1) is solvable for each $f \in C(H)$. $\square$

Corollary 3.6 *Let X be a reflexive embeddable Banach space ($X \subset H \subset X^*$), $K : X^* \to X$ be a positive definite bounded selfadjoint map in H, and $C = K_H^{1/2}$, where K_H is the restriction of K to H, and $T : X^* \to H$ be a bounded linear extension of $K_H^{1/2}$. Let $F = F_1 + F_2 : X \to X^*$ and c be the smallest number such that*

$$(F_1x - F_1y, x - y) \leq c\|x - y\|^2 \text{ for all } \; x, y \in X$$

and F_2 is a Lipshitz map with constant k such that $1 - (c + k)\mu(K) > 0$. Then (1.1) is uniquely approximation solvable in X for each $f \in C(H) \subset X$.

Proof. It suffices to show that $I - TFC : H \to H$ is A-proper. For $x, y \in H$, we have

$$\begin{aligned}
&(x - TFCx - y + TFCy, x - y)\\
&= \|x - y)\|^2 - (TFCx - TFCy, x - y)\\
&= \|x - y\|^2 - (F_1Cx - F_1Cy, Cx - Cy) - (F_2Cx - F_2Cx, Cx - Cy)\\
&\geq \|x - y\|^2 - (c + k)\|Cx - Cy\|^2\\
&\geq (1 - (c + k)\mu(K))\|x - y\|^2.
\end{aligned}$$

Hence, $I - TFC$ is strongly monotone and therefore A-proper with respect to $\Gamma = \{H_n, P_n\}$. □

Next, let us look at case when K is not selfadjoint. Our study of Eq. (1.1) in this case is motivated by the work of Appel-Pascale-Zabrejko [1] for nonlinear Hammerstein integral equations. Following their arguments about the existence of solutions, we present an abstract version of their results and also prove the constructive solvability of Eq. (1.1). We begin by describing the setting of the problem. Let X be an embeddable Banach space, that is, there is a Hilbert space H such that $X \subset H \subset X^*$ so that $\langle y, x\rangle = (y, x)$ for each $y \in H, x \in X$, where $\langle , \rangle$ is the duality pairing of X and X^*.

Let $K : X^* \to X$ be a linear map and K_H be the restriction of K to H such that $K_H : H \to H$. Let $A = (K + K^*)/2$ denote the selfadjoint part of K and $B = (K - K^*)/2$ be the skew-adjoint part of K. Assume that A is positive definite. Under our assumptions on K, both A and B map X^* into X. We know that A can be represented in the form $A = CC^*$, where $C = A^{1/2}$ is the square root of A and $C : H \to X$, and the adjoint map $C^* : X^* \to H$.

We say that K is P-positive definite if $C^{-1}K(C^*)^{-1}$ exists and is bounded in H. It is S-positive if $K(C^*)^{-1}$ exists and is bounded in H. Clearly, P-positivity implies the S-positivity but not conversely. It is easy to see that K is P-positive if and only if $C^{-1}B(C^*)^{-1}$ is bounded in H, and is S-positive if and only if $B(C^*)^{-1}$ is bounded in H. Moreover, K is P-positive if and only if K is angle-bounded, i.e.

$$|(Kx, y) - (y, Kx)| \leq a(Kx, x)^{1/2}(Ky, y)^{1/2} \quad x, y \in H.$$

Denote by M and N the closure of the maps $C^{-1}K(C^*)^{-1}$ and $K(C^*)^{-1}$, respectively, in H. The mappings M and N are defined on the closure (in H) of the range of $C = A^{1/2}$ and suppose that their domains coincide with H. We require the following decompositions to hold

$$K = CMC^*, \ K = NC^*.$$

Note that K, M and N are related as: $N = CM$, $N^* = M^*C^*$ and we have $(Mx, x) = \|x\|^2$ for all $x \in H$. Hence, both M and M^* have trivial nullspaces. Denote by $\mu(K) = \|N\|^2$, which is the positivity constant of K in the sense of Krasnoselskii.

Let $F : X \to X^*$ be a nonlinear map and consider the Hammerstein equation

$$x - KFx = f. \tag{3.7}$$

As in [1], we reduce the solvability of Eq. (3.7) to the solvability of an equivalent equation. For $f \in N(H)$, let $h \in H$ be a solution of

$$M^*h - N^*FNh = M^*k \tag{3.8}$$

where $f = Nk$ for some $k \in H$. Then $M^*(h - C^*FNh - k) = 0$ since $N = CM$ and $N^* = M^*C^*$. Hence, $h = C^*FNh + k$ since M^* is injective and therefore

$$Nh = NC^*FNh + Nk = KFNh + f$$

since $K = NC^*$. Thus, $x = Nh$ is a solution of (3.7). So the solvability of (3.7) is reduced to the solvability of (3.8). We have the following extension of Theorem 3.5. Its unique solvability part is an abstract extension of a result of Appel-Pascale-Zabrejko [1] for Hammerstein integral equations.

Theorem 3.7 *Let $K : X^* \to X$ be P-positive in H, $F : X \to X^*$ and c be the smallest number such that*

$$(Fx - Fy, x - y) \le c\|x - y\|^2 \quad \text{for all } x, y \in X$$

a) If $c\mu(K) < 1$, then (3.7) is uniquely approximation solvable in X for each $f \in N(H) \subset X$.
b) If $c\mu(K) = 1$ and $M^* - N^*FN$ satisfies condition (+) in H, then (3.7) is solvable in X for each $f \in N(H) \subset X$.

Proof. a) As in [1], we shall prove that the map $M^* - N^*FN : H \to H$ is $1 - c\mu(K)$-strongly monotone. Indeed, for each $x, y \in H$ we have

$$\begin{aligned}
&(M^*x - N^*FNx - M^*y + N^*FNy, x - y)\\
&= \|x - y\|^2 - (FNx - FNy, Nx - Ny)\\
&\ge \|x - y\|^2 - c\|Nx - Ny\|^2\\
&\ge (1 - c\mu(K))\|x - y\|^2.
\end{aligned}$$

Hence, $M^* - N^*FN$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ for H and coercive. Hence, (3.8) is uniquely approximation solvable for each $k \in H$ and therefore so is (3.7) for each $f \in N(H)$.
b) If $c\mu(K) = 1$, then as in part a) we get that $M^* - N^*FN$ is monotone and satisfies condition (+). Hence, (3.8) is solvable and therefore so is (3.6) for each $f \in N(H)$. $\square$

Corollary 3.8 *Let $K : X^* \to X$ be P-positive in H, $F = F_1 + F_2 : X \to X^*$ and c be the smallest number such that*

$$(F_1x - F_1y, x - y) \le c\|x - y\|^2 \quad \text{for all } x, y \in X$$

and F_2 is a Lipshitz map with constant k such that $1 - (c + k)\mu(K) > 0$. Then (3.7) is uniquely approximation solvable in X for each $f \in N(H) \subset X$.

Proof. It suffices to show that $M^* - N^*FN : H \to H$ is A-proper and coercive. For $x, y \in H$, we have

$$\begin{aligned}&(M^*x - N^*FNx - M^*y + N^*FNy, x-y)\\ &\geq \|x-y\|^2 - (c+k)\|Nx - Ny\|^2\\ &\geq (1-(c+k)\mu(K))\|x-y\|^2.\end{aligned}$$

Hence, $M^* - N^*FN$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ and coercive. $\square$

Next, we shall look at the case when the selfadjoint part A of K is not positive definite. Suppose that A is quasi-positive definite in H, i.e., it has at most a finite number of negative eigenvalues of finite multiplicity. Let U be the subspace spanned by the eigenvectors of A corresponding to these negative eigenvalues of A and $P : H \to U$ be the orthogonal projection onto U. Then P commutes with A, but not necessarily with B, and acts both in X and X^*. Then the operator $|A| = (I - 2P)A$ is easily seen to be positive definite. Hence, we have the decomposition $|A| = DD^*$, where $D = |A|^{1/2} : H \to X$ and $D^* : X^* \to H$.

As in [1], we call the map K P-quasi-positive if the map $D^{-1}K(D^*)^{-1}$ exists and is bounded in H, and S-quasi-positive if the map $K(D^*)^{-1}$ exists and is bounded in H. Let M and N denote the closure in H of the bounded maps $D^{-1}K(D^*)^{-1}$ and $K(D^*)^{-1}$ respectively. Assume that they are both defined on the whole space H. Suppose that we have the following decompositions

$$K = DMD^*, \quad K = ND^*.$$

Then we have $N = DM$, $N^* = M^*D^*$, and $< Mh, h >= \|h\|^2 - 2\|Ph\|^2$ for all $h \in H$.

Define the number

$$\nu(K) = \sup\{\nu : \nu > 0,\ \|Nh\| \geq (\nu)^{1/2}\|Ph\|,\ \ h \in H\}.$$

Note that for a selfadjoint map K, $\nu(K)$ is the absolute value of the largest negative eigenvalue of K. We have the following extension of Theorem 3.7. Its unique solvability part is an abstact extension of a result in [1] for Hammerstein integral equations.

Theorem 3.9 *Let $K : X^* \to X$ be P-quasi-positive in H, $F : X \to X^*$ and c be the smallest number such that*

$$(Fx - Fy, x-y) \leq c\|x-y\|^2 \quad \textit{for all } x, y \in X$$

a) If $c\nu(K) < -1$, then (1.1) is uniquely approximation solvable in X for each $f \in N(H) \subset X$.
b) If $c\nu(K) = -1$ and $M^* - N^*FN$ satisfies condition (+) in H, then (1.1) is solvable in X for each $f \in N(H) \subset X$.

Proof. a) As in [1], we have that the map $M^* - N^*FN : H \to H$ is $-(1+c\nu(K))$-strongly monotone. Indeed, for each $x, y \in H$ we have

$$\begin{aligned}&(M^*x - N^*FNx - M^*y + N^*FNy, x-y)\\&= \|x-y\|^2 - 2\|P(x-y)\|^2 - (FNx - FNy, Nx - Ny)\\&\geq \|x-y\|^2 - (2+c\nu(K))\|P(x-y)\|^2 \geq -(1+c\nu(K))\|x-y\|^2.\end{aligned}$$

Hence, $M^* - N^*FN$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ for H and coercive. Hence, (3.8) is uniquely approximation solvable for each $k \in H$ and therefore so is (3.7) for each $f \in N(H)$.
b) If $c\nu(K) = -1$, then as in part a) we get that $M^* - N^*FN$ is monotone monotone and satisfies condition (+). Hence, Eq. (3.8) is solvable and therefore so is (3.6) for each $f \in N(H)$. □

Corollary 3.10 *Let $K : X^* \to X$ be P-quasi-positive in H and $F = F_1 + F_2 : X \to X^*$ and c be the smallest number such that*

$$(F_1x - F_1y, x-y) \leq c\|x-y\|^2 \quad \textit{for all } x, y \in X$$

and F_2 is a Lipshitz map with constant k such that $1-(c+k)\mu(K) > 0$. Then (1.1) is uniquely approximation solvable in X for each $f \in N(H) \subset X$.

Proof. It suffices to show that $M^* - N^*FN : H \to H$ is A-proper and coercive. For $x, y \in H$, we have

$$\begin{aligned}&(M^*x - N^*FNx - M^*y + N^*FNy, x-y)\\&= \|x-y\|^2 - (F_1Nx - F_1Ny, Nx-Ny) - (F_2Nx - F_2Ny, Nx - Ny)\\&\geq (1-(c+k)\mu(K))\|x-y\|^2.\end{aligned}$$

Hence, $M^* - N^*FN$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ and coercive. □

4 Hammerstein integral equations

Let $Q \subset R^n$ be a bounded domain, $k(t,s) : Q \times Q \to R$ be measurable and $f(s,u) : Q \times R \to R$ is a Caratheodory function. We consider the problem of a solution $u \in L_2(Q)$ of the Hammerstein integral equation

$$u(t) = \int_Q k(t,s)f(s,u(s))ds + g(t) \tag{4.1}$$

where g is a measurable function. There is a vast literature on the solvability of (4.1) and we just mention the books by Krasnoselskii [5] and Vainberg [16]. Define the linear map

$$Ku(t) = \int_Q k(t,s)u(s)ds$$

in $H = L_2(Q)$. Define $Fu = f(s, u(s))$ and note that (4.1) can be written in the form $u - KFu = g$.

Theorem 4.1 *Let $K : H \to H$ be continuous, $\lambda \notin \Sigma(K)$, and let $d^{-1} = \operatorname{dist}(\lambda, \Sigma(K))$. Let $\delta = \max \|P_n\|$ and for some $k \in (0, d/k)$,*

$$|f(s,u) - \lambda u - (f(s,v) - \lambda v)| \le k|u - v| \quad \text{for } s \in Q,\ u, v \in R.$$

Then (4.1) is uniquely solvable for each $g \in L_2$.

For the proof of this theorem, it is easy to see that the mappings K and F satisfy all conditions of Theorem 3.1. Hence, its conclusions follow from it.

Theorem 4.2 *Let $K : H \to H$ be selfadjoint and for some $\alpha, \beta \in R$,*

$$\alpha|u - v|^2 \le (f(s,u) - f(s,v))(u - v) \le \beta|u - v|^2 \quad \text{for } s \in Q,\ u, v \in R$$

(i) If $-c < \alpha \le \beta < -c + d_c^+$ for some $c \in \Sigma(K) \cap (-\infty, \mu^*)$ or $-c - d_c^+ < \alpha \le \beta < -c$ for some $c \in \Sigma(K) \cap (\mu^*, \infty)$, then (5.1) is uniquely solvable for each $g \in L_2$.

(ii) If $<$ is replaced by $\le$ in (i) and if, for some $a < \lambda$ with $\lambda = c - d_c^-/2$ if $c \le \mu^*$ and $\lambda = c + d_c^+/2$ if $c > \mu^*$, and some $b \in L_2$, we assume

$$|f(s,u) - \lambda u| \le a|u| + b(s) \quad \text{for } s \in Q,\ u \in R$$

then (4.1) is solvable for each $g \in L_2$.

For the proof of this theorem, it is easy to see that the mappings K and F satisfy all conditions of Theorem 3.4. Hence, its conclusions follow from it. Part (i) of this theorem extends a result of Dolph [3]. Assuming compactness of K, we can relax the condition on F.

Theorem 4.3 *Let $K : H \to H$ be selfadjoint and compact and for some $\beta > 0$,*

$$(f(s,u) - f(s,v))(u - v) \le \beta|u - v|^2 \quad \text{for } s \in Q,\ u, v \in R$$

If β is sufficiently small, then (4.1) is uniquely solvable for each $g \in L_2$.

To prove this theorem, it is easy to see that the mappings K and F satisfy all conditions of Theorem 3.5. Hence, its conclusions follow from it.

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