*Electronic Journal of Differential Equations*, Vol. 2002(2002), No. 34, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

# CONSTANTS IN THE OSCILLATION THEORY OF HIGHER ORDER STURM-LIOUVILLE DIFFERENTIAL EQUATIONS

ONDŘEJ DOŠLÝ

Abstract. We find the the exact value of a constant in some oscillation criteria for the higher order Sturm-Liouville differential equation

$$(-1)^n(t^\alpha y^{(n)})^{(n)} = q(t)y.$$

We also study some general aspects in the oscillation theory of this equation.

## 1. Introduction

In this paper we study the oscillatory properties of the higher order Sturm-Liouville differential equations

$$(-1)^n(t^\alpha y^{(n)})^{(n)} = q(t)y. \tag{1.1}$$

A typical phenomenon of the application of the variational principle in the oscillation theory of Sturm-Liouville differential equations is that there is a gap between constants appearing in non-oscillation and oscillation criteria. To explain the situation, consider the simple second order differential equation

$$y'' + q(t)y = 0, \quad q(t) \geq 0. \tag{1.2}$$

The variational approach to the investigation of oscillatory properties of (1.2) is based on the following statement, see [12].

**Proposition 1.1.** *Equation* (1.2) *is non-oscillatory if and only if there exists* $T \in \mathbb{R}$ *such that for every nontrivial* $y \in W_0^{1,2}(T,\infty)$

$$\mathcal{F}(y;T,\infty) := \int_T^\infty \left[y'^2 - q(t)y^2\right] dt > 0. \tag{1.3}$$

Non-oscillation criteria are usually proved using the Wirtinger inequality

$$\int_T^\infty |M'(t)|y^2(t)\,dt \leq 4\int_T^\infty \frac{M^2(t)}{|M'(t)|}y'^2(t)\,dt \tag{1.4}$$

---

2000 *Mathematics Subject Classification.* 34C10.
*Key words and phrases.* Sturm-Liouville differential equation, variational method, oscillation and non-oscillation criteria, conditional oscillation.

Submitted October 25, 2001. Published April 4, 2002.
Supported by Research Project J07/98/143100001 from the Czech Government.

which holds for every $y \in W_0^{1,2}(T,\infty)$, see e.g. [13], here $M$ is a continuously differentiable function such that $M'(t) \neq 0$ for $t \geq T$. Using this inequality it is not difficult to prove that (1.2) is non-oscillatory if

$$\lim_{t\to\infty} t \int_t^\infty q(s)\,ds < \frac{1}{4}. \tag{1.5}$$

Indeed, if (1.5) holds, then for $T$ sufficiently large and any nontrivial $y \in W_0^{1,2}(T,\infty)$ (using integration by parts, the Cauchy inequality and (1.4) with $M(t) = t^{-1}$) we have

$$\begin{aligned} \int_T^\infty q(t)y^2\,dt &\leq -y^2 \int_t^\infty q(s)\,ds\Big|_T^\infty + 2\int_t^\infty |y||y'|\frac{1}{t}t\Big(\int_t^\infty q(s)\,ds\Big)\,dt \\ &< \frac{1}{2}\Big(\int_T^\infty y'^2\,dt\Big)^{1/2}\Big(\frac{y^2}{t^2}\,dt\Big)^{1/2} \leq \int_T^\infty y'^2\,dt. \end{aligned}$$

If we want to establish an oscillation criterion for (1.2), we need to show that for every $T$ there exists a nontrivial $y \in W_0^{1,2}(T,\infty)$ such that $\mathcal{F}(y;T,\infty) \leq 0$. This function we construct as follows. Let $T$ be arbitrary, $T < t_1 < t_2 < t_3$, and define

$$y = \begin{cases} 0 & t \leq T, \\ \frac{t-T}{t_1-T} & T \leq t \leq t_1, \\ 1 & t_1 \leq t \leq t_2, \\ \frac{t_3-t}{t_3-t_2} & t_2 \leq t \leq t_3, \\ 0 & t \geq t_3. \end{cases}$$

Then (using the fact that $q(t) \geq 0$)

$$\begin{aligned} \mathcal{F}(y;T,\infty) &\leq \frac{1}{t_1-T} - \int_{t_1}^{t_2} q(t)\,dt + \frac{1}{t_3-t_2} \\ &= \frac{1}{t_1-T}\Big[1-(t_1-T)\int_{t_1}^{t_2} q(t)\,dt + \frac{t_1-T}{t_3-t_2}\Big]. \end{aligned}$$

Hence, if

$$\lim_{t\to\infty} t\int_t^\infty q(s)\,ds > 1, \tag{1.6}$$

it is not difficult to see that $\mathcal{F}(y;T,\infty) < 0$ if $t_1 < t_2 < t_3$ are sufficiently large, i. e. (1.6) is a sufficient condition for oscillation of (1.2).

Now we see the gap between the constants $\frac{1}{4}$ in (1.5) and 1 in (1.6) obtained by variational method. On the other hand, the application of the so-called Riccati technique, based on the relationship between non-oscillation of (1.2) and the solvability of the Riccati equation

$$w' + q(t) + w^2 = 0,$$

then reveals that the "correct" constant is $\frac{1}{4}$ in the sense that using this method it is possible to show that (1.2) is oscillatory provided the limit in (1.6) is $> \frac{1}{4}$, see e. g. [15].

The above described fact that the variational method gives a gap between constants in oscillation and non-oscillation criteria appears also in the oscillation theory of higher order Sturm-Liouville equations, examples are given in the next section. In contrast to the second order equations, the Riccati technique is not developed properly for higher order equations, so the open problem is what is the "correct" constant for (non)oscillation of the investigated differential equation. The aim of this paper is to show that a suitable modification of the variational method enables to detect a correct oscillation constant and to remove this gap. It turns out, similarly as in the second order case, that this constant is the constant in non-oscillation criteria obtained using inequality (1.4).

This paper is organized as follows. In the next section we recall the relationship between higher order Sturm-Liouville equations and linear Hamiltonian systems, we also recall some known (non)oscillation criteria for Sturm-Liouville equations. Section 3 contains the main results of the paper, new oscillation criteria for (1.1). In the last section we discuss some general aspects of the used approach to the oscillation theory of Sturm-Liouville equations.

## 2. Auxiliary results

We start this section with the basic oscillatory properties of higher order Sturm-Liouville differential equation

$$L(y) := \sum_{k=0}^{n} (-1)^k \big(r_k(t) y^{(k)}\big)^{(k)} = 0, \quad r_n(t) > 0. \tag{2.1}$$

Oscillatory properties of this equation can be investigated within the scope of the oscillation theory of linear Hamiltonian systems (further LHS)

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \tag{2.2}$$

where $A, B, C$ are $n \times n$ matrices with $B, C$ symmetric. Indeed, if $y$ is a solution of (2.1) and we set

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} (-1)^{n-1}(r_n y^{(n)})^{(n-1)} + \dots + r_1 y' \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix},$$

then $(x, u)$ solves (2.2) with $A, B, C$ given by

$$B(t) = \operatorname{diag}\{0, \dots, 0, r_n^{-1}(t)\}, \quad C(t) = \operatorname{diag}\{r_0(t), \dots, r_{n-1}(t)\},$$

$$A = A_{i,j} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

In this case we say that the solution $(x, u)$ of (2.2) *is generated* by the solution $y$ of (2.1). Moreover, if $y_1, \dots, y_n$ are solutions of (2.1) and the columns of the matrix solution $(X, U)$ of (2.2) are generated by the solutions $y_1, \dots, y_n$, we say that the solution $(X, U)$ *is generated by the solutions* $y_1, \dots, y_n$.

Recall that two different points $t_1, t_2$ are said to be *conjugate* relative to system (2.2) if there exists a nontrivial solution $(x, u)$ of this system such that $x(t_1) = 0 = x(t_2)$. Consequently, by the above mentioned relationship between (2.1) and (2.2), these points are conjugate relative to (2.1) if there exists a nontrivial solution $y$ of this equation such that $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, 1, \dots, n-1$. System (2.2) (and hence also equation (2.1)) is said to be *oscillatory* if for every $T \in \mathbb{R}$ there exists a pair of points $t_1, t_2 \in [T, \infty)$ which are conjugate relative to (2.2) (relative to (2.1)), in the opposite case (2.2) (or (2.1)) is said to be *nonoscillatory*. If $w$ is a positive function, the equation

$$L(y) = w(t)y \tag{2.3}$$

with the non-oscillatory operator $L$ given by (2.1) is said to be *conditionally oscillatory* if there exists $\lambda_0 > 0$ such that (2.3) with $\lambda w(t)$ instead of $w(t)$ is oscillatory for $\lambda > \lambda_0$ and non-oscillatory for $\lambda < \lambda_0$. The constant $\lambda_0$ is called the *oscillation constant* of (2.3).

A conjoined basis $(X, U)$ of (2.2) (i.e. a matrix solution of this system with $n \times n$ matrices $X, U$ satisfying $X^T(t)U(t) = U^T(t)X(t)$ and $\operatorname{rank}(X^T, U^T)^T = n$) is said to be the *principal solution* of (2.2) if $X(t)$ is nonsingular for large $t$ and for any other conjoined basis $(\bar X, \bar U)$ such that the (constant) matrix $\bar X^T U - \bar U^T X$ is nonsingular $\lim_{t\to\infty} \bar X^{-1}(t)X(t) = 0$ holds. The last limit equals zero if and only if

$$\lim_{t\to\infty} \Big(\int^t X^{-1}(s)B(s)X^{T-1}(s)\,ds\Big)^{-1} = 0, \tag{2.4}$$

see [14]. A principal solution of (2.2) is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix. If $(X, U)$ is the principal solution, any conjoined basis $(\bar X, \bar U)$ such that the matrix $X^T\bar U - U^T\bar X$ is nonsingular is said to be a *nonprincipal solution* of (2.2). Solutions $y_1, \dots, y_n$ of (2.1) are said to form the *principal (non-principal) system of solutions* if the solution $(X, U)$ of the associated linear Hamiltonian system generated by $y_1, \dots, y_n$ is a principal (non-principal) solution.

Besides the definition of oscillation and non-oscillation of (2.1) by means of the concept of conjugate points, we will use another definition of oscillation and non-oscillation of linear differential equations introduced by Nehari, see [2, Chap. III]. A linear differential equation

$$y^{(n)} + q_{n-1}(x)y^{(n-1)} + \dots + q_0(x)y = 0 \tag{2.5}$$

is said to be disconjugate on an interval $I$ in the sense of Nehari (shortly $N$-disconjugate) if any nontrivial solution of (2.5) has at most $n-1$ zeros on $I$, every zero point counted according to its multiplicity. Equation (2.5) is said to be $N$-non-oscillatory if there exists $T \in \mathbb{R}$ such that (2.5) is $N$-disconjugate on $(T, \infty)$.

Now we recall some known oscillation and non-oscillation criteria for higher order Sturm-Liouville differential equations.

**Proposition 2.1** ([3]). *Suppose that equation* (2.1) *is N-nonoscillatory,* $y_1, \dots, y_n$, $\tilde y_1, \dots, \tilde y_n$ *are its principal and nonprincipal systems of solutions, respectively. Denote by* $(X, U)$ *and* $(\tilde X, \tilde U)$ *the matrix solutions of the associated LHS generated by these systems of solutions. The equation*

$$L(y) = q(t)y \tag{2.6}$$

*is oscillatory provided there exists* $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ *such that one of the following conditions is satisfied:*

(i)

$$\int^\infty q(t)(c_1y_1(t) + \dots + c_ny(t))^2\,dt = \infty,$$

(ii)

$$\lim_{t\to\infty} \frac{\int_t^\infty q(s)(c_1y_1(s)+\dots+c_ny(s))^2\,ds}{c^T\left(\int^t X^{-1}(s)B(s)X^{T-1}\,ds\right)^{-1}c} > 1,$$

*where* $B(t)=\operatorname{diag}\left\{0,\dots,0,\frac{1}{r_n(t)}\right\}$,

(iii)

$$\lim_{t\to\infty} \frac{\int^t q(s)(c_1\tilde y_1(s)+\dots+c_n\tilde y(s))^2\,ds}{c^T\left(\int^t \tilde X^{-1}(s)B(s)\tilde X^{T-1}\,ds\right)^{-1}c} > 1.$$

In the next theorem we specify general conditions of Proposition 2.1 to some special $N$-nonoscillatory differential operators. The oscillation constants appearing in the parts (i) and (ii) of the next theorem were computed in [6] for the Sturm-Liouville difference equation

$$(-1)^n\Delta^n\left(k^{(\alpha)}\Delta^n y_k\right)=q_ky_{k+n},\quad k^{(\alpha)}:=\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)},$$

but the method used there directly extends to differential equations, see also [10, 11].

**Proposition 2.2.** *Equation* (1.1) *is oscillatory provided one of the following conditions is satisfied.*

(i) $\alpha\notin\{1,3,\dots,2n-1\}$, $m\in\{0,\dots,n-1\}$, $\alpha<2n-1$ *and*

$$\lim_{t\to\infty} t^{2n-1-\alpha-2m}\int_t^\infty q(s)s^{2m}\,ds > \mu_{n,\alpha,m}:=\frac{[(n-m-1)!\prod_{j=0}^{m-1}(2n-1-\alpha-j)]^2}{2n-2m-1-\alpha}, \tag{2.7}$$

(ii) $\alpha\notin\{1,3,\dots,2n-1\}$, $m\in\{0,\dots,n-1\}$, $\alpha>2m+1$,

$$\lim_{t\to\infty} t^{2m+1-\alpha}\int_1^t q(s)s^{2(n-m-1)}\,ds > \xi_{n,\alpha,m}:=\frac{[(n-m-1)!\prod_{j=0}^{m-1}(\alpha-m-1-j)]^2}{\alpha-2m-1}, \tag{2.8}$$

(iii) $\alpha\in\{1,3,\dots,2n-1\}$, $m=\frac{2n-1-\alpha}{2}$,

$$\lim_{t\to\infty}\lg t\Big(\int_t^\infty q(s)s^{2m}\,ds\Big) > \rho_{n,m}:=[m!(n-m-1)!]^2. \tag{2.9}$$

Nonoscillatory counterparts of oscillation criteria of the previous theorem read as follows. The proofs of these statements can be found in [1, 12, 13].

**Proposition 2.3.** *Let* $q_+(t):=\max\{0,q(t)\}$ *denotes the nonnegative part of* $q$. *Equation* (1.1) *is nonoscillatory provided one of the following conditions is satisfied.*

(i) $\alpha\notin\{1,3,\dots,2n-1\}$, $\alpha<2n-1$, $m\in\{0,\dots,n-1\}$ *and*

$$\lim_{t\to\infty} t^{2n-1-\alpha-2m}\int_t^\infty q_+(s)s^{2m}\,ds < \nu_{n,\alpha,m}:=\frac{\prod_{j=0}^{n-1}(2n-\alpha-1-j)^2}{4^n(2n-\alpha-1-2m)}, \tag{2.10}$$

(ii) $\alpha \notin \{1,3,\dots,2n-1\}$, $m \in \{0,\dots,n-1\}$, $\alpha > 2m-1$,

$$\lim_{t\to\infty} t^{2m+1-\alpha}\left(\int_1^t q_+(s)s^{2(n-m-1)}\,ds\right) < \zeta_{n,\alpha,m} := \frac{\prod_{j=0}^{n-1}(2n-\alpha-1-j)^2}{4^n(\alpha-2m+1)}, \tag{2.11}$$

(iii) $\alpha \in \{1,3,\dots,2n-1\}$, $m = \frac{2n-1-\alpha}{2}$ *and*

$$\lim_{t\to\infty} \lg t \int_t^\infty q_+(s)s^{2m}\,ds < \frac{\rho_{n,m}}{4} = \frac{[(n-1-m)!m!]^2}{4}. \tag{2.12}$$

Comparing Propositions 2.2, 2.3 we see the gap between constants in oscillation and non-oscillation criteria. For example, if $n=2$, $\alpha=0$ and $m=0$, by the previous propositions the fourth order equation

$$y^{(IV)} = q(t)y \tag{2.13}$$

is oscillatory provided

$$\lim_{t\to\infty} t^3 \int_t^\infty q(s)\,ds > 12$$

and non-oscillatory if

$$\lim_{t\to\infty} t^3 \int_t^\infty q(s)\,ds < \frac{3}{16}.$$

If the last limit lies in the interval $(\frac{3}{16},12)$, Propositions 2.2,2.3 give no information about oscillatory nature of (2.13). Moreover, if $q(t)=\frac{\gamma}{t^4}$, by Propositions 2.2,2.3 the Euler equation

$$y^{(IV)} = \frac{\gamma}{t^4}y \tag{2.14}$$

is oscillatory if $\gamma > 36$ and nonoscillatory if $\gamma < \frac{9}{16}$. Since it is known that (2.14) is actually oscillatory if and only if $\gamma \le \frac{9}{16}$, this leads to the conjecture that the "correct" oscillation constant is the constant appearing in non-oscillation criteria. In this paper we show that it is really the case. In particular, as a consequence of a general result, we get that (2.13) is oscillatory if $\lim_{t\to\infty} t^3\int_t^\infty q(s)\,ds > \frac{3}{16}$.

We finish this section with a statement which we need to treat the exceptional case $\alpha \in \{1,3,\dots,2n-1\}$.

**Proposition 2.4** ([8])**.** *Suppose that $q(t)\ge 0$ for large $t$, $\alpha \in \{1,3,\dots,2n-1\}$, $m := \frac{2n-1-\alpha}{2}$ and $\rho_{n,m} := [(n-1-m)!m!]^2$. If*

$$\int^\infty \big(q(t) - \frac{\rho_{n,m}}{4t^{2n-\alpha}\lg^2 t}\big)t^{2n-1-\alpha}\lg t\,dt = \infty \tag{2.15}$$

*then equation* (1.1) *is oscillatory.*

## 3. Oscillation criteria

The next theorem improves oscillation constants in the parts (i), (ii) of Proposition 2.2 and show that they can be replaced by (less) constants from their nonoscillatory counterparts given in Proposition 2.3.

**Theorem 3.1.** *Equation* (1.1) *is oscillatory provided one of the following two conditions holds:*

(i) $\alpha \notin \{1,3,\dots,2n-1\}$, $\alpha < 2n-1$, $m \in \{0,\dots,n-1\}$ *and*

$$\lim_{t\to\infty} t^{2n-1-\alpha-2m} \int_t^\infty q(s)s^{2m}\,ds > \nu_{n,\alpha,m} = \frac{\prod_{j=0}^{n-1}(2n-1-\alpha-2j)^2}{4^n(2n-\alpha-1-2m)}, \tag{3.1}$$

(ii) $\alpha \notin \{1,3,\dots,2n-1\}$, $m \in \{0,\dots,n-1\}$, $\alpha > 2m+1$, *and*

$$\lim_{t\to\infty} t^{2m+1-\alpha}\Big(\int_1^t q(s)s^{2(n-m-1)}\,ds\Big) > \zeta_{n,\alpha,m} = \frac{\prod_{j=0}^{n-1}(2n-\alpha-1-j)^2}{4^n(\alpha-2m+1)}. \tag{3.2}$$

**Proof.** If the limit in (3.1) or (3.2) equals $\infty$, equation (1.1) is oscillatory by conditions (2.7) and (2.8) of Proposition 2.2. If these limits are finite (and equal $M$), we will use Proposition 2.1, part (i), with

$$L(y) = (-1)^{(n)}(t^\alpha y^{(n)})^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}y, \quad \gamma_{n,\alpha} := \prod_{j=0}^{n-1}\Big(\frac{2n-1-\alpha-j}{2}\Big)^2, \tag{3.3}$$

whereby (1.1) is rewritten into the form

$$(-1)^{(n)}(t^\alpha y^{(n)})^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}y = \Big(q(t) - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}\Big)\,y. \tag{3.4}$$

Observe that the solution $y = t^{\frac{2n-1-\alpha}{2}}$ of the equation $L(y) = 0$ with $L$ given by (3.3) is contained in the principal system of solutions (see [7, 11]) and hence it is sufficient to show that

$$\int^\infty \Big(q(t) - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}\Big)\,t^{2n-1-\alpha}\,dt = \infty.$$

First consider the case (i). Condition (3.1) implies that there exists $\varepsilon > 0$ and $T \in \mathbb{R}$ such that

$$\int_t^\infty q(s)\,ds > \frac{\nu_{n,\alpha,m}+\varepsilon}{t^{2n-1-\alpha-2m}}$$

for $t \ge T$ and hence

$$\int_T^b t^{2n-2-\alpha-2m}\Big(\int_t^\infty q(s)s^{2m}\,ds\Big)\,dt > (\nu_{n,\alpha,m}+\varepsilon)\lg(b/T).$$

Using this inequality and integration by parts we have

$$\begin{aligned}
&\int_T^b \Big(q(t) - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}\Big)\,t^{2n-1-\alpha}\,dt \\
&= \int_T^b q(t)t^{2n-1-\alpha}\,dt - \gamma_{n,\alpha}\lg(b/T) \\
&= -t^{2n-1-\alpha-2m}\int_t^\infty q(s)s^{2m}\,ds\Big|_T^b \\
&\quad +(2n-1-\alpha-2m)\int_T^b t^{2n-2-\alpha-2m}\int_t^\infty q(s)s^{2m}\,ds\,dt\gamma_{n,\alpha}\lg(b/T) \\
&> [(2n-1-\alpha-2m)(\nu_{n,\alpha,m}+\varepsilon) - \gamma_{n,\alpha}]\lg(b/T) \\
&\quad -t^{2n-1-\alpha-2m}\int_t^\infty q(s)s^{2m}\,ds\Big|_T^b \\
&= K - b^{2n-1-\alpha-2m}\int_b^\infty q(s)s^{2m}\,ds + \varepsilon\lg(b/T) \to \infty,
\end{aligned}$$

as $b \to \infty$, where $K = T^{2n-1-\alpha-2m}\int_T^\infty q(s)s^{2m}\,ds$.

Concerning the case (ii), (3.2) implies that there exists $T \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\int_T^b t^{2m-\alpha} \Big( \int^t q(s) s^{2(n-m-1)}\, ds \Big)\, dt > (\zeta_{n,\alpha,m} + \varepsilon) \lg(b/T)$$

and again using integration by parts

$$\begin{aligned} &\int_T^b \Big( q(t) - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} \Big) t^{2n-1-\alpha}\, dt \\ &> \ t^{2n-1-\alpha} \int_1^t q(s)\, ds \Big|_T^b + [(\alpha - 2n + 1)(\zeta_{n,\alpha,m} + \varepsilon) - \gamma_{n,\alpha}] \lg b \to \infty \end{aligned}$$

as $b \to \infty$. □

In the proof of the following theorem which deals with the "critical" case $\alpha \in \{1, 3, \dots, 2n-1\}$ we need to modify slightly the method used in the previous proof. This is due to the fact that for critical values of $\alpha$ the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} = \frac{\lambda}{t^{2n-\alpha}} y \tag{3.5}$$

is no longer conditionally oscillatory (it is non-oscillatory if and only if $\lambda \le 0$), this property (conditional oscillation) has the equation

$$(-1)^n (t^\alpha y^{(n)})^{(n)} = \frac{\lambda}{t^{2n-\alpha} \lg^2 t} y. \tag{3.6}$$

In contrast to the case $\alpha \notin \{1, 3, \dots, 2n-1\}$, we do not know a solution of (3.6) (like $y = t^{(2n-1-\alpha)/2}$ in case of (3.5)). This fact requires also the sign restriction on the function $q$ in Proposition 2.4 which appears also in the next theorem. According to the parts (iii) of Propositions 2.2, 2.3, oscillation constant $\lambda_0$ of (3.6) satisfies $\rho_{n,m}/4 \le \lambda_0 \le \rho_{n,m}$. The next theorem shows that this value of the oscillation constant is $\rho_{n,m}/4$.

**Theorem 3.2.** *Let $\alpha \in \{1, 3, \dots, 2n-1\}$, $m := \frac{2n-1-\alpha}{2}$ and $q(t) \ge 0$ for large $t$. Equation* (1.1) *is oscillatory provided*

$$\lim_{t\to\infty} \lg t \int_t^\infty q(s) s^{2m}\, ds > \frac{\rho_{n,m}}{4} = \frac{[(m!(n-m-1)!]^2}{4}. \tag{3.7}$$

**Proof.** By Proposition 2.4 we need to show that

$$\int^\infty \big( q(t) - \frac{\rho_{n,m}}{4 t^{2n-\alpha} \lg^2 t} \big) t^{2m} \lg t\, dt = \infty.$$

Inequality (3.7) implies the existence of $T \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\frac{1}{t} \int_t^\infty q(s) s^{2m}\, ds > \big( \frac{\rho_{n,m}}{4} + \varepsilon \big) \frac{1}{t \lg t}, \quad t \ge T.$$

Integrating the obtained inequality we get

$$\int_T^b \frac{1}{t} \int_t^\infty q(s) s^{2m}\, ds\, dt > \big( \frac{\rho_{n,m}}{4} + \varepsilon \big) \lg \big( \frac{\lg b}{\lg T} \big)$$

for $b > T$. If the limit in (3.7) equals $\infty$, equation (1.1) is oscillatory by the part (iii) of Proposition 2.2. If this limit is finite, using integration by parts, similarly as in the previous proof

$$
\begin{aligned}
&\int_T^b \big(q(t) - \frac{\rho_{n,m}}{4t^{2n-\alpha}\lg^2 t}\big)t^{2m}\lg t\,dt \\
&= \int_T^b q(t)t^{2m}\lg t\,dt - \frac{\rho_{n,m}}{4}\lg\big(\frac{\lg b}{\lg T}\big) \\
&= -\lg t\int_t^\infty q(s)s^{2m}\,ds\Big|_T^b + \int_T^b \frac{1}{t}\int_t^\infty q(s)s^{2m}\,ds - \frac{\rho_{n,m}}{4}\lg\big(\frac{\lg b}{\lg T}\big) \\
&> \lg T\int_T^\infty q(s)s^{2m}\,ds - \lg b\int_b^\infty q(s)s^{2m}\,ds + \Big(\frac{\rho_{n,m}}{4} + \varepsilon - \frac{\rho_{n,m}}{4}\Big)\lg\big(\frac{\lg b}{\lg T}\big) \\
&= K + \varepsilon\lg\big(\frac{\lg b}{\lg T}\big) \to \infty \quad \text{as } b \to \infty,
\end{aligned}
$$

where $K$ is a real constant. □

## 4. Remarks

In the oscillation and non-oscillation criteria given in Propositions 2.2,2.3 only the so-called *polynomial solutions* of the one-term equation

$$\big(t^\alpha y^{(n)}\big)^{(n)} = 0 \tag{4.1}$$

appeared, i. e. solutions of the form $y = t^j$, $j = 0,\dots,n-1$. In addition to these solutions, equation (4.1) possesses also *nonpolynomial solutions* – the functions satisfying $y^{(n)} = t^{i-\alpha}$, $i = 0,\dots,n-1$. These functions are again powers of $t$ if $\alpha \notin \{1,2\dots,2n\}$ or also of the form $y = t^i\lg t$ if $\alpha \in \{1,2,\dots,2n-1\}$ (the integer $i$ attains the values depending on $n$ and $\alpha$). One can formulate oscillation and non-oscillation criteria containing $\int_t^\infty q(s)y^2(s)\,ds$ or $\int^t q(s)y^2(s)\,ds$ depending on the fact whether a given non-polynomial solution $y$ of (4.1) is in the principal or non-principal system of solutions. The situation is here similar as in the case of polynomial solutions treated in Propositions 2.2, 2.3. The constant in an oscillation criterion is bigger than the corresponding constant in its non-oscillatory counterpart. To illustrate the situation, consider (1.1) with $\alpha \in \{1,3,\dots,2n-1\}$. Then $t^m\lg t$, $m = \frac{2n-1-\alpha}{2}$, is a non-polynomial solution of (4.1) and it is known, see [7], that (1.1) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{\lg t}\int_1^t q(s)s^{2m}\lg^2 s\,ds > \rho_{n,m}$$

and it is non-oscillatory if

$$\lim_{t\to\infty}\frac{1}{\lg t}\int_1^t q(s)_+ s^{2m}\lg^2 s\,ds < \frac{\rho_{n,m}}{4}$$

with $\rho_{n,m}$ given in (2.9). Using exactly the same method as in the proof of Theorem 3.2 one can prove the following statement which shows that the "correct" oscillation constant is $\rho_{n,m}/4$.

**Theorem 4.1.** *Let $\alpha \in \{1,3,\dots,2n-1\}$, $m = \frac{2n-1-\alpha}{2}$ and $q(t) \geq 0$ for large $t$. If*

$$\lim_{t\to\infty}\frac{1}{\lg t}\int_1^t q(s)s^{2m}\lg^2 s\,ds > \frac{\rho_{n,m}}{4} = \frac{[m!(n-m-1)!]^2}{4}$$

*then equation* (1.1) *is oscillatory.*

(ii) In the previous part of the paper equation (1.1) is essentially viewed as a perturbation of the (non-oscillatory) *one-term* equation (4.1). Presented oscillation and non-oscillation criteria show that if the function $q$ is "sufficiently positive" ("not too positive") then (1.1) becomes oscillatory (remains non-oscillatory). More refined criteria can be obtained when (1.1) is regarded as a perturbation of the Euler equation (3.5) in case $\alpha \notin \{1, 3, \dots, 2n-1\}$ and of (3.6) if $\alpha \in \{1, 3, \dots, 2n-1\}$. Also in this case constants in non-oscillation criteria are less than their oscillation counterparts as shows the following statement. Its proof is given in [4, 7], see also [6, 8] for related remarks.

**Proposition 4.2.** *Suppose that* $\alpha \notin \{1, 3, \dots, 2n-1\}$, $\gamma_{n,\alpha}$ *is given in* (3.3) *and*

$$\omega_{n,\alpha} = \frac{\prod_{i=0}^{n-1}(\lambda - i)(\lambda + n - \alpha - i) - \gamma_{n,\alpha}}{\left(\lambda - \frac{2n-1-\alpha}{2}\right)^2}\Big|_{\lambda = \frac{2n-1-\alpha}{2}}.$$

*Equation* (1.1) *is oscillatory if*

$$\lim_{t\to\infty} \lg t \int_t^\infty \Big(q(s) - \frac{\gamma_{n,\alpha}}{s^{2n-\alpha}}\Big) s^{2n-1-\alpha}\, ds > \omega_{n,\alpha}$$

*and it is nonoscillatory if*

$$\lim_{t\to\infty} \lg t \int_t^\infty \Big(q(s) - \frac{\gamma_{n,\alpha}}{s^{2n-\alpha}}\Big)_+ s^{2n-1-\alpha}\, ds < \frac{\omega_{n,\alpha}}{4}.$$

Also in this case we believe that the sharp oscillation constant is $\frac{\omega_{n,\alpha}}{4}$, but at this moment we are able to prove this conjecture only in the special case $n = 2$, $\alpha = 0$. The proof of this statement is based on the following oscillation criterion for the fourth order equation

$$y^{(IV)} - \frac{9}{16t^4} y = p(t)y. \tag{4.2}$$

**Proposition 4.3.** *Suppose that* $p(t) \ge 0$ *for large* $t$. *If*

$$\int^\infty \big(p(s) - \frac{5}{8s^4 \lg^2 s}\big) s^3 \lg s\, ds = \infty \tag{4.3}$$

*then* (4.2) *is oscillatory.*

Using this proposition we can now prove sharpness of the constant $\frac{\omega_{2,0}}{4} = \frac{5}{8}$.

**Theorem 4.4.** *Suppose that* $q(t) \ge \frac{9}{16t^4}$ *for large* $t$. *Equation* (1.1) *with* $n = 2$ *and* $\alpha = 0$ *is oscillatory if*

$$\lim_{t\to\infty} \lg t \int_t^\infty \big(p(s) - \frac{9}{16s^4}\big) s^3 > \frac{\omega_{2,0}}{4} = \frac{5}{8}. \tag{4.4}$$

**Proof.** Denote by $M$ the value of the limit in (4.4). If $M = \infty$, equation (1.1) is oscillatory by Proposition 4.2. In case $\frac{5}{8} < M < \infty$ we use Proposition 4.3 (where the function $q(t) - \frac{9}{16t^4}$ plays the role of $p(t)$).

Inequality (4.4) implies the existence of $\varepsilon > 0$ and $T \in \mathbb{R}$ such that for $t > T$

$$\int_t^\infty \big(q(s) - \frac{9}{16s^4}\big) s^3 > \frac{\frac{5}{8} + \varepsilon}{\lg t},$$

hence

$$\int_T^b \frac{1}{t} \int_t^\infty \big(q(s) - \frac{9}{16s^4}\big) s^3\, ds > \big(\frac{5}{8} + \varepsilon\big) \lg\big(\lg \frac{b}{T}\big)$$

for $b > T$. Integration by parts yields

$$\begin{aligned}
&\int_T^\infty \big(q(s) - \frac{9}{16t^4} - \frac{5}{8t^4 \lg^2 t}\big) t^3 \lg t \, dt \\
= \;& \int_T^b \big(q(t) - \frac{9}{16t^4}\big) t^3 \lg t \, dt - \frac{5}{8} \lg \big(\lg \frac{b}{T}\big) \\
= \;& -\lg t \int_t^\infty \big(q(s) - \frac{9}{16s^4}\big) s^3 \, ds \Big|_T^b + \int_T^b \frac{1}{t}\Big(\int_t^\infty \big(q(s) - \frac{9}{16s^4}\big) s^3 \, ds\Big) dt \\
& -\frac{5}{8} \lg \big(\lg \frac{b}{T}\big) \\
> \;& -\lg t \int_T^\infty \big(q(s) - \frac{9}{16s^4}\big) s^3 \, ds \Big|_T^b + \varepsilon \lg \big(\lg \frac{b}{T}\big) \to \infty
\end{aligned}$$

as $b \to \infty$. Hence, by Proposition 4.3, (1.1) with $n = 2$, $\alpha = 0$ is oscillatory. □

(iii) The reason why we were able to prove Proposition 4.3 and hence also Theorem 4.1 only in the particular case $n = 2$, $\alpha = 0$ is that in this case we are able to compute explicitly all solutions of the equation

$$y^{(IV)} - \frac{9}{16t^4} y = 0 \tag{4.5}$$

and using this information to detect the next logarithmic term for which the resulting equation is conditionally oscillatory. More precisely, we were able to prove that the equation

$$y^{(IV)} - \big(\frac{9}{16t^4} + \frac{\lambda}{t^4 \lg^2 t}\big) y = 0$$

is oscillatory if and only if $\lambda > \frac{5}{8}$. Concerning the general case, it is conjectured in [5] that the equation

$$(-1)^n \big(t^\alpha y^{(n)}\big)^{(n)} - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = \frac{\lambda}{t^{2n-\alpha} \lg^2 t} y \tag{4.6}$$

is conditionally oscillatory. The verification of this conjecture, including the computation of the explicit value of the oscillation constant of (4.6) is the subject of the present investigation.

## References

[1] C. D. Ahlbrandt, D. B. Hinton, R. T. Lewis, *Necessary and sufficient conditions for the discreteness of the spectrum of certain singular differential operators*, Canad. J. Math. **33** (1981), 229–246.

[2] W. A. Coppel, Disconjugacy, Lectures Notes in Math., No. 220, Springer Verlag, Berlin-Heidelberg 1971.

[3] O. Došlý, *Oscillation criteria and the discreteness of the spectrum of self-adjoint, even order, differential operators*, Proc. Roy. Soc. Edinburgh **119A** (1991), 219–232.

[4] O. Došlý, *Oscillation and spectral properties of a class of singular self-adjoint differential operators*, Math. Nachr. **188** (1997), 49-68.

[5] O. Došlý, *Oscillation criteria for fourth order Sturm-Liouville differential equations*, submitted.

[6] O. Došlý, R. Hilscher, *A class of Sturm-Liouville difference equations: (non)oscillation constants and property BD*, to appear in Comput. Appl. Math.

[7] O. Došlý, J. Osička, *Oscillation and non-oscillation of higher order self-adjoint differential equations*, to appear in Czech. Math. J.

[8] O. Došlý, J. Osička, *Oscillatory properties of higher order Sturm-Liouville differential equations*, submitted.

[9] O. Došlý, F. Fiedler, *A remark on Nehari-type criteria for self-adjoint differential equations*, Comment. Math. Univ. Carolinae **32** (1991), 447-462.
[10] F. Fiedler, *Oscillation criteria for a special class of* $2n$ *order ordinary differential operators*, J. Differential Equations **42** (1981), 155-185.
[11] F. Fiedler, *Oscillation criteria for a special class of* $2n$ *order ordinary differential equations*, Math. Nachr. **131** (1987), 205-218.
[12] I. M. Glazman, Direct Methods of Qualitative Analysis of Singular Differential Operators, Davey, Jerusalem 1965.
[13] D. B. Hinton, R. T. Lewis, *Singular differential operators with spectra discrete and bounded below*, Proc. Roy. Soc. Edinburgh **84A** (1979), 117-134.
[14] W. T. Reid, Sturmian Theory for ordinary Differential Equations, Springer, New York-Berlin-Heidelberg, Berlin 1980.
[15] C. A. Swanson, Comparison and Oscillation Theory of Linear Differential Equation, Acad. Press, New York, 1968.

Ondřej Došlý
Department of Mathematics, Masaryk University,
Janaáčkovo nám. 2a, CZ-662 95 Brno.
*E-mail address*: dosly@math.muni.cz