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On plane polynomial vector fields and the Poincaré problem *

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Abstract

In this paper we address the Poincaré problem, on plane polynomial vector fields, under some conditions on the nature of the singularities of invariant curves. Our main idea consists in transforming a given vector field of degree m into another one of degree at most $m+1$ having its invariant curves in projective quasi-generic position. This allows us to give bounds on degree for some well known classes of curves such as the nonsingular ones and curves with ordinary nodes. We also give a bound on degree for any invariant curve in terms of the maximum Tjurina number of its singularities and the degree of the vector field.

1 Introduction

The study of algebraic invariant curves and integrating factors of plane polynomial vector fields goes back at least to Darboux [11] and Poincaré [23]. We refer the reader to [26, 25, 6] for an interesting survey and historical remarks on the problem. For a given polynomial vector field the question of finding invariant algebraic curves reduces mainly to the so-called *Poincaré problem* which consists in finding an upper bound on the degree of such curves. Indeed, any time such bound is found for a given vector field the question of finding its invariant curves can be algorithmically solved by using linear algebra (see e.g. [7, 19, 22]).

Solving the Poincaré problem, and hence finding invariant curves, yields great advances in the algorithmic study of plane polynomial vector fields. Darboux [11] showed that the abundance of invariant algebraic curves of a plane polynomial vector field ensures its integrability. More precisely, he proved that a vector field of degree m with a least $\frac{m(m+1)}{2}+1$ invariant curves has a first integral. Later on Jouanolou [17] showed that any degree m plane polynomial vector field with at least $\frac{m(m+1)}{2}+2$ invariant curves has a rational first integral. In this direction Prelle and Singer studied in [24] another kind of first integrals, namely *elementary first integrals*. They proved that the existence of algebraic integrating factors is necessary for the existence of elementary first integrals,

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and that deciding about the existence of algebraic integrating factors is the main question to be solved in order to decide about the existence of elementary first integrals. Ten years later, Singer [27] used differential algebra techniques to study a wider class of first integrals, namely the *Liouvillian first integrals*, and he proved that they have elementary functions as integrating factors.

It is well known from the work of Jouanolou that a plane polynomial vector field has either a rational first integral or finitely many invariant curves. This gives an indirect proof for the existence of an upper bound for the degree of irreducible invariant curves of a given vector field. As far as we know there is actually no effective method to compute such bound for any given vector field, even more the question promises to be hard. For example, the related question of deciding whether the closure of the set of vector fields with given degree and having invariant curves is an algebraic set is still open (see e.g. [9, 21] for more details on the question). On the other hand, it is well known that the degree of the given vector field is not enough in order to get control on the maximal degree of its irreducible invariant curves. It is for instance easy to find linear vector fields having rational first integrals of arbitrarily high degree. Even more, its is established in [20] and [8] the existence of quadratic plane vector fields without rational first integral and having invariant algebraic curves of any given degree.

Partial answers to the Poincaré problem have been given in recent years. All of them follow the same strategy, which consists in finding an upper bound in terms of the degree of the vector field under some additional conditions on its fixed points or on the nature of the singularities the invariant curves have (see e.g. [5, 3, 4, 28, 29, 2]).

Outline of the paper

In this paper we study the Poincaré problem from "algebraic geometry" point of view. For this purpose it is natural to state the problem in the general setting of a commutative field of characteristic zero. Our main idea consists in reducing the problem , by means of projective transformations, to a situation where invariant curves have no critical points at infinity. This reduction has a double advantage: first it keeps the geometric properties of the invariant curves. Secondly, it allows to use some basic results of projective algebraic geometry such as Bézout theorem.

The paper is structured as follows: in section 2 we define the concept of curves in projective quasi-generic position and we show explicitly how to transform projectively any curve to a curve in such position. Section 3 is devoted to show how to transform vector fields, without loss of control on their degree, into vector fields having their invariant curves in projective quasi-generic position. In section 4 we apply the techniques developed in sections 2 and 3 to the Poincaré problem. We recover the classical bound given in the case of nonsingular curves and we give better bounds than the known ones in the case of curves with ordinary nodes. A bound in terms of the maximum Tjurina number of the singularities of an invariant curve is also given in this section.

The setting of a commutative field of characteristic zero

Polynomial vector fields and invariant algebraic curves are objects of algebraic nature. It is hence natural to study their properties in the general setting of a commutative field of characteristic zero. This gives as well some flexibility to our study of vector fields; we shall for example see that this allows to treat in the same way invariant curves and rational first integrals (lemma 1.1). Another practical reason lies in the study of parameterized vector fields, since any given vector fields $\mathcal{X} \in \mathbb{R}[u, x, y]$, where $u = (u_1, \dots, u_r)$ is a list of parameters, can be viewed as vector field over $\mathbb{R}(u)[x, y]$.

Notation

Let $\mathbb{K}$ be a commutative field of characteristic zero and $\overline{\mathbb{K}}$ its algebraic closure. Let f be a squarefree polynomial in $\mathbb{K}[x, y]$ and $\mathcal{C}(f)$ be the affine plane algebraic curve, over the field $\overline{\mathbb{K}}$, defined by the equation $f(x, y) = 0$.

The zeros in $\overline{\mathbb{K}}^2$ of the ideal $\mathcal{I}(f, \partial_y f)$ are called the *critical points* of the curve $\mathcal{C}(f)$ with respect to the projection on the x-axis. In the same way, the zeros of the ideal $\mathcal{I}(f, \partial_x f)$ are called the critical points of the curve $\mathcal{C}(f)$ with respect to the projection on the y-axis. A critical point of the curve $\mathcal{C}(f)$ with respect to one of the projections is simply called a critical point of the curve.

The multiplicity of a point (α, β) of the curve $\mathcal{C}(f)$ is defined as the smallest integer s such that $\partial^{i+j}_{x^i y^j} f(\alpha, \beta) = 0$ for any $i + j < s$. When $s \geq 2$ the point is called *singular*. The singular points of the curve $\mathcal{C}(f)$ are the critical points for both of the two projections.

If (α, β) is a point of multiplicity s in the curve $\mathcal{C}(f)$ then the Taylor expansion of f around (α, β) writes as

$$f(x, y) = f_s(x - \alpha, y - \beta) + \ldots + f_n(x - \alpha, y - \beta)$$

where the f_i's are homogeneous and $f_s \neq 0$. Since f_s is homogeneous and bivariate it factors over $\overline{\mathbb{K}}$ into a product of linear polynomials. For each linear factor $\ell(x, y)$ of f_s the equation

$$\ell(x, y) = 0$$

gives a tangent line to $\mathcal{C}(f)$ at (α, β). Thus a point of multiplicity s has s tangent lines counted with multiplicities. In the case where f_s is squarefree the point (α, β) is called an *ordinary multiple point* of the curve.

If $\mathcal{I}$ is an ideal of $\mathbb{K}[x, y]$ then to each zero (α, β) of $\mathcal{I}$ corresponds a local ring $\left(\overline{\mathbb{K}}[x, y]/\mathcal{I}\right)_{(\alpha,\beta)}$ obtained by localizing the ring $\overline{\mathbb{K}}[x, y]/\mathcal{I}$ at the maximal ideal $\mathcal{I}(x - \alpha, y - \beta)$. When this local ring is finite dimensional as $\overline{\mathbb{K}}$-vector space we say that (α, β) is an isolated zero of $\mathcal{I}$ and the dimension as vector space of the corresponding local ring is called the *multiplicity* of (α, β) as zero of $\mathcal{I}$.

An important particular case is when two curves $\mathcal{C}(f)$ and $\mathcal{C}(g)$ meet at a point (α, β) and have no one-dimensional common component passing through this point. In this case (α, β) is an isolated zero of the ideal $\mathcal{I}(f, g)$ and its multiplicity is called the *intersection number* of the two curves at this point and denoted by $\mathrm{I}(f, g, (\alpha, \beta))$. The intersection number satisfies the inequality

$$\mathrm{I}(f, g, (\alpha, \beta)) \geq st \tag{1.1}$$

where s (resp. t) is the multiplicity of (α, β) as point of the curve $\mathcal{C}(f)$ (resp. $\mathcal{C}(g)$). The equality holds if and only if the two curves have no common tangent line at the considered point (see [14] for more details).

When the two curves have no one-dimensional common component then they meet at finitely many points. In this case the vector space $\mathbb{K}[x, y]/\mathcal{I}(f, g)$ is finite dimensional and its dimension is the sum of all the intersection numbers at the common points of $\mathcal{C}(f)$ and $\mathcal{C}(g)$ in the affine plane. A fundamental result of algebraic geometry, namely Bézout theorem asserts that

$$\dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(f, g) \leq \deg(f) \deg(g) \tag{1.2}$$

and the equality holds if and only if the two curves have no common points at infinity.

Another important case is the so-called *Milnor number*: given a singular point (α, β) of a curve $\mathcal{C}(f)$, it is easy to see that it is an isolated zero of the ideal $\mathcal{I}(\partial_x f, \partial_y f)$. Its multiplicity as zero of this ideal is called the Milnor number of (α, β) as singular point of $\mathcal{C}(f)$ and is denoted by $\mu(f, (\alpha, \beta))$. For any singular point (α, β) we have the relation

$$\mathrm{I}(f, \partial_y f, (\alpha, \beta)) = \mu(f, (\alpha, \beta)) + r - 1 \tag{1.3}$$

where r is the multiplicity of β as root of the univariate polynomial $f(\alpha, y)$, see e.g. [18].

The multiplicity of a singular point (α, β) of the curve $\mathcal{C}(f)$ as zero of the ideal $\mathcal{I}(f, \partial_x f, \partial_y f)$ is called the *Tjurina number* of the singular point and is denoted by $\tau(f, (\alpha, \beta))$.

By *plane polynomial vector field* with coefficients in the field $\mathbb{K}$, or often a vector field over $\mathbb{K}[x, y]$, we mean a $\mathbb{K}$-derivation $\mathcal{X}$ of the algebra $\mathbb{K}[x, y]$. Any plane polynomial vector field $\mathcal{X}$ with coefficients in $\mathbb{K}$ can be uniquely written in the form $\mathcal{X} = p\partial_x + q\partial_y$, with $p, q \in \mathbb{K}[x, y]$. The maximum of the degrees of p and q is called the degree of $\mathcal{X}$.

Let us recall here that any vector field $\mathcal{X}$ extends uniquely to the fractions field $\mathbb{K}(x, y)$ and as well to any algebraic extension of it. In particular $\mathcal{X}$ extends uniquely as $\overline{\mathbb{K}}$-derivation of $\overline{\mathbb{K}}[x, y]$.

In the sequel we shall also be concerned with transcendent extensions of the field $\mathbb{K}$. In such cases, given a transcendent extension $\mathbb{F}$ of $\mathbb{K}$ the vector field $\mathcal{X}$ do not extend in a unique way to $\mathbb{F}[x, y]$. However, it extends uniquely as $\mathbb{F}$-derivation, and in all the rest this will be our default extension of $\mathcal{X}$.

Invariant curves

Let $\mathcal{X} = p\partial_x + q\partial_y$ be a plane polynomial vector field with coefficients in $\mathbb{R}$ or $\mathbb{C}$ and let

$$\dot{x} = p(x, y), \qquad \dot{y} = q(x, y) \tag{1.4}$$

be the differential system associated with it. We say that an algebraic curve $\mathcal{C}(f)$ is an *invariant curve* of $\mathcal{X}$ if any solution of the differential system (1.4) starting out at a point of $\mathcal{C}(f)$ lies entirely in $\mathcal{C}(f)$. Following the famous Hilbert Nullstellenzatz this can be expressed in terms of algebraic identities as

$$p\partial_x f + q\partial_y f = kf \tag{1.5}$$

This last identity gives a way to define the concept of algebraic invariant curves for polynomial vector fields with coefficients in an abstract field $\mathbb{K}$.

Definition 1.1 Let $\mathcal{X} = p\partial_x + q\partial_y$ be a plane polynomial vector field with coefficients in $\mathbb{K}$. A nonconstant polynomial $f \in \mathbb{K}'[x, y]$, where $\mathbb{K}'$ is a field extension of $\mathbb{K}$, is called an algebraic invariant curve of $\mathcal{X}$ if there exists a polynomial $k \in \mathbb{K}'[x, y]$, called the cofactor of f, such that

$$\mathcal{X}(f) = kf.$$

In the same way, a nonconstant rational function $h = f/g$ with coefficients in an extension $\mathbb{K}'$ of $\mathbb{K}$ is called a rational first integral of the vector field $\mathcal{X}$ if

$$\mathcal{X}(h) = 0.$$

In this case it is easy to verify that f and g are invariant algebraic curves of $\mathcal{X}$ with the same cofactor.

The following lemma shows how rational first integrals can be viewed as invariant algebraic curves over transcendent extensions of the field of coefficients of the vector field.

Lemma 1.1 *Let $\mathcal{X}$ be a plane polynomial vector field with coefficients in the field $\mathbb{K}$. Then the following are equivalent:*

i) The vector field $\mathcal{X}$ has a rational first integral.

ii) There exists an invariant algebraic curve f of $\mathcal{X}$ such that the field generated over $\mathbb{K}$ by the coefficients of f is transcendent over $\mathbb{K}$, and f has no nonconstant factor belonging to $\mathbb{K}[x, y]$.

In particular, any vector field $\mathcal{X}$ with coefficients in a field $\mathbb{K}$ having a first integral has a first integral with coefficients in $\mathbb{K}$.

Proof: "$i \Rightarrow ii$" Let $h = f/g$ be a rational first integral, with coefficients in $\mathbb{K}' \supseteq \mathbb{K}$, of the vector field $\mathcal{X}$ and suppose that $\gcd(f, g) = 1$. Then we have $\mathcal{X}(h) = 0$ which gives the relation

$$g\mathcal{X}(f) - f\mathcal{X}(g) = 0.$$

According to the fact that $\gcd(f,g)=1$ and using Gauss lemma we get

$$\begin{aligned}\mathcal{X}(f)=kf\\ \mathcal{X}(g)=kg\end{aligned}$$

with $k\in\mathbb{K}'[x,y]$.

Let u be an indeterminate over $\mathbb{K}'[x,y]$ and let $h_1=f-ug$. Then an easy computation shows that $\mathcal{X}(h_1)=kh_1$. On the other hand, since u is transcendent over $\mathbb{K}'$ the field generated over $\mathbb{K}$ by the coefficients of h_1 is transcendent over $\mathbb{K}$. The fact that h_1 has no nonconstant factor in $\mathbb{K}[x,y]$ follows immediately from the irreducibility of h_1 in $\mathbb{K}'[u,x,y.]$

"$ii\Rightarrow i$" Let f be an invariant algebraic curve of the vector field $\mathcal{X}$ such that the field $\mathbb{K}'$ generated over $\mathbb{K}$ by the coefficients of f is transcendent over $\mathbb{K}$.

Since $\mathbb{K}'$ is finitely generated over $\mathbb{K}$ we can find $u_1,\dots,u_s\in\mathbb{K}'$ such that $\mathbb{K}(u_1,\dots,u_s)$ is purely transcendent over $\mathbb{K}$ and $\mathbb{K}'$ is algebraic of finite degree over $\mathbb{K}(u_1,\dots,u_s)$.

On the other hand, let us remark that for any $\mathbb{K}(u_1,\dots,u_s)$-isomorphism σ from $\mathbb{K}'$ into its algebraic closure $\overline{\mathbb{K}'}$ the curve $\sigma(f)$ is also invariant for the vector field $\mathcal{X}$. Moreover, there are only finitely many such $\mathbb{K}(u_1,\dots,u_s)$-isomorphisms and their number equals the dimension of $\mathbb{K}'$ as $\mathbb{K}(u_1,\dots,u_s)$-vector space.

By considering the product of all the $\sigma(f)$'s we get a polynomial h with coefficients in the field $\mathbb{K}(u_1,\dots,u_s)$ which is an invariant algebraic curve of $\mathcal{X}$. Without loss of generality we may suppose that h has its coefficients in the ring $\mathbb{K}[u_1,\dots,u_s]$. Indeed, it suffices to multiply h by the common denominator of its coefficients.

Let

$$h=\sum_{|\alpha|\le d}h_\alpha(x,y)u^\alpha$$

with $\alpha=(\alpha_1,\dots,\alpha_s)$, $|\alpha|=\alpha_1+\dots+\alpha_s$ and $u^\alpha=u_1^{\alpha_1}\dots u_s^{\alpha_s}$. We have then

$$\mathcal{X}(h)=\sum_{|\alpha|\le d}\mathcal{X}(h_\alpha)u^\alpha$$

and thus

$$\sum_{|\alpha|\le d}\mathcal{X}(h_\alpha)u^\alpha=k\sum_{|\alpha|\le d}h_\alpha u^\alpha$$

where k is the cofactor of h as invariant curve of $\mathcal{X}$. This last equation should show that k depends on the u_i's, but in fact by comparing the total degrees with respect to the u_i's in this last equality it is easy to see that k must be of degree 0 as polynomial in the u_i's. Thus k is a polynomial in $\mathbb{K}[x,y]$ and moreover we have

$$\mathcal{X}(h_\alpha)=kh_\alpha$$

for any α.

To find a rational first integral of the vector field $\mathcal{X}$ it is enough to prove that for some $\alpha\neq\alpha'$ the rational function $h_\alpha/h_{\alpha'}$ is nonconstant. This last fact is true, otherwise the polynomial h will be of the form $h = ch'$ with $c \in \mathbb{K}(u_1, \ldots, u_s)$ and $h' \in \mathbb{K}[x, y]$, and thus f will have a nonconstant factor in $\mathbb{K}[x, y]$ and this will also be the case for f.

Let us remark that the rational first integral we have constructed has its coefficients in the field $\mathbb{K}$. This same proof shows that starting from a first integral of $\mathcal{X}$ we can always construct another one with coefficients in the field $\mathbb{K}$. □

Remark 1.2 When a vector field $\mathcal{X}$ over $\mathbb{K}[x, y]$ has no rational first integral then essentially the field generated over $\mathbb{K}$ of any algebraic invariant curve is algebraic of finite degree over $\mathbb{K}$, i.e. any invariant algebraic curve f of $\mathcal{X}$ writes as $f = cf_1$ where c is a constant lying in an extension of $\mathbb{K}$ and the coefficients of f_1 generate over $\mathbb{K}$ a finite degree extension. As by-product, in any process of computation of invariant curves there is in general no need to extend the field of coefficients $\mathbb{K}$.

2 Curves in quasi-generic position

In this section we define the concept of curves in quasi-generic position and give some of their properties that will be needed for our purpose. Several aspects of such curves are studied in [15, 12, 1, 13].

Definition 2.1 An affine plane algebraic curve $\mathcal{C}(f)$ is said in quasi-generic position with respect to the projection on the x-axis if the following conditions hold:

i) $\deg(f) = \deg_y(f)$,

ii) the curve $\mathcal{C}(f)$ has no vertical tangent line at its singular points,

iii) the curve $\mathcal{C}(f)$ has no inflexion point with vertical tangent line.

Moreover if the curve $\mathcal{C}(f)$ has no critical points at infinity with respect to the projection on the x-axis then we say that $\mathcal{C}(f)$ is in projective quasi-generic position with respect to the projection on the x-axis. A curve $\mathcal{C}(f)$ is said in quasi-generic position (resp. in projective quasi-generic position) with respect to the projection on the y-axis if the curve defined by the polynomial $f(y, x)$ is in quasi-generic position (resp. in projective quasi-generic position) with respect to the projection on the x-axis.

When the curve $\mathcal{C}(f)$ is in quasi-generic position (resp. in projective quasi-generic position) with respect to both of the projections we simply call it a curve in quasi-generic position (resp. in projective quasi-generic position). Let us notice that if $\mathcal{C}(f)$ is in projective quasi-generic position then for any nonconstant factor f_1 of f the curve $\mathcal{C}(f_1)$ is also in projective quasi-generic position.

The previous definition slightly differs from the one given in [15, 12], and the curves defined there are called in generic position. The concept of curves in generic position is cooked up exactly so that no overlapping occurs in the projections, with respect to the coordinates axes, of the critical points. More precisely, in addition to the conditions given in the previous definition we add the condition that any two distinct critical points have distinct coordinates. This last condition will not be needed for our purpose and adding it will make somewhat involved the proof of theorem 2.3. For these reasons we did not include it in our definition and changed the "generic position" terminology into "quasi-generic position".

The following lemma will be useful to make more explicit projective quasi-generic position.

Lemma 2.1 *Let n be a positive integer and $f_0, \dots, f_n \in \mathbb{K}[x, y]$ be homogeneous polynomials such that $\deg(f_i) = n_i \leq n$. Let $M(x)$ be the matrix of the f_i's in the canonical basis $\{1, y, \dots, y^n\}$ of the $\mathbb{K}[x]$-module $\mathbb{K}[x]_n[y]$. Then*

$$\det(M(x)) = \det(M(1))x^N$$

where $N = (\sum_{i=0}^{n} n_i) - \frac{1}{2}n(n+1)$.

Proof: Let $f_j = \sum_{i=0}^{n} a_{i,j} x^{n_j - i} y^i$, with $a_{i,j} = 0$ if $n_j < i$. The determinant of the matrix $M(x)$ can be written in the form

$$\det(M(x)) = \sum_{\sigma \in S_{n+1}} \varepsilon(\sigma) a_{0,\sigma(0)} \dots a_{n,\sigma(n)} x^{n_{\sigma(0)}} x^{n_{\sigma(1)}-1} \dots x^{n_{\sigma(n)}-n}.$$

If we let $N = (\sum_{i=0}^{n} n_i) - \frac{1}{2}n(n+1)$ then we get $\det(M(x)) = \det(M(1))x^N$. $\square$

As consequence of lemma 2.1 we have the following proposition.

Proposition 2.2 *Let $\mathcal{C}(f)$ be an affine plane algebraic curve given by a degree n polynomial f and write $f = f_n + \dots + f_p$, where f_i is homogeneous of degree i. Then the following are equivalent:*

i) the curve $\mathcal{C}(f)$ has no critical points at infinity.

ii) The polynomial f_n is squarefree and does not have x or y as factors.

iii) The discriminant $\mathrm{Disc}_y(f)$ (resp. $Disc_x(f)$) of the polynomial f with respect to y (resp. x) has degree $n(n-1)$.

Proof: It is sufficient to prove the result for the critical points with respect to the projection on the x-axis.

Since f is squarefree the polynomial $\mathrm{Disc}_y(f)$ vanishes identically if and only if x is a factor of f. Moreover, if this is not the case the degree of $\mathrm{Disc}_y(f)$ equals the number of critical points, with respect to the projection on the x-axis, in the affine plane of the curve $\mathcal{C}(f)$ counted with multiplicities of intersection. Thus the assertions i) and iii) are equivalent according to Bézout theorem.

"$i \Longrightarrow ii$" Since the curve $\mathcal{C}(f)$ has no critical points at infinity the polynomials f and f_n are monic with respect to the variable y, i.e. $\deg(f) = \deg_y(f)$ and $\deg(f_n) = \deg_y(f_n)$. One has then

$$\mathrm{Disc}_y(f) = \det\left(y^{n-2}f, \ldots, yf, f, y^{n-1}\partial_y f, \ldots, \partial_y f\right)$$

and

$$\mathrm{Disc}_y(f_n) = \det\left(y^{n-2}f_n, \ldots, yf_n, f_n, y^{n-1}\partial_y f_n, \ldots, \partial_y f_n\right).$$

Using multilinearity of the function det and lemma 2.1 we obtain

$$\operatorname*{Disc}_y(f) = \mathrm{Disc}_y(f_n) + R = \operatorname*{Disc}_y(f_n(1,y))x^{n(n-1)} + R \tag{2.1}$$

where $\deg(R) \leq n(n-1) - 1$. Since $\mathrm{Disc}_y(f)$ has degree $n(n-1)$ it is so for f_n and this means that f_n is squarefree. The fact that x does not divide f_n follows from $\deg(f_n) = \deg_y(f_n)$.
"$ii \Longrightarrow i$" If f_n is squarefree and does not have x as factor then $\mathrm{Disc}_y(f_n) \neq 0$. This proves that $\deg(\mathrm{Disc}_y(f)) = n(n-1)$ according to the relation (2.1). $\square$

2.1 Transformation into curves in projective quasi-generic position

In this subsection we shall show that for a curve $\mathcal{C}(f)$ and a sufficiently "generic" projective transformation $T \in PGL(3, \mathbb{K})$ of the projective plane $\mathbb{P}^2\mathbb{K}$ the curve defined by $F \circ T(x, y, 1)$ is in projective quasi-generic position. In fact the set of projective transformations T such that the curve defined by $F \circ T(x, y, 1)$ is not in projective quasi-generic position is a projective variety in $PGL(3, \mathbb{K})$, but this result will not be needed in the sequel. For our purpose we shall consider a generic projective transformation which is defined in the following way:

Let u_1, u_2, v, w be new indeterminates, $\mathbb{F} = \mathbb{K}(u_1, u_2, v, w)$ and $\Gamma : \mathbb{F}[x, y] \mapsto \mathbb{F}[x, y]$ be the map defined by

$$f(x, y) \to F(u_1x + u_2y, -u_2x + u_1y, v(u_1x + u_2y) + w(u_1y - u_2x) + 1)$$

The map Γ is bijective and multiplicative (i.e $\Gamma(fg) = \Gamma(f)\Gamma(g)$). Moreover, for any polynomial f with coefficients in $\mathbb{F}$ the curves defined by f and $\Gamma(f)$, over an algebraic closure of $\mathbb{F}$, are projective transformations one of the other. This shows in particular that the two curves have the same geometric invariants (such as multiplicities, Milnor and Tjurina numbers of singularities). On the other hand, the geometric invariants of a curve defined by a polynomial $f \in \mathbb{K}[x, y]$ do not depend on the algebraically closed extension of $\mathbb{K}$ over which we view this curve. Thus, for any polynomial f with coefficients in $\mathbb{K}$ the curve defined by $\Gamma(f)$ conserves the same geometric invariants as $\mathcal{C}(f)$ (defined over $\overline{\mathbb{K}}$) even though these two curves are not defined over the same algebraically closed field.

The following theorem relates the main feature of considering the above generic projective transformation.

Theorem 2.3 *Let $f \in \mathbb{K}[x,y]$ be a squarefree polynomial of degree n defining a curve $\mathcal{C}(f)$, and u_1, u_2, v, w be new indeterminates. Then:*

i) The curve $\mathcal{C}(g)$ defined, over an algebraic closure of $\mathbb{K}(u_1,u_2)$, by the polynomial $g(x,y) = f(u_1x + u_2y, -u_2x + u_1y)$ is in quasi-generic position. Moreover, if the leading homogeneous term of f is squarefree then the curve $\mathcal{C}(g)$ is in projective quasi-generic position.

ii) The curve defined, over an algebraic closure of $\mathbb{F}$, by the polynomial $\Gamma(f)$ is in projective quasi-generic position.

Proof: i) Let us write $f = \sum_i f_i$ where f_i is homogneous of degree i. Then the polynomial g writes as $g = \sum_i g_i$ where $g_i = f_i(u_1x + u_2y, -u_2x + u_1y)$ is the homogeneous term of degree i of g.

In particular, the leading homogeneous term of g is $f_n(u_1x+u_2y, -u_2x+u_1y)$. It is easy to see that this last polynomial has degree n with respect to both x and y.

Let us now prove that the singular points of $\mathcal{C}(g)$ have neither horizontal nor vertical tangent lines. For this, let $[(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)]$ be the list of singular points of the curve $\mathcal{C}(f)$.

Then the list of singular points of the curve $\mathcal{C}(g)$ is given as

$$[\frac{1}{u_1^2+u_2^2}(u_1\alpha_1 - u_2\beta_1, u_2\alpha_1 + u_1\beta_1), \dots, \frac{1}{u_1^2+u_2^2}(u_1\alpha_s - u_2\beta_s, u_2\alpha_s + u_1\beta_s)].$$

On the other hand, let (α,β) be a singular point of multiplicity p in the curve $\mathcal{C}(f)$ and let $(\alpha',\beta') = \dfrac{1}{u_1^2+u_2^2}(u_1\alpha - u_2\beta, u_2\alpha + u_1\beta)$.

Let $f(x,y) = \sum_{i\ge p} h_i(x-\alpha, y-\beta)$ by the Taylor expansion of f around (α,β). Then the Taylor expansion of g around (α',β') writes as

$$g(x,y) = \sum_{i\ge p} h_i(u_1(x-\alpha') + u_2(y-\beta'), -u_2(x-\alpha') + u_1(y-\beta')).$$

Since the homogeneous term $h_p(u_1(x-\alpha')+u_2(y-\beta'), -u_2(x-\alpha')+u_1(y-\beta'))$ is not divisible by $x-\alpha'$ or $y-\beta'$ we deduce that the curve $\mathcal{C}(g)$ has neither horizontal nor vertical tangent lines at the singular point (α',β').

Using similar arguments to the ones used for singular points we can prove that the curve $\mathcal{C}(g)$ has no inflexion point with vertical tangent line. Indeed, if $[(\gamma_1,\delta_1),\dots,(\gamma_s,\delta_s)]$ is the list of inflexion points of the curve $\mathcal{C}(f)$ then the list of inflexion points of the curve $\mathcal{C}(g)$ is given as

$$[\frac{1}{u_1^2+u_2^2}(u_1\gamma_1 - u_2\delta_1, u_2\gamma_1 + u_1\delta_1), \dots, \frac{1}{u_1^2+u_2^2}(u_1\gamma_s - u_2\delta_s, u_2\gamma_s + u_1\delta_s)].$$

Suppose now that f_n is squarefree. Since $f_n(u_1x + u_2y, -u_2x + u_1y)$ is the leading homogeneous term of g and u_1, u_2 are algebraically independent over $\mathbb{K}[x,y]$ we deduce that its is squarefree and not divisible by x or y. Therefore, the curve $\mathcal{C}(g)$ is in projective quasi-generic position according to proposition 2.2.

ii) To prove that $\mathcal{C}(\Gamma(f))$ is in projective quasi-generic position it suffices to show that the leading homogeneous term of $F(x, y, vx + wy + 1)$ is squarefree. This leading homogeneous term is

$$h_n = F(x, y, vx + wy).$$

According to the fact that F is squarefree and u_1, u_2, v, w are algebraically independent over $\mathbb{K}[x, y]$ we deduce that h_n is squarefree and does not have x or y as factors. This proves that $\mathcal{C}(\Gamma(f))$ has no critical points at infinity according to proposition 2.2. □

3 Transformation into vector fields with Invariant curves in projective quasi-generic position

In this section we shall construct a $\mathbb{K}$-linear map $\Gamma_m^\star$ from the $\mathbb{K}$-vector space of vector fields of degree $\leq m$ over $\mathbb{K}[x, y]$ into the space of vector fields of degree $\leq m+1$ over $\mathbb{F}[x, y]$ in such a way that a given vector field $\mathcal{X}$ has $f \in \mathbb{K}[x, y]$ as invariant curve if and only if $\Gamma(f)$ is an invariant curve of the vector field $\Gamma_m^\star(\mathcal{X})$. As by-product all the invariant algebraic curves of the vector field $\Gamma^\star(\mathcal{X})$ are in projective quasi-generic position.

For this aim. we need to define the family of maps $\Gamma_n : \mathbb{F}_n[x, y] \mapsto \mathbb{F}_n[x, y]$,

$$f(x, y) \longrightarrow F_n(u_1x + u_2y, -u_2x + u_1y, v(u_1x + u_2y) + w(u_1y - u_2x) + 1)$$

where $\mathbb{F}_n[x, y]$ denotes the $\mathbb{F}$-vector space of polynomials of degree at most n and F_n stands for the degree n homogenization of f. Contrary to the map Γ all the Γ_n's are $\mathbb{F}$-linear. Moreover, an easy computation shows that

$$\Gamma_n(f) = (v(u_1x + u_2y) + w(-u_2x + u_1y) + 1)^{n-\deg(f)}\Gamma(f).$$

This last relation shows in particular that

$$\Gamma_{n_1+n_2}(f_1f_2) = \Gamma_{n_1}(f_1)\Gamma_{n_2}(f_2) \tag{3.1}$$

for any polynomials f_1, f_2 with $\deg(f_1) \leq n_1$ and $\deg(f_2) \leq n_2$.

We are now able to define in an explicit way the map $\Gamma_m^\star$. Let $\mathcal{X} = p\partial_x + q\partial_y$ be a vector field of degree $\leq m$ over $\mathbb{K}[x, y]$. Then we define $\Gamma_m^\star$ by $\Gamma_m^\star(\mathcal{X}) = r\partial_x + s\partial_y$ with

$$\begin{aligned} r &= \frac{(vu_1^2x + u_1 + u_2^2vx)\Gamma_m(p)}{u_2^2 + u_1^2} + \frac{(wu_2^2x - u_2 + u_1^2wx)\Gamma_m(q)}{u_2^2 + u_1^2} \\ s &= \frac{(vu_2^2y + u_2 + u_1^2vy)\Gamma_m(p)}{u_2^2 + u_1^2} + \frac{(wu_1^2y + u_1 + u_2^2wy)\Gamma_m(q)}{u_2^2 + u_1^2} \end{aligned}$$

The exact relation concerning invariant curves between $\Gamma_m^\star(\mathcal{X})$ and $\mathcal{X}$ is given in the following theorem.

Theorem 3.1 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a vector field of degree m over $\mathbb{K}[x,y]$. Then the following assertions hold:*

i) The vector field $\Gamma^\star_m(\mathcal{X})$ has degree $m+1$ and for any polynomial f of degree at most n we have

$$\Gamma^\star_m(\mathcal{X})(\Gamma_n(f)) = \Gamma_{m+n-1}(\mathcal{X}(f)) + n\Gamma_m(vp+wq)\Gamma_n(f) \tag{3.2}$$

ii) A rational function $\frac{f}{g}$ is a first integral of $\mathcal{X}$ if and only if $\frac{\Gamma_n(f)}{\Gamma_n(g)}$ is a first integral of the vector field $\Gamma^\star_m(\mathcal{X})$, where $n = \max(\deg(f), \deg(g))$.

iii) A polynomial f is an invariant curve of $\mathcal{X}$ if and only if $\Gamma(f)$ is an invariant curve of $\Gamma^\star_m(\mathcal{X})$. In particular, if $\mathcal{X}$ has no rational first integral then all the invariant algebraic curves of $\Gamma^\star_m(\mathcal{X})$ are in projective quasi-generic position.

Proof: i) Since u_1, u_2, v, w are indeterminates over $\mathbb{K}[x,y]$ the map Γ_m sends p and q to polynomials of degree m. Thus the vector field $\Gamma^\star_m(\mathcal{X})$ has degree exactly $m+1$. To simplify the proof of the identity we introduce the following abbreviations

$$\begin{gathered}
X = u_1x + u_2y, \quad Y = -u_2x + u_1y, \\
Z = v(u_1x + u_2y) + w(-u_2x + u_1y) + z, \\
G = F_n(X,Y,Z), \quad A = (\partial_x F_n)(X,Y,Z), \\
B = (\partial_y F_n)(X,Y,Z), \quad C = (\partial_z F_n)(X,Y,Z), \\
P^\star = P_m(X,Y,Z), \quad Q^\star = Q_m(X,Y,Z) \\
R = \frac{(vu_1^2x + u_1z + u_2^2vx)P^\star}{u_2^2 + u_1^2} + \frac{(wu_2^2x - u_2z + u_1^2wx)Q^\star}{u_2^2 + u_1^2}, \\
S = \frac{(vu_2^2y + u_2z + u_1^2vy)P^\star}{u_2^2 + u_1^2} + \frac{(wu_1^2y + u_1z + u_2^2wy)Q^\star}{u_2^2 + u_1^2}
\end{gathered}$$

Now let us consider the expression $R\partial_x G + S\partial_y G$. On the first hand, by letting $z = 1$ in the last expression we get

$$(R\partial_x G + S\partial_y G)(x,y,1) = r\partial_x\Gamma_n(f) + s\partial_y\Gamma_n(f) = \Gamma^\star_m(\mathcal{X})(\Gamma_n(f)).$$

On the other hand, we have

$$\begin{gathered}
\partial_x G = u_1A - u_2B + (u_1v - u_2w)C \\
\partial_y G = u_2A + u_1B + (u_2v + u_1w)C
\end{gathered}$$

which gives after computation (This factorization can be obtained using a symbolic computation software such as Maple)

$$\begin{aligned}
&R\partial_x G + S\partial y G - zP^\star A - zQ^\star B \\
&= \quad (vP^\star + wQ^\star)\Big((wu_1y - u_2wx + u_1vx + u_2vy + z)C \\
&\qquad +(u_1x + u_2y)A + (u_1y - u_2x)B\Big)
\end{aligned}$$

The Euler formula for the homogeneous polynomial G writes as

$$nG = (u_1x + u_2y)A + (-u_2x + u_1y)B + ((u_1x + u_2y)v + (-u_2x + u_1y)w + z)C.$$

Thus

$$R\partial_x G + S\partial y G - zP^\star A - zQ^\star B = n(vP^\star + wQ^\star)G.$$

By letting $z = 1$ in the last equality and taking into account the linearity of the Γ_n's and the equation (3.1) we finally get

$$\Gamma_m^\star(\mathcal{X})(\Gamma_n(f)) = \Gamma_{m+d-1}(\mathcal{X}(f)) + \Gamma_m(k)\Gamma_n(f),$$

where $k = n(vp + wq)$.
ii) Without loss of generality we may suppose that $n = \deg(f)$, we have thus $\Gamma_n(f) = \Gamma(f)$. Let $n_1 = \deg(g)$ and let us simplify the expression

$$\Gamma_m^\star(\mathcal{X})\left(\frac{\Gamma_n(f)}{\Gamma_n(g)}\right)(\Gamma_n(g))^2.$$

Using elementary rules of derivations we have

$$\Gamma_m^\star(\mathcal{X})\left(\frac{\Gamma_n(f)}{\Gamma_n(g)}\right)(\Gamma_n(g))^2 = \Gamma_n(g)\Gamma_m^\star(\mathcal{X})(\Gamma(f)) - \Gamma(f)\Gamma_m^\star(\mathcal{X})(\Gamma_n(g)).$$

Now taking into account the identities (3.1) and (3.2) we obtain

$$\begin{aligned}
&\Gamma_m^\star(\mathcal{X})\left(\frac{\Gamma_n(f)}{\Gamma_n(g)}\right)(\Gamma_n(g))^2 \\
&= \Gamma_n(g)\Gamma_{m+n-1}(\mathcal{X}(f)) - \Gamma(f)\Gamma_{m+n_1-1}(\mathcal{X}(g))\Gamma_{n-n_1}(1) \\
&\quad + \Gamma(fg)((n-n_1)\Gamma_m(vp+wq)\Gamma_{n-n_1}(1) - \Gamma_m^\star(\mathcal{X})(\Gamma_{n-n_1}(1)))
\end{aligned}$$

On the other hand, using the identity (3.1) it is easy to verify that

$$\Gamma_n(g)\Gamma_{m+n-1}(\mathcal{X}(f)) - \Gamma(f)\Gamma_{m+n_1-1}(\mathcal{X}(g))\Gamma_{n-n_1}(1) = \Gamma_{m+2n-1}\big(g^2\mathcal{X}(\frac{f}{g})\big).$$

Moreover, a direct computation shows that

$$(n-n_1)\Gamma_m(vp+wq)\Gamma_{n-n_1}(1) - \Gamma_m^\star(\mathcal{X})(\Gamma_{n-n_1}(1)) = 0$$

and then

$$\Gamma_m^\star(\mathcal{X})\left(\frac{\Gamma_n(f)}{\Gamma_n(g)}\right)(\Gamma_n(g))^2 = \Gamma_{m+2n-1}\big(g^2\mathcal{X}(\frac{f}{g})\big).$$

Thus $\Gamma_m^\star(\mathcal{X})\left(\frac{\Gamma_n(f)}{\Gamma_n(g)}\right) = 0$ if and only if $\mathcal{X}\left(\frac{f}{g}\right) = 0$ according to the fact that Γ_{m+2n-1} is one to one.
iii) Suppose that f is a degree n algebraic invariant curve of the vector field $\mathcal{X}$. Then we have the equality $\mathcal{X}(f) = kf$ with $\deg(k) \le m-1$, and applying Γ_{m+n-1} to it we get

$$\Gamma_{m+n-1}(\mathcal{X}(f)) = \Gamma_{m-1}(k)\Gamma(f).$$

Combining this last equality with the identity (3.2) we obtain

$$\Gamma_m^\star(\mathcal{X})(\Gamma(f)) = k_1\Gamma(f).$$

Conversely, suppose that $\Gamma(f)$ is an invariant algebraic curve of the vector field $\Gamma_m^\star(\mathcal{X})$ and let us write $\Gamma_m^\star(\mathcal{X}) = k_2\Gamma(f)$ with $\deg(k_2) \leq m$. Since Γ_m is bijective we can write $k_2 = \Gamma_m(k_3)$ with $\deg(k_3) \leq m$, and taking into account the identity (3.2) we get

$$\Gamma_{m+n-1}(\mathcal{X}(f)) = \Gamma_m(k_4)\Gamma(f)$$

with $\deg(k_4) \leq m$. Since $\deg(\Gamma_{m+n-1}(\mathcal{X}(f)) \leq m+n-1$ and $\deg(\Gamma(f)) = n$ we have the bound $\deg(\Gamma_m(k_4)) \leq m-1$, and hence $\Gamma_m(k_4) = \Gamma_{m-1}(k_5)$ with $\deg(k_5) \leq m-1$. This gives the relation

$$\Gamma_{m+n-1}(\mathcal{X}(f)) = \Gamma_{m+n-1}(k_5 f)$$

and then

$$\mathcal{X}(f) = k_5 f.$$

Now assume that the given vector field $\mathcal{X}$ has no rational first integral. Then the vector field has finitely many invariant curves. Moreover, following remark 1.2, and without loss of generality, we may assume that all these invariant curves have their coefficients in $\mathbb{K}$. Therefore, all their transforms under Γ are in projective quasi-generic position, and these are exactly the invariant curves of the vector field $\Gamma^\star(\mathcal{X})$. $\square$

A property of the transformed vector field

A remarkable property of the transformed vector field lies in the fact that it has the straight line $v(u_1x + u_2y) + w(-u_2x + u_1y) + 1$ as invariant curve. In this subsection we clear up this question and give some of its consequences.

Lemma 3.2 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a degree m vector field over $\mathbb{K}[x, y]$ and $\Gamma_m^\star(\mathcal{X}) = r\partial_x + s\partial_y$. Then the following holds:*

i) $yr_{m+1} - xs_{m+1} = 0$.

ii) The straight line $\ell(x, y) = v(u_1x+u_2y)+w(-u_2x+u_1y)+1$ is an invariant curve of $\Gamma_m^\star(\mathcal{X})$. If moreover $yp_m - xq_m = 0$ then ℓ is a common factor of r and s.

Proof: i) An easy computation shows that

$$yr - xs = (-u_2x + u_1y)\Gamma_m(p) - (u_1x + u_2y)\Gamma_m(q).$$

This implies in particular that $yr - xs$ is of degree at most $m + 1$ and thus $yr_{m+1} - xs_{m+1} = 0$.

ii) The fact that ℓ is invariant follows immediately from the relation (3.2) by taking $f = 1$ and $n = 1$. Suppose now that $yp_m - xq_m = 0$. First let us note that

$$\begin{aligned}\Gamma_m(p) &= \Gamma_m(p_m) + \ell g\\ \Gamma_m(q) &= \Gamma_m(q_m) + \ell h\end{aligned}$$

where $g, h \in \mathbb{K}[x, y]$. This gives the relation

$$\begin{aligned}(u_1^2 + u_2^2)r = &\quad (vu_1^2x + u_1 + u_2^2vx)\Gamma_m(p_m)\\ &+(wu_2^2x - u_2 + u_1^2wx)\Gamma_m(q_m) + g_1\ell.\end{aligned}$$

On the other hand, a direct computation gives (Here again we used Maple to carry out this division)

$$\begin{aligned}&(-u_2x + u_1y)((vu_1^2x + u_1 + u_2^2vx)\Gamma_m(p_m) + (wu_2^2x - u_2 + u_1^2wx)\Gamma_m(q_m))\\ &= \; x\Gamma_m(q_m)(u_1^2 + u_2^2)\ell + (vu_1^2x + u_1 + vu_2^2x)((u_1x + u_2y)\Gamma_m(q_m)\\ &\quad -(-u_2x + u_1y)\Gamma_m(p_m))\end{aligned}$$

Applying Γ_{m+1} to the identity $yp_m - xq_m = 0$ we get

$$(u_1x + u_2y)\Gamma_m(q_m) - (-u_2x + u_1y)\Gamma_m(p_m) = 0$$

and thus ℓ divides the product

$$(-u_2x + u_1y)((vu_1^2x + u_1 + u_2^2vx)\Gamma_m(p_m) + (wu_2^2x - u_2 + u_1^2wx)\Gamma_m(q_m)).$$

Since on the other hand it is prime with $-u_2x + u_1y$ it must divide the other factor. As by product the polynomial ℓ is a factor of r. The case of s can be done in the same way. $\square$

As consequence of theorem 3.1 and lemma 3.2 we have the following corollary which will be very useful in the sequel.

Corollary 3.3 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a degree m vector field over $\mathbb{K}[x, y]$. Let $\mathbb{K}'$ be a field extension of $\mathbb{K}$ and $f_1, \dots, f_t \in \mathbb{K}'[x, y]$ be squarefree polynomials defining invariant curves of the vector field $\mathcal{X}$. Then there exists a field extension $\mathbb{F}$ of $\mathbb{K}'$, a projective transformation over $\mathbb{F}$ sending $f_1, \dots, f_t$ to $f_1', \dots, f_t'$ and a vector field $\mathcal{Y}$ of degree $m+1$ over $\mathbb{F}[x, y]$ such that:*

i) the curves defined by the f_i''s are in projective quasi-generic position,

ii) the curves defined by the f_i''s are invariant for the vector field $\mathcal{Y}$.

Moreover, if $yp_m - xq_m = 0$ then we can choose $\mathcal{Y}$ of degree m.

Proof: Let $\mathcal{X}'$ be the extension of $\mathcal{X}$ to $\mathbb{K}'[x, y]$ obtained by viewing the elements of $\mathbb{K}'$ as constants. We can then take $\mathbb{F} = \mathbb{K}'(u_1, u_2, v, w)$, $f_i' = \Gamma(f_i)$ and $\mathcal{Y} = \Gamma_m^\star(\mathcal{X}')$.

Suppose now that $yp_m - xq_m = 0$ and let $\mathcal{Y} = r\partial_x + s\partial_y$. Then following lemma 3.2 the linear polynomial $\ell = v(u_1x + u_2y) + w(-u_2x + u_1y) + 1$ is a common factor of r and s, say $r = \ell r_1$ and $s = \ell s_1$. By taking $\mathcal{Y}' = r_1\partial_x + s_1\partial_y$ we obtain a vector field fulfilling the required conditions. $\square$

4 Application to the Poincaré problem

In this section we shall apply the results and tools developed in sections 2 and 3 to the study, under some assumptions on invariant curves, of the Poincaré problem. The main idea in this respect is to reduce the question to the case of invariant curves in projective quasi-generic position, and then use Bézout theorem. We first state some lemmas that we will need in the sequel.

Let $\mathcal{C}(f)$ be a plane algebraic curve given by a squarefree polynomial f and suppose that x and y are not factors of f. In this case the ideals $\mathcal{I}(f, \partial_x f)$ and $\mathcal{I}(f, \partial_y f)$ are zero-dimensional. For a given point (α, β) of the curve $\mathcal{C}(f)$ if we define L_x (resp. L_y) to be the $\mathbb{K}$-linear map of the multiplication by $\partial_x f$ (resp. $\partial_y f$) in the $\mathbb{K}$-algebra $(\mathbb{K}[x,y]/\mathcal{I}(f, \partial_y f))_{(\alpha,\beta)}$ (resp. $(\mathbb{K}[x,y]/\mathcal{I}(f, \partial_x f))_{(\alpha,\beta)}$) its range is $(\mathcal{I}(f, \partial_x f, \partial_y f)/\mathcal{I}(f, \partial_y f))_{(\alpha,\beta)}$ (resp. $(\mathcal{I}(f, \partial_x f, \partial_y f)/\mathcal{I}(f, \partial_x f))_{(\alpha,\beta)}$), and $(\mathcal{I}(f, \partial_y f) : \partial_x f/\mathcal{I}(f, \partial_y f))_{(\alpha,\beta)}$ (resp. $(\mathcal{I}(f, \partial_x f) : \partial_y f/\mathcal{I}(f, \partial_x f))_{(\alpha,\beta)}$) is its kernel.

Now applying the dimension formula, of vector spaces, to L_x (resp. L_y) we get the relations

$$\mathrm{I}(f, \partial_x f, (\alpha, \beta)) = \dim_{\mathbb{K}} \left(\mathbb{K}[x,y]/\mathcal{I}(f, \partial_x f) : \mathcal{I}(\partial_y f)\right)_{(\alpha,\beta)} + \tau(f, (\alpha, \beta)) \quad (4.1)$$

$$\mathrm{I}(f, \partial_y f, (\alpha, \beta)) = \dim_{\mathbb{K}} \left(\mathbb{K}[x,y]/\mathcal{I}(f, \partial_y f) : \mathcal{I}(\partial_x f)\right)_{(\alpha,\beta)} + \tau(f, (\alpha, \beta)) \quad (4.2)$$

This implies in particular that any singular point of the curve $\mathcal{C}(f)$ is a zero of both of the ideals $\mathcal{I}(f, \partial_x f) : \mathcal{I}(f, \partial_x f, \partial_y f)$ and $\mathcal{I}(f, \partial_y f) : \mathcal{I}(f, \partial_x f, \partial_y f)$.

Lemma 4.1 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a vector field of degree m over $\mathbb{K}[x,y]$ and let $\mathcal{C}(f)$ and invariant curve of $\mathcal{X}$. Then the following hold:*

i) Any singular point of the curve $\mathcal{C}(f)$ is a fixed point of $\mathcal{X}$, i.e. a zero of the ideal $\mathcal{I}(p, q)$.

ii) If $\gcd(p, q, f) = 1$ then the number of singular points of $\mathcal{C}(f)$ do not exceed $(m+1)^2$. If moreover $yp_m - xq_m = 0$ then it is bounded by m^2.

Proof: i) Without loss of generality we may assume that x and y are not factors of f. Let (α, β) be a singular point of $\mathcal{C}(f)$. Then (α, β) is a zero of both of the ideals $\mathcal{I}(f, \partial_x f) : \mathcal{I}(f, \partial_x f, \partial_y f)$ and $\mathcal{I}(f, \partial_y f) : \mathcal{I}(f, \partial_x f, \partial_y f)$. Since $p \in \mathcal{I}(f, \partial_y f) : \mathcal{I}(f, \partial_x f, \partial_y f)$ and $q \in \mathcal{I}(f, \partial_x f) : \mathcal{I}(f, \partial_x f, \partial_y f)$ we have $p(\alpha, \beta) = q(\alpha, \beta) = 0$.
ii) The number of singular points of a curve is a projective invariant. Thus according to corollary 3.3 we may assume that $\mathcal{C}(f)$ has no critical points at infinity and $\mathcal{X}$ has degree $m+1$. Let g be the gcd of p and q and let $p = gp_1$, $q = gq_1$. From the relation $p\partial_x f + q\partial_y f$ we deduce that g divides kf. Moreover, according to the assumption $\gcd(p, q, f) = 1$ we have $\gcd(g, f) = 1$ and then g divides k. Therefore, the curve $\mathcal{C}(f)$ is invariant by the vector field $\mathcal{X}_1 = p_1\partial_x + q_1\partial_y$, and following *i)* the singular points of $\mathcal{C}(f)$ are zeros of the zero-dimensional ideal $\mathcal{I}(p_1, q_1)$.

Taking into account the inequalities $\deg(p_1) \leq m+1$, $\deg(q_1) \leq m+1$ and according to Bézout theorem we get the bound $(m+1)^2$. If moreover $yp_m - xq_m = 0$ then following corollary 3.3 we can choose $\mathcal{X}$ of degree m and this gives the bound m^2. $\square$

The case of nonsingular invariant curves

In this subsection we prove that the degree of a nonsingular invariant curve does not exceed $m+1$, where m is the degree of the vector field. This result has been obtained for the first time by Cerveau and Lins Neto in [4] using foliations of the projective plane. Other proofs, more or less involved, of the same result can be found in [28, 16]. It is included here because we can supply an elementary and purely algebraic proof.

Theorem 4.2 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a degree m vector field over $\mathbb{K}[x,y]$ and suppose that in some extension $\mathbb{K}'$ of $\mathbb{K}$ the vector field has a nonsingular invariant curve of degree n given by a squarefree polynomial $f \in \mathbb{K}'[x,y]$. Then the following holds:*

i) The degree of f do not exceed $m+1$. If moreover $yp_m - xq_m = 0$ then $n \leq m$.

ii) If the leading homogeneous term of f is squarefree then $n \leq m$.

Proof: i) It is harmless to assume that $\mathbb{K} = \mathbb{K}'$. Indeed, we can extend the vector field $\mathcal{X}$ to $\mathbb{K}'[x,y]$ by viewing the elements of $\mathbb{K}'$ as constants. On the other hand, According to corollary 3.3 we can reduce to the case where $\mathcal{X}$ has degree $m+1$ and $\mathcal{C}(f)$ is in projective quasi-generic position.

From the relation

$$p\partial_x f + q\partial_y f = kf$$

we deduce that $p \in \mathcal{I}(f, \partial_y f) : \mathcal{I}(f, \partial_x f, \partial_y f)$. Moreover, the fact that $\mathcal{C}(f)$ is nonsingular implies that $\mathcal{I}(f, \partial_x f, \partial_y f) = \mathbb{K}[x,y]$ and hence

$$p \in \mathcal{I}(f, \partial_y f).$$

Let g be the greatest common divisor of p and $\partial_y f$ and let $p = gp_1$ and $\partial_y f = gh$. We have then

$$p_1 \in \mathcal{I}(f, \partial_y f) : g.$$

Now let us prove that $\mathcal{I}(f, \partial_y f) : g = \mathcal{I}(f, h)$. First, it is obvious that $\mathcal{I}(f, \partial_y f) : g \supseteq \mathcal{I}(f, h)$. On the other hand, let $h_1 \in \mathcal{I}(f, \partial_y f) : g$ and let $a, b \in \mathbb{K}[x,y]$ such that

$$gh_1 = af + b\partial_y f.$$

According to the last relation and to the fact that g divides $\partial_y f$ we deduce that g divides the product af. Since f and $\partial_y f$ have no common factors the polynomial g must divide a, say $a = ga_1$.

We have thus the relation $h_1 = a_1 f + bh$ which means that $h_1 \in \mathcal{I}(f, h)$. Therefore we have the inclusion

$$\mathcal{I}(p_1, h) \subseteq \mathcal{I}(f, h)$$

which gives

$$\dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(p_1, h) \geq \dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(f, h).$$

On the other hand, according to the fact that $\mathcal{C}(f)$ is in projective quasi-generic position and that h is a factor of $\partial_y f$ we deduce that f and h have no common zeros at infinity. Thus, by using Bézout theorem we get

$$\begin{aligned} &\dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(f, h) = n \deg(h), \\ &\dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(p_1, h) \leq \deg(p_1) \deg(h). \end{aligned}$$

this finally gives $n \leq \deg(p_1)$ and then $n \leq m + 1$ according to $\deg(p_1) \leq \deg(p) = m + 1$.

if $yp_m - xq_m = 0$ then according to corollary 3.3 we can reduce to the case of a curve in projective quasi-generic position but with a vector field of degree m. This gives the bound $n \leq m$.

ii) Suppose that the leading homogeneous term of f is squarefree. Then applying theorem 3.1 with $v = w = 0$ and according to theorem 2.3 we reduce to the case where $\mathcal{X}$ has degree m and $\mathcal{C}(f)$ is in projective quasi-generic position. This gives the bound $n \leq m$. ☐

The case of invariant curves with ordinary nodes

This subsection deals with invariant algebraic curves having only ordinary singular points. A bound in the case of ordinary double points is given in [5] and also in [28], and it is of order $2m$ where m is the degree of the vector field. Here we give a better bound for this category of curves. A notable feature of the results we present is that no irreducibility assumptions are needed.

Theorem 4.3 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a degree m vector field over $\mathbb{K}[x, y]$ and suppose that in some extension $\mathbb{K}'$ of $\mathbb{K}$ the vector field has an invariant curve of degree n given by a squarefree polynomial $f \in \mathbb{K}'[x, y]$. If the curve $\mathcal{C}(f)$ has only ordinary double points as singularities then the following holds:*

i) the degree of f does not exceed $m + 2$. If moreover $yp_m - xq_m = 0$ then $n \leq m + 1$.

ii) If the leading homogeneous term of f is squarefree then $n \leq m + 1$.

Proof: As in the proof of theorem 4.2 we can reduce to the case where $\mathcal{C}(f)$ is in projective quasi-generic position and $\mathcal{X}$ has degree $m + 1$. If moreover $yp_m - xq_m = 0$ then according to corollary 3.3 we can choose $\mathcal{X}$ of degree m. To treat the two cases at the same time let $m_1 = m$ if $yp_m - xq_m = 0$ and $m_1 = m + 1$ otherwise.

Let g be the gcd of f and p and let $p = gp_1$ and $f = gf_1$. From the relation $p\partial_x f + q\partial_y f = kf$ we deduce that g divides $q\partial_y f$. Since g is a factor of f and $\gcd(f, \partial_y f) = 1$ the polynomial g divides q, say $q = gq_1$. Therefore $\mathcal{C}(f_1)$ is an invariant curve of the vector field $\mathcal{X}_1 = p_1\partial_x + q_1\partial_y$.

On the other hand, since $p_1 \in \mathcal{I}(f_1, \partial_y f_1) : \mathcal{I}(f_1, \partial_x f_1, \partial_y f_1)$ we have the inclusion

$$\mathcal{I}(p_1, f_1) \subseteq \mathcal{I}(f_1, \partial_y f_1) : \mathcal{I}(f_1, \partial_x f_1, \partial_y f_1).$$

Let us note that $\mathcal{I}(f_1, \partial_y f_1)$ and $\mathcal{I}(f_1, \partial_y f_1) : \mathcal{I}(f_1, \partial_x f_1, \partial_y f_1)$ have the same zeros in $\overline{\mathbb{K}}^2$ but with eventually different multiplicities.

Let (α, β) be a zero of $\mathcal{I}(f_1, \partial_y f_1)$. Then one of the following cases occurs:
Case 1: The point (α, β) is nonsingular in the curve $\mathcal{C}(f_1)$. In such situation β is a multiplicity 2 root of the univariate polynomial $f_1(\alpha, y)$ according to the fact that $\mathcal{C}(f_1)$ has no inflexion point with vertical tangent line. This gives $\mathrm{I}(f_1, \partial_y f_1, (\alpha, \beta)) = 1$ and then

$$\mathrm{I}(f_1, p_1, (\alpha, \beta)) \geq \mathrm{I}(f_1, \partial_y f_1, (\alpha, \beta))$$

since $\mathcal{C}(p_1)$ and $\mathcal{C}(f_1)$ meet at (α, β).
Case 2: (α, β) is an ordinary double point of the curve $\mathcal{C}(f_1)$. In this case, according to the fact that the curve has no vertical tangent line at its singular points, we have $\mathrm{I}(f_1, \partial_y f_1, (\alpha, \beta)) = 2$.

On the other hand, the curves $\mathcal{C}(f_1)$ and $\mathcal{C}(p_1)$ meet at the point (α, β) and from relation (1.1) we have $\mathrm{I}(f_1, p_1, (\alpha, \beta)) \geq 2t \geq 2$, where t is the multiplicity of (α, β) as point of $\mathcal{C}(p_1)$. This gives as in the first case

$$\mathrm{I}(f_1, p_1, (\alpha, \beta)) \geq \mathrm{I}(f_1, \partial_y f_1, (\alpha, \beta)).$$

We have thus

$$\sum_{(\alpha,\beta)} \mathrm{I}(f_1, p_1, (\alpha, \beta)) \geq \sum_{(\alpha,\beta)} \mathrm{I}(f_1, \partial_y f_1, (\alpha, \beta))$$

where (α, β) ranges on the zeros of $\mathcal{I}(f_1, \partial_y f_1)$ in the affine plane. Since the curve $\mathcal{C}(f_1)$ has no critical points at infinity we have by using relation (1.2)

$$\sum_{(\alpha,\beta)} \mathrm{I}(f_1, \partial_y f_1, (\alpha, \beta)) = \deg(f_1)(\deg(f_1) - 1).$$

As $\mathcal{I}(p_1, f_1)$ may have other zeros than those of $\mathcal{I}(f_1, \partial_y f_1)$ we have by using once again relation (1.2)

$$\deg(p_1)\deg(f_1) \geq \sum_{(\alpha,\beta)} \mathrm{I}(f_1, p_1, (\alpha, \beta)).$$

This gives $\deg(p_1) + 1 \geq \deg(f_1)$ and then $m_1 + 1 \geq n$ after adding $\deg(g)$ to both sides of the inequality.
ii) Suppose that the leading homogeneous term of f is squarefree. Then applying theorem 3.1 with $v = w = 0$ and according to theorem 2.3 we reduce to the case where $\mathcal{X}$ has degree m and $\mathcal{C}(f)$ is in projective quasi-generic position. This gives the bound $n \leq m + 1$. $\square$

Remark 4.4 Using theorem 4.3 we recover a part of a result given in [7] (see also [10] theorem 1) concerning the case where the invariant curves satisfy genericity conditions so that their product gives a curve having no critical points at infinity and only ordinary nodes as singularities.

A bound in terms of the Tjurina number

This subsection concerns a bound on degree for an invariant curve in terms of the degree of the vector field and an upper bound of the Tjurina numbers of the singularities the curve has.

Theorem 4.5 *Let $\mathcal{X} = p\partial_x + q\partial_y$ be a degree m vector field over $\mathbb{K}[x, y]$ and suppose that in some extension $\mathbb{K}'$ of $\mathbb{K}$ the vector field has an invariant curve of degree n given by a squarefree polynomial $f \in \mathbb{K}'[x, y]$. Let K be the maximum of the Tjurina numbers of the singularities the curve has in the projective plane. Then*

$$n \leq \frac{(1 + \sqrt{(1 + 4K)})(m + 2)}{2}$$

Proof: The Tjurina number is a projective invariant of the singularity. Thus we can reduce to the case where $\mathcal{C}(f)$ is in projective quasi-generic position and $\mathcal{X}$ has degree $m + 1$.

Let g be the gcd of p and f and let $p = gp_1$, $f = gf_1$. In the proof of theorem 4.3 we have shown that g divides q, say $q = gq_1$, and that f_1 is an invariant curve of the vector field $\mathcal{X}_1 = p_1\partial_x + q_1\partial_y$.

On the other hand, since f_1 is a factor of f any upper bound of the Tjurina numbers of the singularities of $\mathcal{C}(f)$ is also an upper bound for those of $\mathcal{C}(f_1)$. Moreover, the curve $\mathcal{C}(f_1)$ is also in projective quasi-generic position.

From the inclusion $\mathcal{I}(p_1, f_1) \subseteq \mathcal{I}(f_1, \partial_y f_1) : \mathcal{I}(f_1, \partial_x f_1, \partial_y f_1)$ we have

$$\begin{aligned} \deg(p_1)\deg(f_1) &\geq \dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(p_1, f_1) \\ &\geq \dim_{\mathbb{K}} \mathbb{K}[x, y]/\mathcal{I}(f_1, \partial_y f_1) : \mathcal{I}(\partial_x f_1). \end{aligned}$$

According to the fact that $\mathcal{C}(f_1)$ is in projective quasi-generic position and following equations (4.2) and (1.2) we get

$$dim_{\mathbb{K}}\mathbb{K}[x, y]/\mathcal{I}(f_1, \partial_y f_1) : \mathcal{I}(\partial_x f_1) = \deg(f_1)(\deg(f_1) - 1) - \sum_{(\alpha,\beta)} \tau(f_1, (\alpha, \beta)).$$

As $\gcd(p_1, q_1, f_1) = 1$ then using lemma 4.1 *ii)* we obtain

$$\sum_{(\alpha,\beta)} \tau(f_1, (\alpha, \beta)) \leq (m + 1 - \deg(g))^2 K.$$

We have then the inequality

$$(m + 1 - \deg(g)) \deg(f_1) \geq \deg(f_1)(\deg(f_1) - 1) - (m + 1 - \deg(g))^2 K.$$

This gives the inequality

$$\deg(f_1) \leq \frac{(1+\sqrt{(1+4K)})(m+2-\deg(g))}{2}.$$

Finally, by adding $\deg(g)$ to both sides of the inequality and taking into account the fact that $(1+\sqrt{1+4K})/2 \geq 1$ we get the required inequality. $\square$

Remark 4.6 If we consider the case of a curve with ordinary nodes as singularities then the previous theorem gives $\frac{(1+\sqrt{5})}{2}(m+2)$ as bound while theorem 4.3 gives $m+2$. This shows that the bound given in theorem 4.5 is not sharp.

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