Electronic Journal of Differential Equations, Vol. 2002(2002), No. 58, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

BOUNDARY-VALUE PROBLEMS FOR THE BIHARMONIC EQUATION WITH A LINEAR PARAMETER

YAKOV YAKUBOV

Abstract. We consider two boundary-value problems for the equation

$$\Delta^2 u(x,y) - \lambda\Delta u(x,y) = f(x,y)$$

with a linear parameter on a domain consisting of an infinite strip. These problems are not elliptic boundary-value problems with a parameter and therefore they are non-standard. We show that they are uniquely solvable in the corresponding Sobolev spaces and prove that their generalized resolvent decreases as $1/|\lambda|$ at infinity in $L_2(\mathbb{R}\times(0,1))$ and $W_2^1(\mathbb{R}\times(0,1))$.

1. Formulation of the problem

The main objective of this paper is to find estimates for the generalized resolvent for the problem

$$L(\lambda, D_x, D_y)u := \Delta^2 u(x,y) - \lambda\Delta u(x,y) = f(x,y), \quad (x,y)\in\Omega, \tag{1.1}$$

$$u(x,0) = u(x,1) = \frac{\partial u(x,0)}{\partial y} = \frac{\partial u(x,1)}{\partial y} = 0, \quad x\in\mathbb{R}, \tag{1.2}$$

where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$, $\lambda\in\mathbb{C}$ and $\Omega := (-\infty,\infty)\times[0,1]\subset\mathbb{R}^2$. Known results on this subject treat elliptic boundary-value problems with a parameter, mostly in bounded domains [1, 2, 4, 5, 8, 9]. Note that (1.1)–(1.2) is not an elliptic boundary-value problem with a parameter [5, p. 98] (one should add a term such as $\lambda^2 u(x,y)$ to get ellipticity with a parameter). Moreover, Ω is an unbounded domain. This fact makes the problem non-standard and known results for boundary-value problems do not apply.

We denote by $F_{x\to\sigma}$ the one-dimensional Fourier transform with respect to x, where σ is the dual variable. Applying the operator $F_{x\to\sigma}$ to problem (1.1)–(1.2) we obtain a boundary value problem for an ordinary differential equation of the

2000 *Mathematics Subject Classification.* 35J40.
Key words and phrases. Biharmonic equation, isomorphism, boundary-value problem.

Submitted April 25, 2002. Published June 18, 2002.

fourth order with 2 parameters

$$L(\lambda, i\sigma, D_y)\widehat{u} := \Big[\frac{d^4}{dy^4} - (2\sigma^2 + \lambda)\frac{d^2}{dy^2} + \sigma^4 + \lambda\sigma^2\Big]\widehat{u}(\sigma, y) = \widehat{f}(\sigma, y), \quad y \in [0, 1], \tag{1.3}$$

$$\widehat{u}(\sigma, 0) = \widehat{u}(\sigma, 1) = \frac{d\widehat{u}(\sigma, 0)}{dy} = \frac{d\widehat{u}(\sigma, 1)}{dy} = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ are parameters, $\widehat{u}(\sigma, y) := (F_{x\to\sigma}u(x, y))(\sigma, y)$.

To solve our main question, we start from the solvability of problem (1.3)–(1.4) and get estimates of its solution depending on the parameters λ and σ.

2. Isomorphism and coerciveness of the equation on the whole axis

Consider equation (1.3) on the whole axis, i.e.,

$$L(\lambda, i\sigma, D_y)u := \Big[\frac{d^4}{dy^4} - (2\sigma^2 + \lambda)\frac{d^2}{dy^2} + \sigma^4 + \lambda\sigma^2\Big]u(y) = f(y), \quad y \in \mathbb{R}. \tag{2.1}$$

Theorem 2.1. *For all complex numbers λ satisfying $|\arg\lambda| \le \pi - \varepsilon$, where $\varepsilon > 0$ is arbitrary, and $\sigma \in \mathbb{R}$, the operator $\mathbb{L}(\lambda, \sigma) : u \to \mathbb{L}(\lambda, \sigma)u := L(\lambda, i\sigma, D_y)u$ from $W_q^4(\mathbb{R})$ onto $L_q(\mathbb{R})$, where $q \in (1, \infty)$, is an isomorphism and for these λ the following estimates hold for solutions of (2.1)*

$$\|u\|_{W_q^4(\mathbb{R})} + \sigma^2\|u\|_{W_q^2(\mathbb{R})} + \sigma^4\|u\|_{L_q(\mathbb{R})} \le C(\varepsilon)\|f\|_{L_q(\mathbb{R})}, \quad |\arg\lambda| \le \pi - \varepsilon, \quad \sigma \in \mathbb{R}, \tag{2.2}$$

$$\|u\|_{W_q^2(\mathbb{R})} + \sigma^2\|u\|_{L_q(\mathbb{R})} \le \frac{C(\varepsilon)}{|\lambda|}\|f\|_{L_q(\mathbb{R})}, \quad |\arg\lambda| \le \pi - \varepsilon, \quad \sigma \in \mathbb{R}. \tag{2.3}$$

Proof. The operator $\mathbb{L}(\lambda, \sigma)$ acts from $W_q^4(\mathbb{R})$ into $L_q(\mathbb{R})$ linearly and continuously. Let us prove that if $f \in L_q(\mathbb{R})$ then (2.1) has a solution u in $W_q^4(\mathbb{R})$ and for this solution estimates (2.2)-(2.3) hold. With the substitution

$$u''(y) - \sigma^2 u(y) = v(y), \quad y \in \mathbb{R}, \tag{2.4}$$

equation (2.1) is reduced to

$$v''(y) - (\sigma^2 + \lambda)v(y) = f(y), \quad y \in \mathbb{R}. \tag{2.5}$$

By a theorem in [9, p. 109], equation (2.5), for $|\arg\lambda| \le \pi - \varepsilon$, $\sigma \in \mathbb{R}$, has a solution $v \in W_q^2(\mathbb{R})$ and

$$\|v\|_{W_q^2(\mathbb{R})} + |\lambda + \sigma^2|\|v\|_{L_q(\mathbb{R})} \le C(\varepsilon)\|f\|_{L_q(\mathbb{R})}, \quad |\arg\lambda| \le \pi - \varepsilon, \ \sigma \in \mathbb{R}. \tag{2.6}$$

Apply now the same theorem [9, p. 109] to (2.4). Then for $\sigma \in \mathbb{R}$, and for $v \in W_q^2(\mathbb{R})$, (2.4) has a solution $u \in W_q^4(\mathbb{R})$ and

$$\|u\|_{W_q^4(\mathbb{R})} + \sigma^2\|u\|_{W_q^2(\mathbb{R})} + \sigma^4\|u\|_{L_q(\mathbb{R})} \le C(\|v\|_{W_q^2(\mathbb{R})} + \sigma^2\|v\|_{L_q(\mathbb{R})}), \quad \sigma \in \mathbb{R}, \tag{2.7}$$

$$\|u\|_{W_q^2(\mathbb{R})} + \sigma^2\|u\|_{L_q(\mathbb{R})} \le C\|v\|_{L_q(\mathbb{R})}, \quad \sigma \in \mathbb{R}. \tag{2.8}$$

Consequently, from (2.6) and (2.7) it follows that for $|\arg\lambda| \le \pi - \varepsilon$ and $\sigma \in \mathbb{R}$, (2.1) has a solution $u \in W_q^4(\mathbb{R})$ and

$$\begin{aligned}\|u\|_{W_q^4(\mathbb{R})} + \sigma^2\|u\|_{W_q^2(\mathbb{R})} + \sigma^4\|u\|_{L_q(\mathbb{R})} &\le C(\varepsilon)(\|f\|_{L_q(\mathbb{R})} + \frac{\sigma^2}{|\lambda+\sigma^2|}\|f\|_{L_q(\mathbb{R})}) \\ &\le C(\varepsilon)\|f\|_{L_q(\mathbb{R})}, \quad |\arg\lambda| \le \pi - \varepsilon, \quad \sigma \in \mathbb{R},\end{aligned}$$

i.e., estimate (2.2) has been proved. In the last inequality we have used that

$$|\lambda + \sigma^2| \ge C(\varepsilon)(|\lambda| + \sigma^2), \quad |\arg\lambda| \le \pi - \varepsilon,\ \sigma \in \mathbb{R},$$

which can be easily checked. In what follows we will often use the fact.

On the other hand, from (2.6) and (2.8), it follows that

$$\|u\|_{W_q^2(\mathbb{R})} + \sigma^2\|u\|_{L_q(\mathbb{R})} \le \frac{C(\varepsilon)}{|\lambda+\sigma^2|}\|f\|_{L_q(\mathbb{R})} \le \frac{C(\varepsilon)}{|\lambda|}\|f\|_{L_q(\mathbb{R})},$$

with $|\arg\lambda| \le \pi - \varepsilon$ and $\sigma \in \mathbb{R}$; i.e., estimate (2.3) holds. $\square$

3. Solvability of the boundary-value problem for the homogeneous equation

Consider a boundary-value problem for the ordinary differential equation of the fourth order on $[0,1]$

$$L(\lambda, i\sigma, D_y)u := \Big[\frac{d^4}{dy^4} - (2\sigma^2 + \lambda)\frac{d^2}{dy^2} + \sigma^4 + \lambda\sigma^2\Big]u(y) = 0, \quad y \in [0,1], \tag{3.1}$$

$$u(0) = f_1, \quad u(1) = f_2, \quad \frac{du(0)}{dy} = f_3, \quad \frac{du(1)}{dy} = f_4, \tag{3.2}$$

where $\lambda \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ are parameters, f_ν are complex numbers. Here (1.3) is homogeneous but the boundary conditions (1.4) are not.

Theorem 3.1. *For each $\varepsilon > 0$ there exists $M > 0$ such that for all complex numbers λ satisfying $|\arg\lambda| \le \pi-\varepsilon$, $|\lambda| \ge M$ and for all real numbers $\sigma \in \mathbb{R}$, $\sigma \ne 0$ problem (3.1)–(3.2) has a unique solution $u(y)$ that belongs to $C^\infty[0,1]$ and for this solution the following inequality holds for $n = 0,1,2,\dots$*

$$\begin{aligned}&\|u^{(n)}\|_{L_q(0,1)} \\ &\le C(\varepsilon,K)\begin{cases} (|\lambda|^{\frac{n-1}{2}-\frac{1}{2q}} + |\sigma|^{n-1})(|f_1|+|f_2|) \\ +(|\lambda|^{\frac{n-1}{2}-\frac{1}{2q}} + \frac{|\sigma|^{n-1}}{|\lambda|^{1/2}})(|f_3|+|f_4|), & 0<|\sigma|\le K, \\ (\frac{(|\lambda|+\sigma^2)^{\frac{n+1}{2}-\frac{1}{2q}}|\sigma|}{|\lambda|} + \frac{(|\lambda|+\sigma^2)|\sigma|^{n-\frac{1}{q}}}{|\lambda|})(|f_1|+|f_2|) \\ +(\frac{(|\lambda|+\sigma^2)^{\frac{n+1}{2}-\frac{1}{2q}}}{|\lambda|} + \frac{(|\lambda|+\sigma^2)^{1/2}|\sigma|^{n-\frac{1}{q}}}{|\lambda|})(|f_3|+|f_4|), & |\sigma|>K.\end{cases}\end{aligned} \tag{3.3}$$

Proof. Let us prove that for any complex numbers f_ν, $\nu = 1,\dots,4$, problem (3.1)–(3.2) has a unique solution $u(y)$ in $C^\infty[0,1]$ and let us estimate this solution.

A characteristic equation of (3.1) has the form

$$\omega^4 - (2\sigma^2+\lambda)\omega^2 + \sigma^4 + \lambda\sigma^2 = 0 \tag{3.4}$$

which has the following roots

$$\omega_1 = -|\sigma^2+\lambda|^{\frac12}e^{i\frac{\arg(\sigma^2+\lambda)}{2}}, \quad \omega_2 = -|\sigma|, \quad \omega_3 = |\sigma^2+\lambda|^{\frac12}e^{i\frac{\arg(\sigma^2+\lambda)}{2}}, \quad \omega_4 = |\sigma|.$$

Then for $|\arg\lambda| \le \pi-\varepsilon$, $\sigma\in\mathbb{R}$, $\sigma\neq 0$, the general solution of (3.1) has the form

$$u(y) = C_1 e^{\omega_1 y} + C_2 e^{\omega_2 y} + C_3 e^{\omega_3 (y-1)} + C_4 e^{\omega_4 (y-1)}. \tag{3.5}$$

Substituting (3.5) into (3.2), we obtain a system for finding C_i, $i=1,\dots,4$,

$$\begin{gathered} C_1 + C_2 + C_3 e^{-\omega_3} + C_4 e^{-\omega_4} = f_1,\\ C_1 e^{\omega_1} + C_2 e^{\omega_2} + C_3 + C_4 = f_2,\\ C_1\omega_1 + C_2\omega_2 + C_3\omega_3 e^{-\omega_3} + C_4\omega_4 e^{-\omega_4} = f_3,\\ C_1\omega_1 e^{\omega_1} + C_2\omega_2 e^{\omega_2} + C_3\omega_3 + C_4\omega_4 = f_4. \end{gathered} \tag{3.6}$$

Since for $|\arg\lambda| \le \pi-\varepsilon$, $\sigma\in\mathbb{R}$ we have $\frac{\pi}{2}+\frac{\varepsilon}{2} < \arg\omega_1 < \frac{3\pi}{2}-\frac{\varepsilon}{2}$ and $|\arg\omega_3| < \frac{\pi}{2}-\frac{\varepsilon}{2}$, then $\operatorname{Re}\omega_1 < -\delta(\varepsilon)(|\lambda|^{\frac12}+|\sigma|)$ and $-\operatorname{Re}\omega_3 < -\delta(\varepsilon)(|\lambda|^{\frac12}+|\sigma|)$, where $\delta(\varepsilon)>0$. The determinant of system (3.6) is

$$D(\lambda,\sigma) = \begin{vmatrix} 1 & 1 & e^{-\omega_3} & e^{-\omega_4}\\ e^{\omega_1} & e^{\omega_2} & 1 & 1\\ \omega_1 & \omega_2 & \omega_3 e^{-\omega_3} & \omega_4 e^{-\omega_4}\\ \omega_1 e^{\omega_1} & \omega_2 e^{\omega_2} & \omega_3 & \omega_4 \end{vmatrix}.$$

Calculating this determinant and taking into account that $\omega_3=-\omega_1$ and $\omega_4=-\omega_2$, we obtain

$$D(\lambda,\sigma) = \omega_2\Big[(\omega_1^2+\omega_2^2)(1-e^{2\omega_1})\frac{1-e^{2\omega_2}}{\omega_2} - 2\omega_1(1+e^{2(\omega_1+\omega_2)}+e^{2\omega_2}+e^{2\omega_1}-4e^{\omega_1+\omega_2})\Big].$$

Let $0<|\sigma|\le K$. Because of $\lim_{\sigma\to 0}\frac{1-e^{2\omega_2}}{\omega_2} = -2\neq 0$, one can choose M such a big that for all $|\arg\lambda|\le\pi-\varepsilon$, $|\lambda|\ge M$ the following true ($\omega_1^2=\omega_2^2+\lambda=\sigma^2+\lambda$)

$$|(\omega_1^2+\omega_2^2)(1-e^{2\omega_1})\frac{1-e^{2\omega_2}}{\omega_2}| \ge C(K,\varepsilon)(|\lambda|+\sigma^2)$$

and

$$|2\omega_1(1+e^{2(\omega_1+\omega_2)}+e^{2\omega_2}+e^{2\omega_1}-4e^{\omega_1+\omega_2})| \le \frac{C(K,\varepsilon)}{2}(|\lambda|+\sigma^2).$$

Then $|D(\lambda,\sigma)| \ge \frac{C(K,\varepsilon)}{2}|\sigma|(|\lambda|+\sigma^2)$.

In the case of $|\sigma|>K$, we write the determinant as

$$\begin{aligned} D(\lambda,\sigma) =&[\omega_2(1-e^{\omega_1})-\omega_1(1-e^{\omega_2})][\omega_2(1+e^{\omega_1})-\omega_1(1+e^{\omega_2})]\\ &+(\omega_2-\omega_1)^2 e^{2(\omega_1+\omega_2)} + R(\lambda,\sigma), \end{aligned}$$

where $|R(\lambda,\sigma)| \le C(|\lambda|^{1/2}+|\sigma|)/e^{C(\varepsilon)|\sigma|}$. Using that $\omega_1^2=\omega_2^2+\lambda$ we have

$$\begin{aligned} D(\lambda,\sigma) =& \frac{-\lambda(1-e^{\omega_2})^2 + 2\omega_2^2 e^{\omega_2} - \omega_2^2 e^{2\omega_2} - 2\omega_2^2 e^{\omega_1} + \omega_2^2 e^{2\omega_1}}{\omega_2(1-e^{\omega_1})+\omega_1(1-e^{\omega_2})}\\ &\times \frac{-\lambda(1+e^{\omega_2})^2 - 2\omega_2^2 e^{\omega_2} - \omega_2^2 e^{2\omega_2} + 2\omega_2^2 e^{\omega_1} + \omega_2^2 e^{2\omega_1}}{\omega_2(1+e^{\omega_1})+\omega_1(1+e^{\omega_2})}\\ &+ \frac{\lambda^2}{(\omega_2+\omega_1)^2} e^{2(\omega_1+\omega_2)} + R(\lambda,\sigma). \end{aligned}$$

We have $|\omega_2+\omega_1|^2 = (\operatorname{Re}(\omega_2+\omega_1))^2 + (\operatorname{Im}(\omega_2+\omega_1))^2 = (-|\sigma|+\operatorname{Re}\omega_1)^2 + (\operatorname{Im}\omega_1)^2 = \sigma^2+|\omega_1|^2-2|\sigma|\operatorname{Re}\omega_1 \ge \sigma^2+|\lambda+\sigma^2| \ge C(\varepsilon)(|\lambda|+\sigma^2)$, because $\operatorname{Re}\omega_1<0$. Therefore, for all $|\arg\lambda| \le \pi-\varepsilon$, $|\lambda| \ge M$ and $|\sigma| > K$,

$$\begin{aligned}|D(\lambda,\sigma)| &\ge C(K)\frac{|\lambda|^2}{|\lambda|+\sigma^2} - C(\varepsilon)\frac{|\lambda|^2}{(|\lambda|+\sigma^2)\mathrm{e}^{C(\varepsilon)(|\lambda|^{1/2}+|\sigma|)}} - C\frac{|\lambda|^{1/2}+|\sigma|}{\mathrm{e}^{C(\varepsilon)|\sigma|}}\\ &\ge \frac{C(K)}{2}\frac{|\lambda|^2}{|\lambda|+\sigma^2}.\end{aligned}$$

Hence, for $|\arg\lambda| \le \pi-\varepsilon$, $|\lambda| \ge M$ and $\sigma \in \mathbb{R}$, $\sigma \ne 0$, system (3.6) has a unique solution

$$C_1 = \frac{\begin{vmatrix} f_1 & 1 & \mathrm{e}^{-\omega_3} & \mathrm{e}^{-\omega_4}\\ f_2 & \mathrm{e}^{\omega_2} & 1 & 1\\ f_3 & \omega_2 & \omega_3\mathrm{e}^{-\omega_3} & \omega_4\mathrm{e}^{-\omega_4}\\ f_4 & \omega_2\mathrm{e}^{\omega_2} & \omega_3 & \omega_4\end{vmatrix}}{D(\lambda,\sigma)},\quad C_2 = \frac{\begin{vmatrix} 1 & f_1 & \mathrm{e}^{-\omega_3} & \mathrm{e}^{-\omega_4}\\ \mathrm{e}^{\omega_1} & f_2 & 1 & 1\\ \omega_1 & f_3 & \omega_3\mathrm{e}^{-\omega_3} & \omega_4\mathrm{e}^{-\omega_4}\\ \omega_1\mathrm{e}^{\omega_1} & f_4 & \omega_3 & \omega_4\end{vmatrix}}{D(\lambda,\sigma)},$$

$$C_3 = \frac{\begin{vmatrix} 1 & 1 & f_1 & \mathrm{e}^{-\omega_4}\\ \mathrm{e}^{\omega_1} & \mathrm{e}^{\omega_2} & f_2 & 1\\ \omega_1 & \omega_2 & f_3 & \omega_4\mathrm{e}^{-\omega_4}\\ \omega_1\mathrm{e}^{\omega_1} & \omega_2\mathrm{e}^{\omega_2} & f_4 & \omega_4\end{vmatrix}}{D(\lambda,\sigma)},\quad C_4 = \frac{\begin{vmatrix} 1 & 1 & \mathrm{e}^{-\omega_3} & f_1\\ \mathrm{e}^{\omega_1} & \mathrm{e}^{\omega_2} & 1 & f_2\\ \omega_1 & \omega_2 & \omega_3\mathrm{e}^{-\omega_3} & f_3\\ \omega_1\mathrm{e}^{\omega_1} & \omega_2\mathrm{e}^{\omega_2} & \omega_3 & f_4\end{vmatrix}}{D(\lambda,\sigma)},$$

where

$$|D(\lambda,\sigma)| \ge C(\varepsilon,K)\begin{cases}|\sigma|(|\lambda|+\sigma^2), & 0<|\sigma|\le K,\\ \frac{|\lambda|^2}{|\lambda|+\sigma^2}, & |\sigma|>K.\end{cases}$$

Calculating these determinants one can obtain that

$$|C_{1,3}| \le C(\varepsilon,K)\begin{cases}\frac{|\lambda|}{(|\lambda|+\sigma^2)^{\frac{3}{2}}}[|f_1|+|f_2|+(|f_3|+|f_4|)\frac{|\lambda|+\sigma^2}{|\lambda|}], & 0<|\sigma|\le K,\\ \frac{(|\lambda|+\sigma^2)^{1/2}}{|\lambda|}[(|f_1|+|f_2|)|\sigma|+|f_3|+|f_4|], & |\sigma|>K,\end{cases}$$

and

$$|C_{2,4}| \le C(\varepsilon,K)\begin{cases}\frac{|\lambda|}{|\sigma|(|\lambda|+\sigma^2)^{\frac{3}{2}}}[(|f_1|+|f_2|)(|\lambda|+\sigma^2)^{1/2}+|f_3|+|f_4|], & 0<|\sigma|\le K,\\ \frac{(|\lambda|+\sigma^2)^{1/2}}{|\lambda|}[(|f_1|+|f_2|)(|\lambda|+\sigma^2)^{1/2}+|f_3|+|f_4|], & |\sigma|>K,\end{cases}$$

or

$$|C_{1,3}| \le C(\varepsilon,K)\begin{cases}\frac{1}{|\lambda|^{1/2}}[|f_1|+|f_2|+|f_3|+|f_4|], & 0<|\sigma|\le K,\\ \frac{(|\lambda|+\sigma^2)^{1/2}}{|\lambda|}[(|f_1|+|f_2|)|\sigma|+|f_3|+|f_4|], & |\sigma|>K,\end{cases}$$

and

$$|C_{2,4}| \le C(\varepsilon,K)\begin{cases}\frac{1}{|\sigma|}[|f_1|+|f_2|+\frac{1}{|\lambda|^{1/2}}(|f_3|+|f_4|)], & 0<|\sigma|\le K,\\ \frac{(|\lambda|+\sigma^2)^{1/2}}{|\lambda|}[(|f_1|+|f_2|)(|\lambda|+\sigma^2)^{1/2}+|f_3|+|f_4|], & |\sigma|>K.\end{cases}$$

From (3.5) for $n = 0, 1, 2, \dots$, we have

$$
\begin{aligned}
&\|u^{(n)}\|_{L_q(0,1)} \\
&\quad \le |C_1||\omega_1|^n \|\mathrm{e}^{\omega_1 \cdot}\|_{L_q(0,1)} + |C_2||\omega_2|^n \|\mathrm{e}^{\omega_2 \cdot}\|_{L_q(0,1)} \\
&\qquad + |C_3||\omega_3|^n \|\mathrm{e}^{\omega_3(\cdot - 1)}\|_{L_q(0,1)} + |C_4||\omega_4|^n \|\mathrm{e}^{\omega_4(\cdot - 1)}\|_{L_q(0,1)} \\
&\quad \le C(\varepsilon)[(|C_1| + |C_3|)(|\lambda| + \sigma^2)^{\frac{n}{2} - \frac{1}{2q}} + (|C_2| + |C_4|)|\sigma|^n \big(\frac{1 - \mathrm{e}^{-|\sigma| q}}{|\sigma|}\big)^{1/q}] \\
&\quad \le C(\varepsilon, K) \begin{cases} (|\lambda|^{\frac{n-1}{2} - \frac{1}{2q}} + |\sigma|^{n-1})(|f_1| + |f_2|) \\ +(|\lambda|^{\frac{n-1}{2} - \frac{1}{2q}} + \frac{|\sigma|^{n-1}}{|\lambda|^{1/2}})(|f_3| + |f_4|), & 0 < |\sigma| \le K, \\ (\frac{(|\lambda| + \sigma^2)^{\frac{n+1}{2} - \frac{1}{2q}} |\sigma|}{|\lambda|} + \frac{(|\lambda| + \sigma^2)|\sigma|^{n - \frac{1}{q}}}{|\lambda|})(|f_1| + |f_2|) \\ +(\frac{(|\lambda| + \sigma^2)^{\frac{n+1}{2} - \frac{1}{2q}}}{|\lambda|} + \frac{(|\lambda| + \sigma^2)^{1/2}|\sigma|^{n - \frac{1}{q}}}{|\lambda|})(|f_3| + |f_4|), & |\sigma| > K. \end{cases}
\end{aligned}
$$

□

From (3.3), in particular, follows

$$
\|u(\cdot)\|_{L_q(0,1)} \le C(\varepsilon, K) \begin{cases} \frac{1}{|\sigma|}(|f_1| + |f_2|) + \frac{1}{|\sigma||\lambda|^{\frac{1}{2}}}(|f_3| + |f_4|), & 0 < |\sigma| \le K, \\ \frac{|\lambda| + \sigma^2}{|\lambda||\sigma|^{1/q}}(|f_1| + |f_2|) \\ + \frac{(|\lambda| + \sigma^2)^{1/2}}{|\lambda||\sigma|^{1/q}}(|f_3| + |f_4|), & |\sigma| > K, \end{cases} \tag{3.7}
$$

and

$$
\|u'(\cdot)\|_{L_q(0,1)} \le C(\varepsilon, K) \begin{cases} |f_1| + |f_2| + \frac{1}{|\lambda|^{\frac{1}{2q}}}(|f_3| + |f_4|), & 0 < |\sigma| \le K, \\ \frac{(|\lambda| + \sigma^2)|\sigma|^{1 - \frac{1}{q}}}{|\lambda|}(|f_1| + |f_2|) \\ + \frac{(|\lambda| + \sigma^2)^{1 - \frac{1}{2q}}}{|\lambda|}(|f_3| + |f_4|), & |\sigma| > K. \end{cases} \tag{3.8}
$$

4. Isomorphism of the boundary value problem with a linear parameter and estimates for its solution

Now we consider the main problem (1.1)–(1.2).

Theorem 4.1. *For each $\varepsilon > 0$ there exists $M > 0$ such that for all complex numbers λ satisfying $|\arg \lambda| \le \pi - \varepsilon$, $|\lambda| \ge M$, the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := L(\lambda, D_x, D_y)u$ from $W_2^{s+4}(\mathbb{R} \times (0,1), u(x,0) = u(x,1) = u_y'(x,0) = u_y'(x,1))$ onto $W_2^s(\mathbb{R} \times (0,1))$, where $s \ge 0$, is an isomorphism and for these λ the following estimates hold*

$$
\|u\|_{W_2^k(\mathbb{R} \times (0,1))} \le C(\varepsilon) \frac{1}{|\lambda|} \|f\|_{W_2^k(\mathbb{R} \times (0,1))}, \quad k = 0, 1,
$$

where $u(x,y)$ is a solution of (1.1)–(1.2).

Proof. Fix λ such that $|\arg \lambda| \le \pi - \varepsilon$, $|\lambda| \ge M$. In this case problem (1.1)–(1.2) becomes an elliptic boundary value problem with constant coefficients (for the definition see, e.g., [6]). Then, the required isomorphism follows from [6, Theorem 1.1, p.44]. Indeed, to check the condition there, one should prove that there is no eigenvalue μ on the imaginary axis of the spectral problem

$$u^{(4)}(y) + (2\mu^2 - \lambda)u''(y) + (\mu^4 - \lambda\mu^2)u(y) = 0, \quad y \in (0,1),$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

Let us prove this by contradiction. If there is such μ then $\mu^2 \le 0$ and there is an eigenfunction u, $u \not\equiv 0$, i.e., $\int_0^1 |u(y)|^2 dy > 0$. Moreover, $\int_0^1 |u'(y)|^2 dy > 0$, otherwise $u(y)$ would be constant and, taking into account $u(0) = 0$, $u(y) \equiv 0$.

Multiply the first equation of the above spectral problem by $\overline{u(y)}$ and integrate by parts on $(0,1)$. Then, using boundary conditions, we get

$$\int_0^1 |u''(y)|^2 dy + (\lambda - 2\mu^2) \int_0^1 |u'(y)|^2 dy + \mu^2(\mu^2 - \lambda) \int_0^1 |u(y)|^2 dy = 0,$$

i.e.,

$$\int_0^1 |u''(y)|^2 dy + (\operatorname{Re}\lambda - 2\mu^2) \int_0^1 |u'(y)|^2 dy + \mu^2(\mu^2 - \operatorname{Re}\lambda) \int_0^1 |u(y)|^2 dy = 0,$$
$$\operatorname{Im}\lambda \int_0^1 |u'(y)|^2 dy - \mu^2 \operatorname{Im}\lambda \int_0^1 |u(y)|^2 dy = 0.$$

But from the second equation follows that $\operatorname{Im}\lambda = 0$. Then $\operatorname{Re}\lambda \ge M > 0$ and this contradicts to the first equation.

Now prove estimates of the theorem. First, consider a solution of problem (1.3)–(1.4). We find the solution of problem (1.3)–(1.4) in the form $\widehat{u} = u_1 + u_2$, where u_1 is a restriction on [0,1] of a solution $\tilde{u}_1$ of the equation

$$\Big[\frac{d^4}{dy^4} - (2\sigma^2 + \lambda)\frac{d^2}{dy^2} + \sigma^4 + \lambda\sigma^2\Big]\tilde{u}_1(\sigma, y) = \tilde{f}(\sigma, y), \quad y \in \mathbb{R},$$

where $\tilde{f} \in L_q(\mathbb{R})$ is an extension of $\widehat{f} \in L_q(0,1)$ such that the extension operator $\widehat{f} \to \tilde{f} : L_q(0,1) \to L_q(\mathbb{R})$ is bounded [7, p.314]. Then for u_1, from Theorem 2.1, estimates (2.2)-(2.3) hold, i.e.,

$$\begin{gathered}\|u_1(\sigma,\cdot)\|_{W_q^4(0,1)} + \sigma^2\|u_1(\sigma,\cdot)\|_{W_q^2(0,1)} + \sigma^4\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \le C(\varepsilon)\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)},\\ \|u_1(\sigma,\cdot)\|_{W_q^2(0,1)} + \sigma^2\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \le \frac{C(\varepsilon)}{|\lambda|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)},\end{gathered} \tag{4.1}$$

where $|\arg\lambda| \le \pi - \varepsilon$, $\sigma \in \mathbb{R}$. The second summand u_2 in the form of $\widehat{u}$ is, by virtue of Theorem 3.1, a unique solution of the problem

$$\begin{gathered}\Big[\frac{d^4}{dy^4} - (2\sigma^2 + \lambda)\frac{d^2}{dy^2} + \sigma^4 + \lambda\sigma^2\Big]u_2(\sigma, y) = 0, \quad y \in [0,1],\\ u_2(\sigma,0) = -u_1(\sigma,0), \quad u_2(\sigma,1) = -u_1(\sigma,1),\\ \frac{du_2(\sigma,0)}{dy} = -\frac{du_1(\sigma,0)}{dy}, \quad \frac{du_2(\sigma,1)}{dy} = -\frac{du_1(\sigma,1)}{dy},\end{gathered} \tag{4.2}$$

where $|\arg\lambda|\le\pi-\varepsilon$, $|\lambda|\ge M$, and $\sigma\in\mathbb{R}$, $\sigma\neq 0$. Then from (3.7) for the solution u_2 of problem (4.2) we have

$$\|u_2(\sigma,\cdot)\|_{L_q(0,1)}\le C(\varepsilon,K)\begin{cases}\frac{1}{|\sigma|}(|u_1(\sigma,0)|+|u_1(\sigma,1)|)\\ +\frac{1}{|\sigma||\lambda|^{1/2}}(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|), & 0<|\sigma|\le K,\\ \frac{|\lambda|+\sigma^2}{|\lambda||\sigma|^{1/q}}(|u_1(\sigma,0)|+|u_1(\sigma,1)|)\\ +\frac{(|\lambda|+\sigma^2)^{1/2}}{|\lambda||\sigma|^{1/q}}(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|), & |\sigma|>K,\end{cases} \tag{4.3}$$

and from (3.8),

$$\|\frac{du_2(\sigma,\cdot)}{dy}\|_{L_q(0,1)}\le C(\varepsilon,K)\begin{cases}|u_1(\sigma,0)|+|u_1(\sigma,1)|\\ +\frac{1}{|\lambda|^{\frac{1}{2q}}}(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|), & 0<|\sigma|\le K,\\ \frac{(|\lambda|+\sigma^2)|\sigma|^{1-\frac1q}}{|\lambda|}(|u_1(\sigma,0)|+|u_1(\sigma,1)|)\\ +\frac{(|\lambda|+\sigma^2)^{1-\frac{1}{2q}}}{|\lambda|}(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|), & |\sigma|>K.\end{cases} \tag{4.4}$$

From [3, Ch.3, §10, Theorem 10.4] it follows that

$$\|u^{(j)}\|_{C[0,1]}\le C(h^{1-\gamma}\|u^{(\ell)}\|_{L_q(0,1)}+h^{-\gamma}\|u\|_{L_q(0,1)}),$$

where $j<\ell$, $0<h<h_0$, $\gamma=(j+\frac1q)/\ell$. Choose $j=0,1$, $\ell=2$, $h=\mu^{-2}$, and $\mu\gg 1$. Then

$$\begin{aligned}|u(y)|&\le C(\mu^{-2+\frac1q}\|u''\|_{L_q(0,1)}+\mu^{1/q}\|u\|_{L_q(0,1)}),\quad y\in[0,1],\\ |u'(y)|&\le C(\mu^{-1+\frac1q}\|u''\|_{L_q(0,1)}+\mu^{1+\frac1q}\|u\|_{L_q(0,1)}),\quad y\in[0,1].\end{aligned} \tag{4.5}$$

From (4.3) and (4.5) we have for $|\sigma|>K$,

$$\begin{aligned}\|u_2(\sigma,\cdot)\|_{L_q(0,1)}\le&C(\varepsilon,K)\frac{(|\lambda|+\sigma^2)^{1/2}|\sigma|^{1-\frac1q}}{|\lambda||\sigma|}[(|u_1(\sigma,0)|+|u_1(\sigma,1)|)(|\lambda|+\sigma^2)^{1/2}\\ &+|u_1'(\sigma,0)|+|u_1'(\sigma,1)|]\\ \le&C(\varepsilon,K)\frac{|\sigma|^{1-\frac1q}}{|\lambda|^{1/2}}[(|u_1(\sigma,0)|+|u_1(\sigma,1)|)|\lambda|^{1/2}\\ &+(|u_1(\sigma,0)|+|u_1(\sigma,1)|)|\sigma|+|u_1'(\sigma,0)|+|u_1'(\sigma,1)|]\\ \le&C(\varepsilon,K)[(|u_1(\sigma,0)|+|u_1(\sigma,1)|)|\sigma|^{1-\frac1q}\\ &+(|u_1(\sigma,0)|+|u_1(\sigma,1)|)\frac{|\sigma|^2}{|\lambda|^{1/2}}+(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|)\frac{|\sigma|^{1-\frac1q}}{|\lambda|^{1/2}}]\\ \le&C(\varepsilon,K)[\mu_1^{-2+\frac1q}|\sigma|^{1-\frac1q}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)}\\ &+\mu_1^{1/q}|\sigma|^{1-\frac1q}\|u_1(\sigma,\cdot)\|_{L_q(0,1)}+\mu_2^{-2+\frac1q}\frac{|\sigma|^2}{|\lambda|^{1/2}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)}\\ &+\mu_2^{1/q}\frac{|\sigma|^2}{|\lambda|^{1/2}}\|u_1(\sigma,\cdot)\|_{L_q(0,1)}+\mu_3^{-1+\frac1q}\frac{|\sigma|^{1-\frac1q}}{|\lambda|^{1/2}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)}\\ &+\mu_3^{1+\frac1q}\frac{|\sigma|^{1-\frac1q}}{|\lambda|^{1/2}}\|u_1(\sigma,\cdot)\|_{L_q(0,1)}].\end{aligned}$$

Choose $\mu_1 = |\sigma|^{1+q}$, $\mu_2 = |\lambda|^{\frac{q}{2}}$, $\mu_3^{1+\frac{1}{q}} = |\lambda|^{1/2}|\sigma|^{1+\frac{1}{q}}$, where $|\sigma| > K \gg 1$ and $|\lambda| \geq M \gg 1$. Then, by virtue of (4.1), we obtain

$$\begin{aligned}\|u_2(\sigma,\cdot)\|_{L_q(0,1)} \leq & C(\varepsilon,K)[\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \sigma^2\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \\ & + |\lambda|^{-q}\sigma^2\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \sigma^2\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \\ & + |\lambda|^{-\frac{q}{q+1}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \sigma^2\|u_1(\sigma,\cdot)\|_{L_q(0,1)}] \\ \leq & C(\varepsilon,K)\frac{1}{|\lambda|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \quad |\sigma| > K.\end{aligned} \tag{4.6}$$

On the other hand, for $0 < |\sigma| \leq K$, from (4.3) and (4.5) we have ($|\lambda| \geq M$)

$$\begin{aligned}\|u_2(\sigma,\cdot)\|_{L_q(0,1)} \leq & C(\varepsilon,K)[\frac{1}{|\sigma|}(|u_1(\sigma,0)| + |u_1(\sigma,1)|) \\ & + \frac{1}{|\sigma||\lambda|^{1/2}}(|u_1'(\sigma,0)| + |u_1'(\sigma,1)|)] \\ \leq & C(\varepsilon,K)\frac{1}{|\sigma|}[\mu_1^{-2+\frac{1}{q}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \mu_1^{1/q}\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \\ & + \mu_2^{-1+\frac{1}{q}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \mu_2^{1+\frac{1}{q}}\|u_1(\sigma,\cdot)\|_{L_q(0,1)}].\end{aligned}$$

Choose $\mu_1 = \mu_2 = R \gg 1$. It implies, by virtue of (4.1), that

$$\|u_2(\sigma,\cdot)\|_{L_q(0,1)} \leq C(\varepsilon,K)\frac{1}{|\sigma||\lambda|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \quad 0 < |\sigma| \leq K.$$

The last inequality with (4.6) taking into account gives us the following estimate for a solution of (4.2) for $\sigma \in \mathbb{R}$, $\sigma \neq 0$

$$\|u_2(\sigma,\cdot)\|_{L_q(0,1)} \leq C(\varepsilon)\frac{1}{|\lambda|}\big(\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)} + \frac{1}{|\sigma|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}\big). \tag{4.7}$$

Then from (4.1) and (4.7) for a solution $\widehat{u}(\sigma,y) = u_1(\sigma,y) + u_2(\sigma,y)$ of problem (1.3)–(1.4) we have

$$\|\widehat{u}(\sigma,\cdot)\|_{L_q(0,1)} \leq C(\varepsilon)\frac{1}{|\lambda|}\big(\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)} + \frac{1}{|\sigma|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}\big),$$

where $\sigma \in \mathbb{R}$, $\sigma \neq 0$, $|\arg\lambda| \leq \pi - \varepsilon$, $|\lambda| \geq M$. Multiplying the last inequality on $|\sigma|$ we obtain

$$\|\widehat{u_x'}(\sigma,\cdot)\|_{L_q(0,1)} \leq C(\varepsilon)\frac{1}{|\lambda|}(\|\widehat{f_x'}(\sigma,\cdot)\|_{L_q(0,1)} + \|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}), \tag{4.8}$$

where $\sigma \in \mathbb{R}$, $\sigma \neq 0$, $|\arg\lambda| \leq \pi - \varepsilon$, $|\lambda| \geq M$.

From (4.4) and (4.5) we have for $0 < |\sigma| \leq K$ and $|\lambda| \geq M$

$$\begin{aligned}\|\frac{du_2(\sigma,\cdot)}{dy}\|_{L_q(0,1)} \leq & C(\varepsilon,K)[|u_1(\sigma,0)| + |u_1(\sigma,1)| + |u_1'(\sigma,0)| + |u_1'(\sigma,1)|] \\ \leq & C(\varepsilon,K)[\mu_1^{-2+\frac{1}{q}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \mu_1^{1/q}\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \\ & + \mu_2^{-1+\frac{1}{q}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \mu_2^{1+\frac{1}{q}}\|u_1(\sigma,\cdot)\|_{L_q(0,1)}].\end{aligned}$$

Choose $\mu_1 = \mu_2 = R \gg 1$. Then (4.1) implies

$$\|\frac{du_2(\sigma,\cdot)}{dy}\|_{L_q(0,1)} \leq C(\varepsilon,K)\frac{1}{|\lambda|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \quad 0 < |\sigma| \leq K. \tag{4.9}$$

On the other hand, from (4.4) and (4.5), we have for $|\sigma| > K$

$$\begin{aligned}
\|\frac{du_2(\sigma,\cdot)}{dy}\|_{L_q(0,1)} \le & C(\varepsilon,K)[|\sigma|^{1-\frac{1}{q}}(|u_1(\sigma,0)|+|u_1(\sigma,1)|) \\
&+ \frac{|\sigma|^{3-\frac{1}{q}}}{|\lambda|}(|u_1(\sigma,0)|+|u_1(\sigma,1)|) + |\lambda|^{-\frac{1}{2q}}(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|) \\
&+ \frac{|\sigma|^{2-\frac{1}{q}}}{|\lambda|}(|u_1'(\sigma,0)|+|u_1'(\sigma,1)|)] \\
\le & C(\varepsilon,K)[\mu_1^{-2+\frac{1}{q}}|\sigma|^{1-\frac{1}{q}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \mu_1^{1/q}|\sigma|^{1-\frac{1}{q}}\|u_1(\sigma,\cdot)\|_{L_q(0,1)} + \mu_2^{-2+\frac{1}{q}}\frac{|\sigma|^{3-\frac{1}{q}}}{|\lambda|}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \mu_2^{1/q}\frac{|\sigma|^{3-\frac{1}{q}}}{|\lambda|}\|u_1(\sigma,\cdot)\|_{L_q(0,1)} + \mu_3^{-1+\frac{1}{q}}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \mu_3^{1+\frac{1}{q}}\|u_1(\sigma,\cdot)\|_{L_q(0,1)} + \mu_4^{-1+\frac{1}{q}}\frac{|\sigma|^{2-\frac{1}{q}}}{|\lambda|}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \mu_4^{1+\frac{1}{q}}\frac{|\sigma|^{2-\frac{1}{q}}}{|\lambda|}\|u_1(\sigma,\cdot)\|_{L_q(0,1)}].
\end{aligned}$$

Choose $\mu_1 = \mu_2 = |\sigma|^{1+q}$, $\mu_3 = R \gg 1$, $\mu_4 = |\sigma|^{\frac{2q+1}{q+1}}$, where $|\sigma| > K \gg 1$. Then, by (4.1),

$$\begin{aligned}
\|\frac{du_2(\sigma,\cdot)}{dy}\|_{L_q(0,1)} \le & C(\varepsilon,K)[\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \sigma^2\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \frac{1}{|\lambda|}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \frac{1}{|\lambda|}\sigma^4\|u_1(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \|u_1''(\sigma,\cdot)\|_{L_q(0,1)} + \|u_1(\sigma,\cdot)\|_{L_q(0,1)} + \frac{|\sigma|}{|\lambda|}\|u_1''(\sigma,\cdot)\|_{L_q(0,1)} \\
&+ \frac{1}{|\lambda|}\sigma^4\|u_1(\sigma,\cdot)\|_{L_q(0,1)}] \\
\le & C(\varepsilon,K)\frac{1}{|\lambda|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \quad |\sigma| > K.
\end{aligned} \tag{4.10}$$

Therefore, for a solution $\widehat{u}(\sigma,y) = u_1(\sigma,y) + u_2(\sigma,y)$ of problem (1.3)–(1.4) from (4.1), (4.9), (4.10), and $\widehat{u_y'}(\sigma,y) = \widehat{u}_y'(\sigma,y)$ we have

$$\|\widehat{u_y'}(\sigma,\cdot)\|_{L_q(0,1)} = \|\widehat{u}_y'(\sigma,\cdot)\|_{L_q(0,1)} \le C(\varepsilon)\frac{1}{|\lambda|}\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \tag{4.11}$$

where $\sigma \in \mathbb{R}$, $\sigma \ne 0$, $|\arg\lambda| \le \pi - \varepsilon$, $|\lambda| \ge M$.

Consider now a solution $u(x,y)$ of the main problem (1.1)–(1.2). Since $u(x,0) = u(x,1) = 0$, for all $x \in \mathbb{R}$,

$$\begin{gathered}
\|u\|_{L_q(\mathbb{R}\times(0,1))} \le C\|u_y'\|_{L_q(\mathbb{R}\times(0,1))}, \\
\|u\|_{W_q^1(\mathbb{R}\times(0,1))} \le C(\|u_y'\|_{L_q(\mathbb{R}\times(0,1))} + \|u_x'\|_{L_q(\mathbb{R}\times(0,1))}).
\end{gathered} \tag{4.12}$$

Finally, taking $q = 2$, from (4.8), (4.11), (4.12) and the Parseval equality, for a solution $u(x, y)$ of problem (1.1)–(1.2) we have

$$\|u\|_{L_2(\mathbb{R}\times(0,1))} \le C\|u'_y\|_{L_2(\mathbb{R}\times(0,1))} = C\|\widehat{u'_y}\|_{L_2(\mathbb{R}\times(0,1))} \le C(\varepsilon)\frac{1}{|\lambda|}\|f\|_{L_2(\mathbb{R}\times(0,1))},$$

and

$$\begin{aligned}\|u\|_{W_2^1(\mathbb{R}\times(0,1))} &\le C(\|\widehat{u'_y}\|_{L_2(\mathbb{R}\times(0,1))} + \|\widehat{u'_x}\|_{L_2(\mathbb{R}\times(0,1))})\\ &\le C(\varepsilon)\frac{1}{|\lambda|}(\|f'_x\|_{L_2(\mathbb{R}\times(0,1))} + \|f\|_{L_2(\mathbb{R}\times(0,1))})\\ &\le C(\varepsilon)\frac{1}{|\lambda|}\|f\|_{W_2^1(\mathbb{R}\times(0,1))},\end{aligned}$$

where $|\arg\lambda| \le \pi - \varepsilon,\ |\lambda| \ge M$. □

5. A boundary value problem for the biharmonic equation with the second order derivative in boundary conditions

Consider now the following problem in the strip $\Omega := (-\infty, \infty) \times [0, 1] \subset \mathbb{R}^2$,

$$L(\lambda, D_x, D_y)u := \Delta^2 u(x, y) - \lambda\Delta u(x, y) = f(x, y), \quad (x, y) \in \Omega, \tag{5.1}$$

$$\begin{gathered} L_1 u := u(x, 0) = 0, \quad L_2 u := u(x, 1) = 0\\ L_3 u := \frac{\partial^2 u(x, 0)}{\partial y^2} = 0, \quad L_4 u := \frac{\partial^2 u(x, 1)}{\partial y^2} = 0, \quad x \in \mathbb{R}.\end{gathered} \tag{5.2}$$

Applying the Fourier transform operator $F_{x\to\sigma}$ to (5.1)–(5.2), we obtain a boundary-value problem for an ordinary differential equation of the fourth order with 2 parameters

$$L(\lambda, i\sigma, D_y)\widehat{u} := \Big[\frac{d^4}{dy^4} - (2\sigma^2 + \lambda)\frac{d^2}{dy^2} + \sigma^4 + \lambda\sigma^2\Big]\widehat{u}(\sigma, y) = \widehat{f}(\sigma, y), \quad y \in [0, 1], \tag{5.3}$$

$$\widehat{u}(\sigma, 0) = \widehat{u}(\sigma, 1) = \frac{d^2\widehat{u}(\sigma, 0)}{dy^2} = \frac{d^2\widehat{u}(\sigma, 1)}{dy^2} = 0, \tag{5.4}$$

where $\lambda \in \mathbb{C}$ and $\sigma \in \mathbb{R}$ are parameters, $\widehat{u}(\sigma, y) := (F_{x\to\sigma}u(x, y))(\sigma, y)$.

These two problems, (5.1)–(5.2) and (5.3)–(5.4), are much more easier to handle than (1.1)–(1.2) and (1.3)–(1.4), respectively. Moreover, we can get here a more complete result.

Theorem 5.1. *For each $\varepsilon > 0$ there exists $M > 0$ such that for all complex numbers λ satisfying $|\arg\lambda| \le \pi - \varepsilon,\ |\lambda| \ge M$, the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := L(\lambda, D_x, D_y)u$ from $W_2^4(\mathbb{R} \times (0, 1), L_\nu u = 0, \nu = 1, \dots, 4)$ onto $L_2(\mathbb{R} \times (0, 1))$ is an isomorphism and for these λ the following estimates hold*

$$\|u\|_{W_2^4(\mathbb{R}\times(0,1))} \le C(\varepsilon)\|f\|_{L_2(\mathbb{R}\times(0,1))}, \tag{5.5}$$

and

$$\|u\|_{W_2^2(\mathbb{R}\times(0,1))} \le C(\varepsilon)\frac{1}{|\lambda|}\|f\|_{L_2(\mathbb{R}\times(0,1))}, \tag{5.6}$$

for a solution $u(x, y)$ of problem (5.1)–(5.2).

Proof. The required isomorphism follows from [6, Theorem 1.1, p.44] (the proof is done as in the proof of Theorem 4.1). To get estimates (5.5) and (5.6), first consider a solution $\widehat{u}(\sigma,y)$ of problem (5.3)–(5.4). Substituting $v(\sigma,y) := \frac{d^2\widehat{u}(\sigma,y)}{dy^2} - \sigma^2\widehat{u}(\sigma,y)$, one can consider, instead of (5.3)–(5.4), the two problems

$$\begin{gathered} \frac{d^2\widehat{u}(\sigma,y)}{dy^2} - \sigma^2\widehat{u}(\sigma,y) = v(\sigma,y),\ y\in[0,1], \\ \widehat{u}(\sigma,0) = \widehat{u}(\sigma,1) = 0, \end{gathered} \tag{5.7}$$

and

$$\begin{gathered} \frac{d^2v(\sigma,y)}{dy^2} - (\sigma^2+\lambda)v(\sigma,y) = \widehat{f}(\sigma,y),\ y\in[0,1], \\ v(\sigma,0) = v(\sigma,1) = 0. \end{gathered} \tag{5.8}$$

From a theorem in [9, p. 110] for problem (5.7) for each fixed σ, such that $|\sigma| > K$ an isomorphism from $W_q^4(0,1)$ onto $W_q^2(0,1)$ follows and for a solution $\widehat{u}(\sigma,y)$ of problem (5.7) the following estimate holds

$$\sum_{k=0}^{4} |\sigma|^{4-k}\|\widehat{u}(\sigma,\cdot)\|_{W_q^k(0,1)} \le C(\|v(\sigma,\cdot)\|_{W_q^2(0,1)} + |\sigma|^2\|v(\sigma,\cdot)\|_{L_q(0,1)}), \quad |\sigma| > K. \tag{5.9}$$

For $|\sigma| \le K$ one can easily obtain that for a solution of (5.7),

$$\|\widehat{u}(\sigma,\cdot)\|_{W_q^4(0,1)} \le C\|v(\sigma,\cdot)\|_{W_q^2(0,1)},\ |\sigma| \le K. \tag{5.10}$$

From the same theorem [9, p. 110] for problem (5.8) for each fixed $\sigma\in\mathbb{R}$ an isomorphism from $W_q^2(0,1)$ onto $L_q(0,1)$ follows and for a solution $v(\sigma,y)$ of problem (5.8) the following estimate holds

$$\|v(\sigma,\cdot)\|_{W_q^2(0,1)} + |\lambda+\sigma^2|\|v(\sigma,\cdot)\|_{L_q(0,1)} \le C(\varepsilon)\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \tag{5.11}$$

where $|\arg\lambda| \le \pi-\varepsilon$, $|\lambda| \ge M$. We have $|\lambda+\sigma^2| \ge C(\varepsilon)(|\lambda|+\sigma^2)$ (see section 2). Then, from (5.9), (5.10), and (5.11) for problem (5.3)–(5.4) an isomorphism from $W_q^4((0,1), L_\nu\widehat{u}=0, \nu=1,\dots,4)$ onto $L_q(0,1)$ follows for $|\arg\lambda| \le \pi-\varepsilon,\ |\lambda| \ge M$ for each fixed $\sigma\in\mathbb{R}$. Moreover, for a solution $\widehat{u}(\sigma,y)$ of problem (5.3)–(5.4) the following estimate holds

$$\sum_{k=0}^{4} |\sigma|^{4-k}\|\widehat{u}(\sigma,\cdot)\|_{W_q^k(0,1)} \le C(\varepsilon)\|\widehat{f}(\sigma,\cdot)\|_{L_q(0,1)}, \tag{5.12}$$

where $\sigma\in\mathbb{R}$, $|\arg\lambda| \le \pi-\varepsilon,\ |\lambda| \ge M$. Taking now $q=2$ and using the Parseval equality to (5.12) we obtain for a solution $u(x,y)$ of problem (5.1)–(5.2) estimate (5.5). From (5.5) and equation (5.1) follows

$$|\lambda|\|\Delta u\|_{L_2(\mathbb{R}\times(0,1))} \le C(\varepsilon)\|f\|_{L_2(\mathbb{R}\times(0,1))}. \tag{5.13}$$

On the other hand, from [6, Theorem 1.1, p.44] it follows that

$$\|u\|_{W_2^2(\mathbb{R}\times(0,1))} \le C\|\Delta u\|_{L_2(\mathbb{R}\times(0,1))}. \tag{5.14}$$

From (5.13) and (5.14) we obtain estimate (5.6). □

References

[1] Agmon, S. and Nirenberg, L., Properties of solutions of ordinary differential equations in Banach spaces, *Comm. Pure Appl. Math.*, **16** (1963), 121–239.

[2] Agranovich, M. S. and Vishik, M. I., Elliptic problems with a parameter and parabolic problems of general type, *Uspekhi Mat. Nauk*, **19**, 3 (1964), 53–161 (Russian; English translation in *Russian Math. Surveys*, **19**, 3 (1964), 53–159).

[3] Besov, O. V., Ilin, V. P. and Nikolskii, S. M., *Integral Representations of Functions and Embedding Theorems,* Halsted Press, New York, v.I, 1978.

[4] Kozlov, V. A. and Maz'ya, V. G., *Differential Equations with Operator Coefficients*, Springer, 1999.

[5] Kozlov, V. A., Maz'ya, V. G. and Rossmann, J., *Elliptic Boundary Value Problems in Domains with Point Singularities*, AMS, Math. Surv. and Monogr., v.52, 1997.

[6] Nazarov, S. A. and Plamenevskii, B. A., *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, Walter de Gruyter, Berlin, 1994.

[7] Triebel, H., *Interpolation Theory. Function Spaces. Differential Operators*, North-Holland, Amsterdam, 1978.

[8] Yakubov, S., *Completeness of Root Functions of Regular Differential Operators,* Longman, Scientific and Technical, New York, 1994.

[9] Yakubov, S. and Yakubov, Ya., *Differential-Operator Equations. Ordinary and Partial Differential Equations*, Chapman and Hall/CRC, Boca Raton, 2000.

Yakov Yakubov
Raymond and Beverly Sackler Faculty of Exact Sciences
School of Mathematical Sciences, Tel-Aviv University
Ramat-Aviv 69978, Israel
E-mail address: yakubov@post.tau.ac.il