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Boundary behavior of blow-up solutions to some weighted non-linear differential equations *

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Abstract

We investigate, under appropriate conditions on the weight g and the non-linearity f, the boundary behavior of solutions to

$$\left(r^\alpha (u')^{p-1}\right)' = r^\alpha g(r) f(u),$$

$0 < r < R$, $u'(0) = 0$, $u(r) \to \infty$ as $r \to R$. The results obtained here generalize, and in some cases improve known results.

1 Introduction

Let $\alpha \geq 0$, $p > 1$, $R > 0$. For $0 < r < R$ we consider positive and increasing solutions of the problem

$$\begin{gathered}\left(r^\alpha (u')^{p-1}\right)' = r^\alpha g(r) f(u), \\ u'(0) = 0, \quad u(r) \to \infty \quad \text{as} \quad r \to R,\end{gathered} \tag{1.1}$$

where $f(s)$ is assumed to satisfy

$$\begin{gathered}f(0) = 0, \quad f(s) > 0, \quad \int_0^1 \Big(\int_0^s f(t)dt\Big)^{-1/p} ds = \infty, \\ f'(s) \geq 0 \quad \text{for } s > 0.\end{gathered} \tag{1.2}$$

Solutions of (1.1) are called blow-up solutions. Note that radial solutions of the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = g(|x|) f(u),$$

in the ball $B := B(0, R) \subseteq \mathbb{R}^N$ satisfy the differential equation (1.1) with $\alpha = N - 1$. It will be convenient to rewrite (1.1) as

$$\left((u')^{p-1}\right)' + \frac{\alpha}{r}(u')^{p-1} = g(r) f(u), \quad u'(0) = 0, \quad u(R) = \infty. \tag{1.3}$$

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Throughout the paper we shall assume the following condition on the non-identically-zero weight function:

$$g \geq 0, \quad g \in C([0,R)), \quad \text{and } g \text{ is non-decreasing near } R. \tag{1.4}$$

Whenever the monotonicity condition on g is not required, we will state it explicitly.

A necessary and sufficient condition for the existence of a solution to (1.1), when $g(r) = C > 0$, is the Keller-Osserman condition [6, 9]:

$$\int_1^\infty \frac{ds}{F(s)^{1/p}} < \infty, \quad F(s) = \int_0^s f(t)dt. \tag{1.5}$$

If this condition holds for f, then the function

$$\psi(t) := \int_t^\infty \frac{1}{(qF(s))^{1/p}}\, ds, \qquad t > 0, \tag{1.6}$$

is well-defined, decreasing and convex. Here $q := p/(p-1)$. Let ϕ be the inverse function of ψ. Then $\lim_{s\to 0}\phi(s) = \infty$, $\lim_{s\to\infty}\phi(s) = 0$, and ϕ is known to satisfy the 1-dimentional equation $(-(-\phi')^{p-1})' = f(\phi)$.

The case $g = C > 0$ has been investigated extensively. For $p = 2$, $\alpha = N-1$, asymptotic boundary estimates have been obtained in [8, 2]. For $p > 1$ such estimates were obtained in [5]. The question of existence of blow-up solutions when $g(r)$ is bounded and vanishes on a set of positive measure has been considered in [7], when $p = 2$ and $\alpha = N-1$. Again for $p = 2$ and $\alpha = N-1$, the situation when $g(r)$ is unbounded near $r = R$ has recently been discussed in [4]. In [10], equation (1.1) was investigated in the general case when $g(r)$ is unbounded near R.

Our purpose in this paper is to study the boundary behavior of blow-up solutions of (1.1) when g is unbounded near R and f satisfies the condition (1.5).

For later reference, let us recall the following two results from [11] and [5]. The first is a comparison Lemma (see a proof in [11]). For notational convenience in stating the Lemma, we let L denote the differential operator on the left hand side of equation (1.3) above.

Lemma 1.1 (Comparison) *Let $0 \leq a < b$. Suppose $u, w \in C^1([a,b])$ with $(u')^{p-1}, (w')^{p-1} \in C^1((a,b])$ satisfy*

$$\begin{gathered} Lu - g(r)f(u) \leq Lw - g(r)f(w) \quad \text{in } (a,b] \\ u(a+) < w(a+), \quad u'(a) \leq w'(a). \end{gathered}$$

Then $u' \leq w'$ in $[a,b]$, which implies $u < w$ in $(a,b]$.

In this Lemma $u(a+) < w(a+)$ means that there exists $\epsilon > 0$ such that $u < w$ in $(a, a+\epsilon)$.

The second result we shall need is stated below (see [5] for a proof).

Lemma 1.2 *If conditions* (1.2) *and* (1.5) *hold for some* $p > 1$*, then*

$$\lim_{t\to\infty} \frac{t^p}{F(t)} = 0.$$

2 Existence of blow-up solutions

Let us first make a few remarks about solutions of (1.3). Starting with the inequality (which follows from (1.3))

$$((u')^{p-1})' \le g(r)f(u(r)), \quad 0 < r < R,$$

we multiply both sides of the inequality by $u'(r)$ and integrate the resulting inequality on (r_0, r), to arrive at

$$(u'(r))^p \le q \int_{r_0}^{r} g(s)f(u(s))u'(s)\,ds + (u'(r_0))^p, \quad r_0 < r < R.$$

Here $0 < r_0 < R$ is such that g is increasing on (r_0, R). Since $u(r) \to \infty$ as $r \to R$ we find that

$$u'(r) \le 2g^{1/p}(r)(qF(u(r)))^{1/p}, \quad r_1 < r < R, \tag{2.1}$$

for some $r_0 \le r_1 < R$, depending on u.

Since $g(r)$, and $f(u(r))$ are increasing in (r_0, R) it easily follows, on integrating both sides of equation (1.1) on (r_0, r), that for a given $\epsilon > 0$ there is r_2 that depends on u and ϵ such that

$$(u')^{p-1} < \frac{r(1+\epsilon)}{\alpha+1} g(r)f(u), \quad r_2 < r < R\,. \tag{2.2}$$

The use of this inequality in (1.3) gives

$$((u')^{p-1})' > \frac{1-\alpha\epsilon}{\alpha+1} g(r)f(u(r)), \quad r_2 < r < R. \tag{2.3}$$

Note that this last inequality implies that u' is increasing near R.

Remark 2.1 *Suppose* g *is non-decreasing on* $(0, R)$*, and* u *is a solution of (1.1). Taking note of the fact that* $u'(0) = 0$*, the argument leading up to the inequality (2.3) shows that this inequality holds for* $0 < r < R$ *with* $\epsilon = 0$.

To restate some of the main results that were proved in [10], let us consider the following two conditions:

$$g^{1/p} \in L^1(0, R) \tag{2.4}$$

$$\liminf_{r\to R} (R - r)g(r)^{1/p} > 0\,. \tag{2.5}$$

We recall the following two results from [10].

Theorem 2.2 *Assume that (2.4) holds. Then (1.1) has a solution if and only if condition (1.5) holds.*

Theorem 2.3 *Assume that (2.5) holds. Then (1.1) has a solution if and only if condition (1.5) fails to hold.*

Let us give a proof of the sufficiency of the (1.5) condition for the existence of a blow-up solution in Theorem 2.2. For the proof of the rest of the assertions in Theorems 2.2 and 2.3, we refer the reader to [10].

Proof. Let $0 < r_0 < R$ be such that g is increasing on (r_0, R) and let us fix a positive integer m with $1/m < R - r_0$. For each $k \geq m$, let w_k be a blow-up solution of (1.3) on $(0, R - 1/k)$. Since g is bounded on this interval, such a solution exists by the (1.5) condition. Then we must have $w_{k+1}(0) \leq w_k(0)$, for if $w_{k+1}(0) > w_k(0)$ then by the Comparison Lemma, we would have $w_k(r) \leq w_{k+1}(r)$, $0 < r < R - 1/k$. But this is not possible as w_k blows up at $R - 1/k$, and w_{k+1} does not. In fact we must have

$$w_{k+1}(r) \leq w_k(r), \quad 0 \leq r < R - \frac{1}{k}.$$

To see this, suppose on the contrary $w_{k+1}(r) > w_k(r)$ for some $0 < r < R - 1/k$. Then since $w_{k+1}(0) \leq w_k(0)$ the function $w_{k+1} - w_k$ takes on a positive maximum in the interior of $[0, r_1]$ where r_1 is chosen sufficiently close to $R - 1/k$ so that $w_{k+1}(r_1) \leq w_k(r_1)$. If $0 < r^* < r_1$ is such a maximum point, then we have $w_{k+1}(r^*) > w_k(r^*)$ and $w'_{k+1}(r_*) = w'_k(r_*)$. But then by the Comparison Lemma again, we would have $w_{k+1} > w_k$ on $(r_*, R - 1/k)$ which is obviously not possible. Therefore the claimed inequality holds.

Using this and the fact that w_k and w_{k+1} satisfy equation (1.1) we obtain

$$\begin{aligned}
(w'_{k+1}(r))^{p-1} &= r^{-\alpha} \int_0^r s^\alpha g(s) f(w_{k+1}(s))\, ds \\
&\leq r^{-\alpha} \int_0^r s^\alpha g(s) f(w_k(s))\, ds \\
&= (w'_k(r))^{p-1}, \qquad 0 < r < R - 1/k,
\end{aligned}$$

thus showing that

$$w'_{k+1}(r) \leq w'_k(r), \quad 0 \leq r < R - 1/k. \tag{2.6}$$

For $t, r \in (0, R - 1/k)$, and $n > k$ we have

$$\begin{aligned}
|w_n(r) - w_n(t)| &= \Big| \int_t^r w'_n(s)\, ds \Big| \\
&\leq w'_n(\zeta) |r - t| \\
&\leq w'_{k+1}(R - 1/k) |r - t|,
\end{aligned}$$

where $\zeta = \max\{r, t\}$. Thus $\{w_n\}_{n=k+1}^{\infty}$ is a bounded equicontinuous family in $C([0, R-1/k])$, and hence has a convergent subsequence. Let u be the limit. We will show that u is the desired blow-up solution of (1.3) in $(0, R)$. To see this, note that for $r \in [0, R-1/k]$ and $n > k$ the solution w_n satisfies the integral equation

$$w_n(r) = w_n(0) + \int_0^r \left(\int_0^t \left(\frac{s}{t}\right)^{\alpha} g(s) f(w_n(s))\, ds \right)^{\frac{1}{p-1}} dt.$$

Letting $n \to \infty$ we see that u satisfies the same integral equation. Since k is arbitrary we conclude that u satisfies equation (1.3) in $(0, R)$. We now show that $u(r) \to \infty$ as $r \to R$. Let r_1 such that (2.1) holds for w_m in $r_1 < r < R - 1/m$. Because of the inequality (2.6), the same r_1 can be used for w_k in the interval $(r_1, R-1/k)$ for any $k \geq m$. Thus using the inequality (2.1) for w_k we see that

$$\frac{w_k'(s)}{(qF(w_k(s)))^{1/p}} \leq 2g^{1/p}(s), \quad r_1 < s < R - 1/k.$$

Integrating this on $(r, R-1/k)$ for any $r_1 < r < R - 1/k$ leads to

$$\int_{w_k(r)}^{\infty} \frac{1}{(qF(s))^{1/p}}\, ds \leq 2 \int_r^{R-1/k} g^{1/p}(s)\, ds \leq 2 \int_r^{R} g^{1/p}(s)\, ds.$$

Now letting $k \to \infty$ we obtain the integral estimate

$$\int_{u(r)}^{\infty} \frac{1}{(qF(s))^{1/p}}\, ds \leq 2 \int_r^{R} g^{1/p}(s)\, ds, \quad r_1 \leq r < R.$$

Letting $r \to R$, we notice that the left hand side integral tends to zero, since by hypothesis $g \in L^{1/p}(0, R)$. Thus we conclude that $u(r) \to \infty$ when $r \to R$, as desired. □

Remark 2.4 The above theorems do not cover the situation when g satisfies neither condition (2.4) nor (2.5). The following example shows that neither theorem is true in this case. Let $p > 1$ and $0 < R \leq 1$ be fixed. Then $u(r) = -\log(R-r) - r/R$ satisfies the equations, $(j = 0, 1)$,

$$(u'(r)^{p-1})' = g_j(r) f_j(u), \quad u'(0) = 0, \quad u(r) \to \infty \quad \text{as } r \to R,$$

where

$$g_j(r) = (p-1)(rR^{-1})^{p-2}(R-r)^{-p}(-\log(R-r) - r/R)^{-p+j},$$

and $f_j(t) = t^{p-j}, \quad j = 0, 1$. Observe that

$$\lim_{r \to R} (R-r) g_j(r)^{1/p} = 0, \quad \text{and} \quad g_j^{1/p} \notin L^1(0, R), \quad j = 0, 1.$$

Furthermore it is clear that f_0 satisfies (1.5), but f_1 does not. Both satisfy all the hypotheses in (1.2).

Remark 2.5 Suppose f satisfies (1.2) and (1.5). If (1.3) admits a blow-up solution, then it is known (Theorem 2.2 of [10]) that

$$\lim_{r\to R} g(r)^{1/p}(R-r) = 0.$$

However it remains unclear whether this limit condition is also sufficient for existence of blow-up solutions.

3 Boundary Behavior of Blow-up Solutions

The following condition, introduced in [3], will be useful in investigating some boundary behavior of solutions to (1.3). Let the function ψ defined in (1.6) satisfy

$$\liminf_{t\to\infty} \frac{\psi(\beta t)}{\psi(t)} > 1, \quad \text{for any } 0<\beta<1. \tag{3.1}$$

This condition implies the following Lemma given in [3] without proof. Since subsequent results rely on this Lemma, we shall include the short proof for the readers' convenience as well as for completeness.

Lemma 3.1 *Let $\psi \in C[t_0,\infty)$. Suppose that ψ is strictly monotone decreasing and satisfies (3.1). Let $\phi := \psi^{-1}$. Given a positive number γ there exist positive numbers $\eta_\gamma, \delta_\gamma$ such that the following hold:*

1. *If $\gamma > 1$, then $\phi((1-\eta)\delta) \le \gamma\phi(\delta)$ for all $\eta \in [0,\eta_\gamma]$, $\delta \in [0,\delta_\gamma]$*
2. *If $\gamma < 1$, then $\phi((1+\eta)\delta) \ge \gamma\phi(\delta)$ for all $\eta \in [0,\eta_\gamma]$, $\delta \in [0,\delta_\gamma]$.*

Proof. We prove (1) only, as (2) is an easy consequence of (1). Let $\gamma > 1$ be given. By hypothesis,

$$\liminf_{t\to\infty} \frac{\psi(\gamma^{-1}t)}{\psi(t)} = \alpha, \quad \text{for some } \alpha = \alpha(\gamma) > 1.$$

Let us fix $1 < \theta < \alpha$. Then for some t_γ we have

$$\psi(\gamma^{-1}t) \ge \theta\psi(t), \quad \text{for all } t \ge t_\gamma. \tag{3.2}$$

Now let $\delta_\gamma := \theta\psi(t_\gamma)$, and $\eta_\gamma := (\theta-1)/\theta$. If $0 < \delta \le \delta_\gamma$, then $\phi(\delta/\theta) \ge t_\gamma$, and hence by (3.2), we obtain $\phi(\delta/\theta) \le \gamma\phi(\delta)$. Thus if $0 < \eta \le \eta_\gamma$ so that $(1-\eta)\delta \ge \delta/\theta$ then we conclude that $\phi((1-\eta)\delta) \le \phi(\delta/\theta) \le \gamma\phi(\delta)$ as desired. $\square$

We will also need the following consequence of the above Lemma.

Lemma 3.2 *Let ψ and ϕ be as in Lemma 3.1, and let $\gamma > 0$, $C > 0$ be given. Then there is a positive constant $\delta_0 = \delta_0(C,\gamma)$ and a positive integer $m = m(C,\gamma)$ such that the following hold:*

1. *If $\gamma > 1$, then $\phi(C\delta) \le \gamma^m\phi(\delta)$ for all $0 < \delta < \delta_0$*
2. *If $\gamma < 1$, then $\phi(C\delta) \ge \gamma^m\phi(\delta)$ for all $0 < \delta < \delta_0$.*

Proof. We prove (1) only, as (2) is an immediate consequence of (1). If $C \geq 1$, then there is nothing to prove. So assume that $0 < C < 1$. Corresponding to $\gamma > 1$, let η_γ and δ_γ be the positive constants given in Lemma 3.1. Choose $0 < \eta < \eta_\gamma$ such that $0 < C < 1 - \eta$, and let $\delta_0 = (1-\eta)^m \delta_\gamma / C$, where m is the smallest positive integer such that $(1-\eta)^m < C$. With such a choice of δ_0 we clearly see that $0 < C\delta/(1-\eta)^j < \delta_\gamma$ for all $j = 1, \cdots m$ whenever $0 < \delta < \delta_0$. Thus if $0 < \delta < \delta_0$, we apply Lemma 3.1 repeatedly to obtain

$$\phi(C\delta) \leq \gamma^m \phi\big(\frac{C}{(1-\eta)^m}\delta\big).$$

Recalling that ϕ is decreasing we obtain the claimed inequality. $\square$

Note that if condition (1.5) holds for f, and equation (1.1) admits a solution, then $g^{1/p}(r)(R-r) \to 0$ as $r \to R$ (see Remark 2.5). Thus $\phi\big(g^{1/p}(r)(R-r)\big) \to \infty$ as $r \to R$.

Theorem 3.3 *Suppose that f satifies (1.2), and (1.5) condition together with condition (3.1). If u is a solution of (1.1), then*

1. $\limsup_{r \to R} \dfrac{u(r)}{\phi\left(g^{1/p}(r)(R-r)\right)} \leq 1.$
2. *If, moreover, g satisfies (2.4), then* $\liminf_{r \to R} \dfrac{u(r)}{\phi\big(\int_r^R g^{1/p}(s)\,ds\big)} \geq 1.$

Proof. (1) Let k be the smallest positive integer such that $1/k < R$, and for each positive integer $j \geq k$ let $g(M_j) := \max\{g(r) : 0 \leq r \leq R - 1/j\}$. Furthermore, let w_j be a solution of

$$((w')^{p-1})' + \frac{\alpha}{r}(w')^{p-1} = (\alpha+1)g(M_j)f(w), \quad w'(0) = 0, \quad w(R-1/j) = \infty,$$

in $(0, R-1/j)$. Since for $j > m$ we have

$$((w_j')^{p-1})' + \frac{\alpha}{r}(w_j')^{p-1} \geq (\alpha+1)g(M_m)f(w_j), \quad \text{on} \quad (0, R-1/m),$$

by the Comparison Lemma we conclude that $w_j(0) \leq w_m(0)$. In particular we have $w_j(0) \leq w_k(0)$ for all $j > k$. On using Remark 2.1 we get

$$\begin{aligned} w_j'(r) &\geq (qg(M_j))^{1/p}(F(w_j(r)) - F(w_j(0)))^{1/p} \\ &= (g(M_j))^{1/p}(qF(w_j(r)))^{1/p}\Big(1 - \frac{F(w_j(0))}{F(w_j(r))}\Big)^{1/p}. \end{aligned}$$

Rewriting this expression, and using the inequality $w_j(0) \leq w_k(0)$ for all $j > k$ we find that

$$\frac{w_j'(r)}{(qF(w_j(r)))^{1/p}} \geq (g(M_j))^{1/p}\Big(1 - \frac{F(w_k(0))}{F(w_j(r))}\Big)^{1/p}.$$

Integrating this on $(r, R-1/j)$ we obtain

$$\begin{aligned}&\int_{w_j(r)}^{\infty} \frac{1}{(qF(t))^{1/p}}\,dt\\ &\geq (g(M_j))^{1/p}(R-1/j-r)\Big[\frac{1}{R-1/j-r}\int_r^{R-1/j}\Big(1-\frac{F(w_k(0))}{F(w_j(s))}\Big)^{1/p}\,ds\Big].\end{aligned}$$

Note that the expression in the bracket on the right tends to one as r approaches $R-1/j$. Thus we have obtained

$$\psi(w_j(r)) \geq (g(M_j))^{1/p}(R-1/j-r)(1-\eta_j(r)),$$

where

$$\eta_j(r) = 1 - \Big[\frac{1}{R-1/j-r}\int_r^{R-1/j}\big(1-\frac{F(w_k(0))}{F(w_j(s))}\big)^{1/p}\,ds\Big],$$

so that $\eta_j(r) \to 0$ as $r \to R-1/j$.

Hence we obtain, by Lemma 3.1, that for any $\epsilon > 0$ there is $r_0(\epsilon, j)$ such that (recall that $g(M_j) \geq g(r)$, $0 \leq r \leq R-1/j$)

$$\begin{aligned}w_j(r) &\leq \phi\big((g(M_j))^{1/p}(R-1/j-r)(1-\eta_j(r))\big)\\ &\leq (1+\epsilon)\phi\left(g^{1/p}(r)(R-1/j-r)\right), \quad r_0(\epsilon,j) < r < R-1/j.\end{aligned}$$

But then given any $j \geq k$, the inequality $u(r) \leq w_j(r)$ holds for r sufficiently close to $R-1/j$ so that

$$u(r) \leq (1+\epsilon)\phi\left(g^{1/p}(r)(R-1/j-r)\right).$$

Therefore, we conclude that

$$\limsup_{r\to R-1/j} \frac{u(r)}{\phi\left(g^{1/p}(r)(R-1/j-r)\right)} \leq 1.$$

Since j can be taken arbitrarily large, the claim follows.

To prove (2), suppose that g is non-decreasing on (r_0, R) for some $0 < r_0 < R$. On multiplying both sides of the inequality $((u')^{p-1})' < g(r)f(u)$ by u' and integrating the resulting inequality on (r_0, r), we obtain

$$\frac{u'(r)}{(qF(u))^{1/p}} \leq g(r)^{1/p}\Big(1+\frac{u'(r_0)^p}{qg(r)F(u(r))}\Big)^{1/p}. \tag{3.3}$$

Integrating this inequality on (r, R) for $r > r_0$,

$$\begin{aligned}\int_{u(r)}^{\infty}\frac{1}{(qF(t))^{1/p}}\,dt &\leq \Big(1+\frac{u'(r_0)^p}{qg(r)F(u(r))}\Big)^{1/p}\int_r^R g(t)^{1/p}\,dt\\ &= (1+\vartheta(r))\int_r^R g(t)^{1/p}\,dt,\end{aligned}$$

where $\vartheta(r) \to 0$ as $r \to R$. Therefore,

$$u(r) \geq \phi\Big((1+\vartheta(r)) \int_r^R g(t)^{1/p}\,dt\Big).$$

By Lemma 3.1, we see that for any $\epsilon > 0$ we can find $r_1(\epsilon) > r_0$ such that

$$u(r) > (1-\epsilon)\phi\Big(\int_r^R g(t)^{1/p}\,dt\Big), \quad r_1(\epsilon) < r < R.$$

Since ϵ is arbitrary, this proves the desired inequality. □

Remark 3.4 (a) For the conclusion in (1) of the above theorem to hold, g need not be non-decreasing near R.
(b) If g is assumed to be non-decreasing on $(0, R)$, then by taking r_0 to be 0 in (3.3) it immediately follows that

$$u(r) \geq \phi\Big(\int_r^R g(\zeta)^{1/p}\,d\zeta\Big), \quad 0 < r < R.$$

Remark 3.5 Without further condition on the weight g the limits in the above Theorem may not be 1. In fact the limit supremum in (1) of Theorem 3.3 could be zero, while the limit infimum in (2) of the same Theorem could be infinity. See [4] for such examples.

Let us now introduce a condition on g that will ensure that the limits in Theorem 3.3 are unity for any blow-up solution.

$$\lim_{r\to R} \frac{1}{R-r} \int_r^R \Big(\frac{g(s)}{g(r)}\Big)^{1/p} ds = 1. \tag{3.4}$$

Theorem 3.6 *Let f satisfy* (1.2), (1.5), *and* (3.1). *Suppose also that g satisfies* (3.4). *Then for any blow-up solution u of* (1.1) *the following limits hold.*

$$\lim_{r\to R} \frac{u(r)}{\phi(\int_r^R g^{1/p}(s)\,ds)} = 1, \quad \lim_{r\to R} \frac{u(r)}{\phi(g^{1/p}(r)(R-r))} = 1. \tag{3.5}$$

Proof. Let $\epsilon > 0$ be given. Then by Lemma 3.1 we pick η_ϵ and δ_ϵ such that

$$\phi((1-\eta)\delta) \leq (1+\epsilon)\phi(\delta), \quad \text{and} \quad \phi((1+\eta)\delta) \geq (1-\epsilon)\phi(\delta), \tag{3.6}$$

for all $\eta \in [0, \eta_\epsilon]$, $\delta \in [0, \delta_\epsilon]$. By (3.4), we choose $r_\epsilon > 0$ such that for all $r_\epsilon < r < R$ we have

$$\big(1 - \frac{1}{2}\eta_\epsilon\big) g^{1/p}(r)(R-r) \leq \int_r^R g^{1/p}(s)\,ds \leq \big(1 + \frac{1}{2}\eta_\epsilon\big) g^{1/p}(r)(R-r).$$

That is

$$\begin{aligned}\phi\big((1+\frac{1}{2}\eta_\epsilon)g^{1/p}(r)(R-r)\big) &\le \phi\Big(\int_r^R g^{1/p}(s)\,ds\Big)\\ &\le \phi\big((1-\frac{1}{2}\eta_\epsilon)g^{1/p}(r)(R-r)\big).\end{aligned}$$

From these last inequalities and (3.6), we conclude that

$$(1-\epsilon)\phi\left(g^{1/p}(r)(R-r)\right) \le \phi\Big(\int_r^R g^{1/p}(s)\,ds\Big) \le (1+\epsilon)\phi\left(g^{1/p}(r)(R-r)\right),$$

for $r_* < r < R$. Thus we have shown that

$$\lim_{r\to R}\frac{\phi\big(g^{1/p}(r)(R-r)\big)}{\phi\big(\int_r^R g^{1/p}(s)\,ds\big)} = 1. \tag{3.7}$$

This limit and (1) from Theorem 3.3, then show that

$$\limsup_{r\to R}\frac{u(r)}{\phi\big(\int_r^R g^{1/p}(s)\,ds\big)} \le 1.$$

This last inequality with (2) of Theorem 3.3 would then give the desired limit. The second limit in (3.5) follows on similar lines. This concludes the proof of the Theorem. □

Remark 3.7 The proof of Theorem 3.6 shows that the weight g need not be non-decreasing near R for (3.7) to hold. Consequently the first limit in (3.5) holds for any non-negative continuous weight g. (See Remark (3.4)).

Now, we show a class of weights g that satisfy the condition (3.4). For this let $h \in C([0,R])$ be a non-decreasing positive function. Given $0 \le \beta \le p$, let

$$g(s) := h(s)\log^\beta(1/(R-s)), \quad 0 \le s < R.$$

Then g is non-decreasing near R, and for r sufficiently close to R

$$\begin{aligned}1 \le& \frac{1}{R-r}\int_r^R \big(\frac{g(s)}{g(r)}\big)^{1/p}\,ds\\ \le& \frac{1}{R-r}\big(\frac{h(R)}{h(r)}\big)^{1/p}\int_r^R \Big(\frac{\log(1/(R-s))}{\log(1/(R-r))}\Big)^{\beta/p}\,ds\\ \le& \frac{1}{R-r}\big(\frac{h(R)}{h(r)}\big)^{1/p}\int_r^R \frac{\log(1/(R-s))}{\log(1/(R-r))}\,ds\\ =& \big(\frac{h(R)}{h(r)}\big)^{1/p}\big(1-\frac{1}{\log(R-r)}\big).\end{aligned}$$

Therefore, it follows that g satisfies (3.4).

For convenience we will use the following notation in our subsequent considerations.

$$\lambda := \liminf_{r\to R} \frac{1}{R-r}\int_r^R \big(\frac{g(s)}{g(r)}\big)^{1/p}\,ds, \quad \Lambda := \limsup_{r\to R} \frac{1}{R-r}\int_r^R \big(\frac{g(s)}{g(r)}\big)^{1/p}\,ds.$$

For our next result, we consider the following condition on f:

$$\lim_{r\to R} \frac{sf(s)}{F(s)} = E \tag{3.8}$$

where E is an extended real number.

In the following two Theorems, we do not require g to be non-decreasing. The first theorem provides a converse to Theorem 3.6 under some conditions on f.

Theorem 3.8 *Suppose f satisfies* (1.2), (1.5), *and* (3.8). *Assume also that* (2.4) *holds for the weight g. If u is a solution of* (1.3) *such that the limits in* (3.5) *hold, then the weight g satisfies* (3.4), *or the limit E in (3.8) is ∞.*

Proof. Suppose that g fails to satisfy condition (3.4). Using the notation given above, we then have $\lambda < 1$ or $\Lambda > 1$. Let $\delta > 0$ such that $\lambda < 1-\delta$ or $1+\delta < \Lambda$. Then we pick some sequence r_m that converges to R so that

$$(1+\delta)g^{1/p}(r_m)(R-r_m) < \int_{r_m}^R g^{1/p}(s)\,ds, \quad \text{if } \Lambda > 1$$

or

$$(1-\delta)g^{1/p}(r_m)(R-r_m) > \int_{r_m}^R g^{1/p}(s)\,ds, \quad \text{if } \lambda < 1.$$

If the limits in (3.5) hold, then clearly we have the limit

$$\lim_{m\to\infty} \frac{\phi\left(g^{1/p}(r_m)(R-r_m)\right)}{\phi\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)} = 1.$$

By the mean value theorem, and using that $-\phi'$ is decreasing, we have

$$\begin{aligned}
&\Big|\frac{\phi\left(g^{1/p}(r_m)(R-r_m)\right)}{\phi\big(\int_{r_m}^R g^{\frac{1}{p}}(s)\,ds\big)} - 1\Big| \\
&= \frac{-\phi'(\vartheta(r_m))\big|\int_{r_m}^R g^{\frac{1}{p}}(s)\,ds - g^{1/p}(r_m)(R-r_m)\big|}{\phi\big(\int_{r_m}^R g^{\frac{1}{p}}(s)\,ds\big)} \\
&\geq \frac{-\phi'\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)\cdot\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)}{\phi\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)}\Big|1 - \frac{g^{1/p}(r_m)(R-r_m)}{\int_{r_m}^R g^{1/p}(s)\,ds}\Big|
\end{aligned} \tag{3.9}$$

From (3.9) and the above limit, we conclude that

$$\lim_{m\to\infty} \frac{-\phi'\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)\cdot\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)}{\phi\big(\int_{r_m}^R g^{1/p}(s)\,ds\big)}\Big|1-\frac{g^{1/p}(r_m)(R-r_m)}{\int_{r_m}^R g^{1/p}(s)\,ds}\Big| = 0.$$

Since

$$\Big|1-\frac{g^{1/p}(r_m)(R-r_m)}{\int_{r_m}^R g^{1/p}(s)\,ds}\Big| \ge \frac{\delta}{1+\delta} > 0,$$

we conclude that the limit of the other factor must be zero. Recalling that

$$-\phi'(t) = (qF(\phi(t)))^{1/p},$$

and hence letting $s=\phi(t)$ so that $s\to\infty$ if and only if $t\to 0$, we have the following chain of equalities.

$$\begin{aligned}
\lim_{t\to 0}\frac{-\phi'(t)t}{\phi(t)} &= \lim_{t\to 0}\frac{(qF(\phi(t)))^{1/p}t}{\phi(t)}\\
&= \lim_{s\to\infty}\frac{\psi(s)}{s/(qF(s))^{1/p}}\\
&= \lim_{s\to\infty}\frac{-1}{1-(sf(s))/(pF(s))} = 1/(E/p-1).
\end{aligned}$$

In computing the above limit, we have used L'Hôpital's rule which is justified by Lemma 1.2. Going back to (3.9), since the first term on the right side of this inequality tends to zero as $m\to\infty$ we thus conclude that the limit in (3.8) is $E=\infty$. $\square$

Theorem 3.9 *Suppose f satisfies* (1.2), (1.5), (3.1), (3.8) *and g satisfies* (2.4). *If for any solution u of* (1.3), *we have either*

$$\limsup_{r\to R}\frac{u(r)}{\phi(g^{1/p}(r)(R-r))} = 0, \quad \text{or} \quad \liminf_{r\to R}\frac{u(r)}{\phi(\int_r^R g^{1/p}(s)\,ds)} = \infty,$$

then either $\lambda=0$ or $\Lambda=\infty$.

Proof. We will prove the case when the limit superior is zero, the other case being similar. Suppose contrary to our conclusion we have $\lambda>0$ and $\Lambda<\infty$. Let us fix λ_0 and Λ_0 with $0<\lambda_0<\lambda$ and $\Lambda_0>\Lambda$. Then for some r_0 we have

$$\int_r^R g^{1/p}(s)\,ds > \lambda_0 g^{1/p}(r)(R-r), \qquad r_0<r<R.$$

By Lemma 3.2, we know that $\phi(\lambda_0 g^{1/p}(r)(R-r)) \ge (1/2)^m\phi(g^{1/p}(r)(R-r))$ for r sufficiently close to R. Consequently, by hypothesis and by (2) of Theorem

3.3 we conclude that

$$\begin{aligned}
&\liminf_{r\to R}\frac{\phi(\lambda_0 g^{1/p}(r)(R-r))}{\phi(\int_r^R g^{1/p}(s)\,ds)}\\
&\quad\ge \liminf_{r\to R}\frac{u(r)}{\phi(\int_r^R g^{1/p}(s)\,ds)}\liminf_{r\to R}\frac{\phi(\lambda_0 g^{1/p}(r)(R-r))}{u(r)}\ge\infty.
\end{aligned}$$

By the mean value theorem, and since $-\phi'$ is decreasing, for $r_0<r<R$, we have

$$\begin{aligned}
&\Big|\frac{\phi\left(\lambda_0 g^{1/p}(r)(R-r)\right)}{\phi\big(\int_r^R g^{\frac1p}(s)\,ds\big)}-1\Big|\\
&=\frac{-\phi'(\vartheta(r))\big|\int_r^R g^{1/p}(s)\,ds-\lambda_0 g^{\frac1p}(r)(R-r)\big|}{\phi\big(\int_r^R g^{1/p}(s)\,ds\big)}\\
&\le\frac{-\phi'\left(\lambda_0 g^{1/p}(r)(R-r)\right)\cdot\left(g^{1/p}(r)(R-r)\right)}{\phi\big(\int_r^R g^{1/p}(s)\,ds\big)}\Big|\frac{\int_r^R g^{1/p}(s)\,ds}{g^{1/p}(r)(R-r)}-\lambda_0\Big|
\end{aligned}\tag{3.10}$$

From the definition of Λ, we can choose r_1 such that

$$\int_r^R g^{1/p}(s)\,ds\le\Lambda_0 g^{1/p}(r)(R-r),\quad r_1<r<R,$$

Then by Lemma 3.2, there is some positive integer m, such that for some r_2 and all $r_2<r<R$, we have

$$\phi\Big(\int_r^R g^{1/p}(s)\,ds\Big)\ge\phi\big(\Lambda_0 g^{1/p}(r)(R-r)\big)\ge\big(\frac12\big)^m\phi\big(\lambda_0 g^{1/p}(r)(R-r)\big).$$

Applying this inequality to the last inequality in (3.10), we find

$$\begin{aligned}
&\Big|\frac{\phi\left(\lambda_0 g^{1/p}(r)(R-r)\right)}{\phi\big(\int_r^R g^{1/p}(s)\,ds\big)}-1\Big|\\
&\le\frac{-2^m\phi'\left(\lambda_0 g^{1/p}(r)(R-r)\right)\cdot\left(g^{1/p}(r)(R-r)\right)}{\phi\left(\lambda_0 g^{1/p}(r)(R-r)\right)}\Big|\frac{\int_r^R g^{1/p}(s)\,ds}{g^{1/p}(r)(R-r)}-\lambda_0\Big|,
\end{aligned}\tag{3.11}$$

for all $\max\{r_0,r_1,r_2\}\le r<R$. Recall from the computations in the proof of the above theorem that

$$\lim_{t\to0}\frac{-\phi'(t)t}{\phi(t)}=\lim_{s\to\infty}\frac{-1}{1-(sf(s))/(pF(s))}=1/(E/p-1),$$

where we have used the condition (and the notation) in (3.8) to write the last limit. This shows that the limit inferior of the right hand side of (3.11) is finite which contradicts our earlier conclusion that the left hand side has infinite limit. □

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