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Periodic solutions of a piecewise linear beam equation with damping and nonconstant load *

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Abstract

Using the Lyapunov-Schmidt reduction method, the authors discuss the existence and multiplicity of periodic solutions for a piecewise linear beam equation with damping and nonconstant load when the nonlinearities cross the eigenvalues. The result answers partially an open question possed by Lazer and McKenna [12].

1 Introduction

Choi and Jung [4] considered the piecewise linear one-dimensional beam equation

$$\begin{gathered} u_{tt} + u_{xxxx} + bu^+ - au^- = h(x,t), \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \quad t \in \mathbb{R} \end{gathered} \tag{1.1}$$

with u being π-periodic in t and even in x and t. The authors assumed that the upward and the downward restoring coefficients in the vibrating beam are constant and different.

Let $h = s_1\phi_{00} + s_2\phi_{01}$, where $\phi_{00} = \cos x$ and $\phi_{01} = \cos x \cos 2t$ are the eigenfunctions of $u_{tt} + u_{xxxx}$. In [4], by using of Lyapunov-Schmidt reduction method, the authors transformed problem (1.1) into a 2-dimensional problem on the subspace $V = \text{span}\{\phi_{00}, \phi_{01}\}$ and investigated the multiplicity of solutions for (1.1) with the nonlinearity $-(bu^+ - au^-)$ crossing finitely many eigenvalues.

When $u_{tt}+u_{xxxx}$ is replaced by $u_{tt}-u_{xx}$ in (1.1), the problem is the piecewise linear wave equation, for which there is a large body of literature concerning the existence and multiplicity of periodic solutions; see for example [5, 6, 14, 18] and references therein. Problem (1.1) originates from a simple mathematical model of the suspension bridge presented by Lazer and McKenna in [12]:

$$\begin{gathered} u_{tt} + u_{xxxx} + \delta u_t + ku^+ = W(x) + \epsilon h(x,t), \quad \text{in } (0, L) \times \mathbb{R}, \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad t \in \mathbb{R}, \end{gathered} \tag{1.2}$$

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where $u(x,t)$ denotes the displacement of the road bed in downward direction at position x and t, $W(x)$ is the weight per unit length at x, and $\epsilon h(x,t)$ is an external forcing term. The constant k represents the restoring force of the cables, and δ is the viscous damping.

For (1.2), in an earlier paper, McKenna and Walter [16] investigated the simplified situation

$$\begin{gathered} u_{tt} + u_{xxxx} + ku^+ = 1 + \epsilon h(x,t), \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \quad t \in \mathbb{R}, \end{gathered} \tag{1.3}$$

where u is π-periodic in t and even in x and t. Using degree theory, they proved that (1.3) has at least two solutions when $3 < k < 15$ and the one is sign-changing. In [3], using degree theory and with critical point theory, Choi and Jung gave the relationship between multiplicity of solutions of (1.3) and the source term $s_1 \cos x + s_2 \cos 2t \cos x$. Some other relevant studies can be found in [1, 2, 4, 8, 9, 10]. In all these papers $\delta = 0$; i.e., there is no damping present. As remarked in [16], the methods used in all these papers do not seem to be valid for the case $\delta \neq 0$. Meanwhile the later case is more interesting as studied in [1, 2, 11, 17]. In [12], an open question was stated as Problem 6 which is relevant to this case. The question is

> Is there a non-degeneracy condition on $h(x,t)$, which will ensure that solutions of (1.3) persist if damping is present?

In this paper, we assume that there is a damping term; i.e., $\delta \neq 0$ and the source term is

$$h(x,t) = \alpha \cos x + \beta \cos 2t \cos x + \gamma \sin 2t \cos x.$$

As in [4], we study the equation

$$\begin{gathered} u_{tt} + u_{xxxx} + \delta u_t + bu^+ - au^- = \alpha \cos x + \beta \cos 2t \cos x + \gamma \sin 2t \cos x, \\ \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \quad t \in \mathbb{R}, \end{gathered} \tag{1.4}$$

where u is π-periodic in t and even in x.

The main goal of this paper is to establish the relationship between the constant a, b, δ and the source term h, so that there exists a sign-changing periodic solutions when $h(x,t)$ is of single-sign. We also guarantee the existence of multiple periodic solutions. The results extend the corresponding results in [4] and [2] and answer partially the open problem mentioned above.

2 Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the set of positive integers, integers, and reals, respectively. Let $\Lambda = \mathbb{Z} \times \mathbb{N}$, $\Omega = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$, and $L^2(\Omega)$ be the usual space of square integrable functions with usual inner product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. For the Sobolev space $H^1(\Omega)$, we denote the standard inner product by $\langle u,v\rangle_1 = (u,v)+(u_x,v_x)+(u_t,v_t)$, and norm by $\|u\|_1$.

It is known that the eigenvalue problem

$$\begin{gathered}
u_{tt}+u_{xxxx}=\lambda u, \quad \text{in } (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R},\\
u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0, \quad t\in\mathbb{R},\\
u(x,t)=u(-x,t)=u(x,t+\pi), \quad \text{in } (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}
\end{gathered}$$

has infinitely many eigenvalues and corresponding eigenfunctions:

$$\begin{gathered}
\lambda_{mn}=(2n-1)^4-4m^2,\\
\phi_{mn}=e^{2mti}\cos(2n-1)x,
\end{gathered}$$

where $(m,n)\in\Lambda$. Then we define the Hilbert space

$$H=\big\{u\in L^2(\Omega): u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0, u(x,t)=u(-x,t)\big\},$$

for which the set of functions $\{\phi_{mn}:(m,n)\in\Lambda\}$ forms an orthogonal base.

For $\delta\neq 0$ and functions $u(x,t)=\sum_\Lambda u_{mn}\phi_{mn}$, we define the selfadjoint linear operator

$$Au=\sum_\Lambda((2n-1)^4-4m^2)u_{mn}\phi_{mn}$$

on the domain

$$D(A)=\{u\in H: u(x,t)=\sum_\Lambda u_{mn}\phi_{mn} \text{ and } \sum_\Lambda((2n-1)^4-4m^2)^2|u_{mn}|^2<\infty\}$$

where $u_{-mn}=\overline{u}_{mn}$. We also define the closed linear operator

$$A_\delta u = Au+\delta u_t$$

on the domain

$$\begin{aligned}
D(A_\delta)=\Big\{u\in H: u(x,t)=&\sum_\Lambda u_{mn}\phi_{mn}\\
&\text{and } \sum_\Lambda(((2n-1)^4-4m^2)^2+4m^2\delta^2)|u_{mn}|^2<\infty\Big\}.
\end{aligned}$$

It is easy to check that set of eigenvalues for A and A_δ are

$$\begin{gathered}
\sigma(A)=\{\lambda_{mn}=(2n-1)^4-4m^2:(m,n)\in\Lambda\},\\
\sigma(A_\delta)=\{\overline{\lambda}_{mn}=(2n-1)^4-4m^2+2m\delta i:(m,n)\in\Lambda\}.
\end{gathered}$$

We consider the weak solutions of (1.4), which are $u\in D(A_\delta)$ satisfying

$$A_\delta u+bu^+-au^-=h(x,t), \tag{2.1}$$

where $h(x,t) = \alpha\cos x + \beta\cos 2t\cos x + \gamma\sin 2t\cos x$. Assuming that δ satisfies

$$\sqrt{9+4\delta^2} < 17,$$

we set

$$a_0 = \min\{17, \sqrt{225+16\delta^2}\}, \quad b_0 = \sqrt{9+4\delta^2}. \tag{2.2}$$

For section 4 in this paper we assume that

$$-1 < a < b_0 < b < a_0. \tag{2.3}$$

For Section 5, we assume that

$$-a_0 < a < -1, \quad b_0 < b < a_0. \tag{2.4}$$

To find solutions of (2.1), we first investigate the properties of operator A_δ. We have the following lemma.

Lemma 2.1 *For all $u \in H$, the operator A_δ satisfies*

$$\|A_\delta^{-1}u\| \le \|u\|, \quad \|A_\delta^{-1}u\|_1 \le 2\|u\|.$$

Proof. Note that

$$A_\delta^{-1}u = \sum_\Lambda \frac{1}{(2n-1)^4 - 4m^2 + 2m\delta i} u_{mn}\phi_{mn}$$

for $u = \sum_\Lambda u_{mn}\phi_{mn}$, $u \in H$. For any $(m,n) \in \Lambda$, we have

$$|(2n-1)^4 - 4m^2 + 2m\delta i|^2 \ge 1,$$

then

$$\|A_\delta^{-1}u\|^2 = \sum_\Lambda |\frac{1}{(2n-1)^4 - 4m^2 + 2m\delta i} u_{mn}|^2 \le \sum_\Lambda |u_{mn}|^2 = \|u\|^2.$$

On the other hand,

$$\|A_\delta^{-1}u\|_1^2 = \sum_\Lambda \frac{1+4m^2+(2n-1)^2}{((2n-1)^4-4m^2)^2 + 4m^2\delta^2}|u_{mn}|^2,$$

while

$$\begin{aligned} &\frac{1+4m^2+(2n-1)^2}{((2n-1)^4-4m^2)^2+4m^2\delta^2} \\ &= \frac{1+4m^2+(2n-1)^2}{((2n-1)^2+2m)^2((2n-1)^2-2m)^2+4m^2\delta^2} \\ &\le \frac{1+4m^2+(2n-1)^2}{((2n-1)^2+2|m|)^2+4m^2\delta^2} \\ &\le \frac{((2n-1)^2+2|m|)^2+1}{((2n-1)^2+2|m|)^2} \le 2. \end{aligned}$$

Therefore,

$$\|A_\delta^{-1}u\|_1^2 \le 2\sum_\Lambda |u_{mn}|^2 = 2\|u\|^2$$

which completes the proof. □

By Lemma 2.1, it follows that the operator $A_\delta^{-1} : H \to H$ is compact since the embedding $H^1 \hookrightarrow L^2$ is compact. Let

$$\begin{aligned}
\phi_1(x,t) &= \cos x,\\
\phi_2(x,t) &= \cos 2t \cos x,\\
\phi_3(x,t) &= \sin 2t \cos x.
\end{aligned}$$

Let V_3 be the three-dimensional subspace of H spanned by ϕ_1, ϕ_2, ϕ_3, W_3 be the orthogonal complement of V_3 in H. Let P_3 be the orthogonal projection H onto V_3. Then every element $u \in H$ is expressed by $u = v + w$, where $v = P_3u$, $w = (I - P_3)u$. Hence, equation (2.1) is equivalent to the system

$$A_\delta w + (I - P_3)(b(v+w)^+ - a(v+w)^-) = 0, \tag{2.5}$$
$$A_\delta v + P_3(b(v+w)^+ - a(v+w)^-) = \alpha\phi_1 + \beta\phi_2 + \gamma\phi_2. \tag{2.6}$$

Using the Lyapunov-Schmidt reduction method and the contraction mapping principle, we can easily obtain the following lemma.

Lemma 2.2 . *Suppose that condition (2.3) or (2.4) hold, then, for fixed $v \in V_3$, equation (2.5) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the L^2-norm) in terms of v.*

By this lemma, the study of existence of solutions for equation (2.1) is reduced to that of equation (2.6). Namely, we need to study only the problem

$$A_\delta v + P_3(b(v+\theta(v))^+ - a(v+\theta(v))^-) = \alpha\phi_1 + \beta\phi_2 + \gamma\phi_2. \tag{2.7}$$

on the three dimensional subspace V_3 spanned by $\{\phi_1, \phi_2, \phi_3\}$.

Now, define the mapping $\Phi : V_3 \to V_3$ by

$$\Phi(v) = A_\delta v + P_3(b(v+\theta(v))^+ - a(v+\theta(v))^-),$$

and the mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(s_1, s_2, s_3) = (t_1, t_2, t_3)$$

where $v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3$, and $\Phi(v) = t_1\phi_1 + t_2\phi_2 + t_3\phi_3$. By Lemma 2.2, we know that Φ and F are continuous on V_3 and $\mathbb{R}^3$, respectively. On the other hand, we have the following Lemma.

Lemma 2.3 $\Phi(cv) = c\Phi(v)$ *for $c \ge 0$.*

Proof. Let $c \geq 0$. If v satisfies

$$A_\delta \theta(v) + (I - P_3)k(v + \theta(v))^+ = 0,$$

then

$$A_\delta (c\theta(v)) + (I - P_3)k(cv + c\theta(v))^+ = 0,$$

and hence, $\theta(cv) = c\theta(v)$. Therefore, we have

$$\begin{aligned} \Phi(cv) &= A_\delta(cv) + P_3 k(cv + \theta(cv))^+ \\ &= A_\delta(cv) + P_3 k(cv + c\theta(v))^+ \\ &= c\Phi(v). \end{aligned}$$

Lemma 2.3 implies that Φ or F maps a ray to a ray and a cone with vertex 0 onto a cone with vertex 0. Set

$$\begin{aligned} &C_1 = \{S = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1 \geq 0, s_2^2 + s_3^2 \leq s_1^2\}, \\ &C_2 = \{S = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1 \leq 0, s_2^2 + s_3^2 \leq s_1^2\}, \\ &C_3 = \mathbb{R}^3 \backslash (C_1 \cup C_2). \end{aligned}$$

and

$$\begin{aligned} &D_1 = \{v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3 : S = (s_1, s_2, s_3) \in C_1\}, \\ &D_2 = \{v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3 : S = (s_1, s_2, s_3) \in C_2\}, \\ &D_3 = V_3 \backslash (D_1 \cup D_2). \end{aligned}$$

Lemma 2.4 *If $v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3$, then*

1. *$v \geq 0$ if only if $v \in D_1$,*
2. *$v \leq 0$ if only if $v \in D_2$.*

The proof of this lemma is simple and we omit it.

Lemma 2.5 *Suppose that v and $\theta(v)$ satisfy (2.5) and (2.7). If v is a sign-changing solution of (2.7), then $u = v + \theta(v)$ is a sign-changing solution of (2.1).*

Proof. If u is a single-sign solution, then $u^+ = u$ or $u^+ = 0$. By (2.5) and Lemma 2.2, we know $\theta(v) = 0$, and hence $v = u$ is a single-sign solution, which is a contradiction. □

3 Uniqueness of the solution

In this section, we consider the general equation

$$A_\delta u + bu^+ - au^- = h, \tag{3.1}$$

where $h(x, t) \in H$. By the contraction mapping principle, we obtain the following uniqueness result for (3.1).

Theorem 3.1 *If the constant $\delta \neq 0$ and the constants a, b satisfy*

$$-1 < a, b < 3 + \delta^2, \tag{3.2}$$

then equation (3.1) has a unique solution.

The proof of this theorem is standard and we omit it here.

Remark. Condition (3.2) should be replaced by $-1 < k < 3$ when $\delta \to 0$ and $a = 0, b = k$. The condition $-1 < k < 3$ was used in [16] to ensure the uniqueness of solution to (1.3) when $\delta = 0$.

4 The nonlinearity crosses one eigenvalue

In this section, we consider equation (2.1) under condition (2.3). From the discussions in section 2, to consider equation (2.1) we need firstly to investigate the image of the cones C_1, C_2 and C_3 under F and the image of the cones D_1, D_2 and D_3 under Φ. Set

$$\begin{aligned}
\Theta_1 &= \big\{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 \geq 0, t_2^2 + t_3^2 \leq \frac{(b-3)^2 + 4\delta^2}{(b+1)^2} t_1^2\big\}, \\
\Theta_2 &= \big\{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 \leq 0, t_2^2 + t_3^2 \leq \frac{(a-3)^2 + 4\delta^2}{(a+1)^2} t_1^2\big\}, \\
\Theta_3 &= \mathbb{R}^3 \backslash (\Theta_1 \cup \Theta_2).
\end{aligned}$$

and

$$\begin{aligned}
\Omega_1 &= \{v = t_1\phi_1 + t_2\phi_2 + t_3\phi_3 : (t_1, t_2, t_3) \in \Theta_1\}, \\
\Omega_2 &= \{v = t_1\phi_1 + t_2\phi_2 + t_3\phi_3 : (t_1, t_2, t_3) \in \Theta_2\}, \\
\Omega_3 &= V_3 \backslash (\Omega_1 \cup \Omega_2).
\end{aligned}$$

Suppose that a, b, δ satisfy (2.3) and

$$b > 1 + \frac{1}{2}\delta^2 > a, \tag{4.1}$$

then we have

$$\frac{(b-3)^2 + 4\delta^2}{(b+1)^2} < 1 \quad \text{and} \quad \frac{(a-3)^2 + 4\delta^2}{(a+1)^2} > 1.$$

Denote

$$\begin{aligned}
C_4 &= C_1 \backslash \overline{\Theta_1}, \quad C_5 = \overline{\Theta_2} \backslash C_2, \\
D_4 &= D_1 \backslash \overline{\Omega_1}, \quad D_5 = \overline{\Omega_2} \backslash D_2.
\end{aligned}$$

By Lemma 2.4, we have that $v \geq 0$ for all $v \in D_4$ and v changes sign for every $v \in D_5$.

Lemma 4.1

$$\begin{aligned} F(C_1)=\Theta_1, \quad F(C_2)=\Theta_2, \quad C_4\subset\Theta_3\subset F(C_3),\\ \Phi(D_1)=\Omega_1, \quad \Phi(D_2)=\Omega_2, \quad D_4\subset\Omega_3\subset \Phi(D_3). \end{aligned}$$

Proof. Suppose $(s_1,s_2,s_3)\in C_1$, then $v=s_1\phi_1+s_2\phi_2+s_3\phi_3\in D_1$. By Lemmas 2.2 and 2.4, we know $v\geq 0$ and $\theta(v)=0$. At the same time, we obtain

$$\begin{aligned} A_\delta v+P_3k(v+\theta(v))^+ &= A_\delta(s_1\phi_1+s_2\phi_2+s_3\phi_3)+k(s_1\phi_1+s_2\phi_2+s_3\phi_3)\\ &= (\phi_1,\phi_2,\phi_3)\begin{pmatrix} b+1 & 0 & 0\\ 0 & b-3 & 2\delta\\ 0 & -2\delta & b-3\end{pmatrix}\begin{pmatrix} s_1\\ s_2\\ s_3\end{pmatrix}; \end{aligned}$$

therefore, for all $(s_1,s_2,s_3)\in C_1$,

$$F(s_1,s_2,s_3)=\begin{pmatrix} b+1 & 0 & 0\\ 0 & b-3 & 2\delta\\ 0 & -2\delta & b-3\end{pmatrix}\begin{pmatrix} s_1\\ s_2\\ s_3\end{pmatrix}.$$

Assuming that $(s_1,s_2,s_3)\in C_2$, in the same way, we obtain

$$\begin{aligned} A_\delta v+P_3k(v+\theta(v))^+ &= A_\delta(s_1\phi_1+s_2\phi_2+s_3\phi_3)\\ &= (\phi_1,\phi_2,\phi_3)\begin{pmatrix} a+1 & 0 & 0\\ 0 & a-3 & 2\delta\\ 0 & -2\delta & a-3\end{pmatrix}\begin{pmatrix} s_1\\ s_2\\ s_3\end{pmatrix}. \end{aligned}$$

Therefore, for all $(s_1,s_2,s_3)\in C_1$,

$$F(s_1,s_2,s_3)=\begin{pmatrix} a+1 & 0 & 0\\ 0 & a-3 & 2\delta\\ 0 & -2\delta & a-3\end{pmatrix}\begin{pmatrix} s_1\\ s_2\\ s_3\end{pmatrix}.$$

It follows that $F(C_1)=\Theta_1$, $F(C_2)=\Theta_2$ and $\Phi(D_1)=\Omega_1$, $\Phi(D_2)=\Omega_2$. Moreover, By Lemma 2.3 and the continuity of F, we have

$$\Theta_3\subset F(C_3), \quad \Omega_3\subset\Phi(D_3).$$

The proof of this Lemma is complete. $\square$

Now, combining Lemmas 2.2, 2.4, 2.5, and 4.1, we obtain the following result.

Theorem 4.2 *Let $h(x,t)=\alpha\cos x+\beta\cos 2t\cos x+\gamma\sin 2t\cos x$. Suppose that δ,a,b satisfy $\delta>0$, (2.3) and (4.1), then equation (2.1), and hence problem (1.4) has a solution satisfying:*

1. *If $\alpha>0$ and*
$$\beta^2+\gamma^2\leq\frac{(b-3)^2+4\delta^2}{(b+1)^2}\alpha^2,$$
then $h>0$ and the corresponding solution is positive.

2. *If $\alpha > 0$ and*
$$\frac{(b-3)^2+4\delta^2}{(b+1)^2}\alpha^2 < \beta^2+\gamma^2 < \alpha^2,$$
then $h > 0$ but the corresponding solution changes sign.

3. *If $\alpha < 0$ and*
$$\alpha^2 < \beta^2+\gamma^2 \le \frac{(a-3)^2+4\delta^2}{(a+1)^2}\alpha^2,$$
then h changes sign but the corresponding solution is negative.

4. *If $\alpha < 0$ and $\beta^2+\gamma^2 \le \alpha^2$, then $h < 0$ and the corresponding solution is negative.*

5. *Elsewhere, the function h changes sign and the corresponding solution changes sign.*

Remark. By Theorem 4.2, if $a = 0$, $b = k \to 3$, and $\delta \to 0$, then for almost every $h > 0$, problem (1.4) has sign-changing periodic solution. If the damping δ is large enough such that $\delta^2 > 2k-2$ and condition (2.3) is satisfied, then when
$$\alpha^2 < \beta^2+\gamma^2 \le \frac{(k-3)^2+4\delta^2}{(k+1)^2}\alpha^2,$$
the corresponding solution is positive though h changes sign.

This result answers partially the problem mentioned in section 1.

5 The nonlinearity crosses two eigenvalues

In this section, we consider equation (2.1) under condition (2.4). In addition, we suppose
$$b > 1+\frac{1}{2}\delta^2. \tag{5.1}$$
As in section 4, we first investigate the image of the cones C_1, C_2 and C_3 under F and the image of the cones D_1, D_2 and D_3 under Φ. Set
$$\begin{aligned}
\Theta_1 &= \big\{(t_1,t_2,t_3)\in\mathbb{R}^3 : t_1\ge 0, t_2^2+t_3^2 \le \frac{(b-3)^2+4\delta^2}{(b+1)^2}t_1^2\big\},\\
\Theta_6 &= \big\{(t_1,t_2,t_3)\in\mathbb{R}^3 : t_1\ge 0, t_2^2+t_3^2 \le \frac{(a-3)^2+4\delta^2}{(a+1)^2}t_1^2\big\},\\
\Theta_7 &= \overline{\Theta_6}\backslash\overline{\Theta_1}.
\end{aligned}$$
and
$$\begin{aligned}
\Omega_1 &= \{v = t_1\phi_1+t_2\phi_2+t_3\phi_3 : (t_1,t_2,t_3)\in\Theta_1\},\\
\Omega_6 &= \{v = t_1\phi_1+t_2\phi_2+t_3\phi_3 : (t_1,t_2,t_3)\in\Theta_6\},\\
\Omega_7 &= \overline{\Omega_6}\backslash\overline{\Omega_1}.
\end{aligned}$$

Lemma 5.1 *For every $v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3 \in V_3$, there exists a constant $d > 0$ such that $(\Phi(v), \phi_1) \geq d|s_2 + s_3|$.*

Proof. Note that

$$bu^+ - au^- + u = (b+1)u^+ - (a+1)u^- \geq \min\{b+1, -(a+1)\}|u| = c|u|$$

and

$$\phi_1 \geq \frac{1}{\sqrt{2}}|\phi_2 + \phi_3|.$$

Let $v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3$. Then we have

$$\begin{aligned}
(\Phi(v), \phi_1) =& (A_\delta(s_1\phi_1 + s_2\phi_2 + s_3\phi_3) + P_3(b(v+\theta(v))^+ - a(v+\theta(v))^-), \phi_1) \\
=& (\phi_1, v + \theta(v) + b(v+\theta(v))^+ - a(v+\theta(v))^-) \\
\geq& (\phi_1, c|v + \theta(v)|) \\
=& c\int_\Omega \phi_1 |s_1\phi_1 + s_2\phi_2 + s_3\phi_3 + \theta(v)| dtdx \\
\geq& \frac{c}{\sqrt{2}} \int_\Omega |\phi_2 + \phi_3||s_1\phi_1 + s_2\phi_2 + s_3\phi_3 + \theta(v)| dtdx \\
\geq& \frac{c_1}{\sqrt{2}}|s_2 + s_3|.
\end{aligned}$$

Taking $d = c_1/\sqrt{2}$, the conclusion follows. $\square$

Note that Lemma 5.1 implies $t_1 \geq 0$ for every $v = s_1\phi_1 + s_2\phi_2 + s_3\phi_3 \in V_3$ and $\Phi(v) = t_1\phi_1 + t_2\phi_2 + t_3\phi_3$.

Lemma 5.2

$$\begin{aligned}
&F(C_1) = \Theta_1, \quad F(C_2) = \Theta_6, \quad \Theta_7 \subset F(C_3), \\
&\Phi(D_1) = \Omega_1, \quad \Phi(D_2) = \Omega_6, \quad \Omega_7 \subset \Phi(D_3).
\end{aligned}$$

The proof of this lemma follows by similar calculations as in Lemma 4.1 with $a < -1$. Now we can obtain the main result in this section.

Theorem 5.3 *Let $h(x,t) = \alpha \cos x + \beta \cos 2t \cos x + \gamma \sin 2t \cos x$. Suppose that δ, a, b satisfy $\delta > 0$, (2.4) and (5.1), then we have:*

1. *If $h \in \overline{\Omega_1}$, (2.1) has a positive solution, and a negative solution.*
2. *If h belongs to interior of Ω_7, (2.1) has a negative solution and at least one sign-changing solution.*
3. *If $h \in \partial\Omega_6$, (2.1) has a negative solution.*

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