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A REDUCTION METHOD FOR PROVING THE EXISTENCE OF SOLUTIONS TO ELLIPTIC EQUATIONS INVOLVING THE p-LAPLACIAN

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Abstract. We introduce a reduction method for proving the existence of solutions to elliptic equations involving the p-Laplacian operator. The existence of solutions is implied by the existence of a positive essentially weak subsolution on a manifold and the existence of a positive supersolution on each compact domain of this manifold. The existence and nonexistence of positive supersolutions is given by the sign of the first eigenvalue of a nonlinear operator.

1. Introduction

Let (M, g) be a complete non-compact Riemannian manifold of dimension $n \geq 3$. On this manifold, we consider the elliptic quasilinear equation

$$\Delta_p u + ku^{p-1} - Ku^q = 0, \tag{1.1}$$

with $q > p - 1$, where $K \geq 0$ and $k \leq K$ are smooth functions on the manifold M and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-laplacian operator of u.

Under some positivity assumption on the function K, we reduce the existence of a weak positive solution to (1.1) on M to the existence of a positive essentially weak subsolution on M together with the existence of a positive supersolution on each compact subdomain of M. The difficulty we face using the method of sub and supersolutions resides in seeking a positive subsolution $\underline{u}$ and a positive supersolution $\overline{u}$ that at the same time satisfy the condition $\underline{u} \leq \overline{u}$. Our reduction method makes easier the analysis of (1.1) on general complete non-compact manifolds. This result extends the case studied by Peter Li et al [2] for the Laplace-Beltrami operator (i.e. $p = 2$).

In the third section, we show that the existence and the nonexistence of positive supersolutions to (1.1) on arbitrary bounded subdomains of M is completely determined by the sign of the first eigenvalue of the non-linear operator $L_p u = -\Delta_p u - k|u|^{p-2} u$ on the zero set $Z_o = \{x \in M : K(x) = 0\}$ of the function K. This property was also obtained in [2] for the Laplace-Beltrami operator.

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2. Reduction Result

Definition 2.1. A positive and smooth function K is said to be essentially positive if there exists an exhaustion by compact domains $\{\Omega_i\}_{i\geq 0}$ such that

$$M = \cup_{i\geq 0}\Omega_i \quad \text{and} \quad K\big|_{\partial\Omega_i} > 0 \quad \forall i \geq 0.$$

Furthermore, If there is a positive weak supersolution $u_i \in H_1^p(\Omega_i) \cap C^o(\Omega_i)$ on each Ω_i, then K is called permissible.

Definition 2.2. A positive solution u of the equation (1.1) is said to be maximal if for every positive solution v, we have $v \leq u$.

In this section, we prove the following theorem.

Theorem 2.3. *Suppose that K is permissible and $k \leq K$. If there exists a positive subsolution $\underline{u} \in H^p_{1,\mathrm{loc}}(M) \cap L^\infty(M) \cap C^o(M)$ of (1.1) on M, then it has a weak positive and maximal solution $u \in H_1^p(M)$. Moreover u is of class $C^{1,\alpha}$ on each compact set for some $\alpha \in (0,1)$.*

To prove this theorem, we show the following lemmas.

Lemma 2.4. *Let $\Omega \subset M$ be a bounded domain. Assume that (1.1) has a positive subsolution $\underline{u} \in H^p_{1,\mathrm{loc}}(\Omega) \cap C^o(\Omega)$ and a positive supersolution $\overline{u} \in H^p_{1,\mathrm{loc}}(\Omega)$. If $(\overline{u} - \underline{u})\big|_{\partial\Omega} \geq 0$ then $\overline{u} \geq \underline{u}$ on Ω.*

Proof. First, we note that multiplying a positive supersolution $\overline{u}$ of (1.1) by a constant $a \geq 1$ we get a supersolution. Indeed,

$$\begin{aligned}\Delta_p(au) + k(au)^{p-1} - K(au)^q &= a^{p-1}\left(\Delta_p u + ku^{p-1}\right)u^q - K(au)^q \\ &\leq a^{p-1}Ku^q\left(1 - a^{q-p+1}\right) \\ &\leq 0.\end{aligned}$$

So we can assume without loss of generality that $\overline{u} \geq 1$ on a compact domain. Suppose that the set $S = \{x \in \Omega : \overline{u}(x) < \underline{u}(x)\}$ is not empty. Let $\phi = \max(\underline{u} - \overline{u}, 0)$ be the test function which is positive and belongs to $H^p_{1,0}(\Omega)$. We have,

$$\begin{aligned}&\int_S \left\langle |\nabla\underline{u}|^{p-2}\nabla\underline{u} - |\nabla\overline{u}|^{p-2}\nabla\overline{u}, \nabla(\underline{u} - \overline{u})\right\rangle dv_g \\ &\leq \int_S (k(\underline{u}^{p-1} - \overline{u}^{p-1})(\underline{u} - \overline{u}) - K(\underline{u}^q - \overline{u}^q))(\underline{u} - \overline{u})dv_g \\ &\leq \int_S K(\underline{u}^{p-1} - \overline{u}^{p-1} - \underline{u}^q + \overline{u}^q))(\underline{u} - \overline{u})dv_g \\ &\leq \int_S K(\underline{u}^{p-1}(1 - \underline{u}^{q-p+1}) - \overline{u}^{p-1}(1 - \overline{u}^{q-p+1}))(\underline{u} - \overline{u})dv_g \\ &\leq \int_S K(\underline{u}^{p-1}(1 - \underline{u}^{q-p+1}) - \overline{u}^{p-1}(1 - \underline{u}^{q-p+1}))(\underline{u} - \overline{u})dv_g \\ &\leq \int_S K(1 - \underline{u}^{q-p+1})(\underline{u}^{p-1} - \overline{u}^{p-1})(\underline{u} - \overline{u})dv_g \\ &\leq 0 \quad (q - p + 1 > 0).\end{aligned}$$

If $p \geq 2$, by Simon inequality there exists a positive constant $C_p > 0$ such that

$$C_p \int_S |\nabla(\underline{u} - \overline{u})|^p dv_g \leq \int_S \left\langle |\nabla\underline{u}|^{p-2}\nabla\underline{u} - |\nabla\overline{u}|^{p-2}\nabla\overline{u}, \nabla(\underline{u} - \overline{u})\right\rangle dv_g \leq 0.$$

Hence,

$$\left\|(\underline{u}-\overline{u})^{+}\right\|_{H^{p}_{1,0}(\Omega)}=\int_{\Omega}\left|\nabla(\underline{u}-\overline{u})^{+}\right|^{p}dv_{g}=0$$

i.e. $(\underline{u}-\overline{u})^{+}=0$, or $\underline{u}\le\overline{u}$ on Ω.

For $1<p<2$, there exists by the same inequality there exists a positive constant $C'_p>0$ such that

$$C'_p\int_S\frac{|\nabla(\underline{u}-\overline{u})|^2}{(|\nabla\underline{u}|+|\nabla\overline{u}|)^{2-p}}dv_g\le\int_S\big\langle\,|\nabla\underline{u}|^{p-2}\,\nabla\underline{u}-|\nabla\overline{u}|^{p-2}\,\nabla\overline{u},\nabla(\underline{u}-\overline{u})\big\rangle dv_g\le 0$$

that is

$$\int_S\frac{|\nabla(\underline{u}-\overline{u})|^2}{(|\nabla\underline{u}|+|\nabla\overline{u}|)^{2-p}}\,dv=0. \tag{2.1}$$

It follows from the Hölder inequality that,

$$\begin{aligned}\int_S|\nabla(\underline{u}-\overline{u})|^p\,dv_g&=\int_S\frac{|\nabla(\underline{u}-\overline{u})|^p}{(|\nabla\underline{u}|+|\nabla\overline{u}|)^{p(1-\frac{p}{2})}}((|\nabla\underline{u}|+|\nabla\overline{u}|)^{p(1-\frac{p}{2})}dv_g\\&\le\Big(\int_S\frac{|\nabla(\underline{u}-\overline{u})|^2}{(|\nabla\underline{u}|+|\nabla\overline{u}|)^{2-p}}dv_g\Big)^{p/2}\Big(\int_S(|\nabla\underline{u}|+|\nabla\overline{u}|)^p)^{1-\frac{p}{2}}dv_g\Big).\end{aligned}$$

By (2.1), we get

$$\left\|(\underline{u}-\overline{u})^{+}\right\|_{H^{p}_{1,0}(\Omega)}=\int_{\Omega}\left|\nabla(\underline{u}-\overline{u})^{+}\right|^{p}dv_{g}=0\,.$$

Hence $\underline{u}\le\overline{u}$ on Ω. □

Let $H^n(-1)$ be the n-dimensional simply connected hyperbolic space of sectional curvature equals to -1.

Lemma 2.5. *Let $\varepsilon>0$, $\beta>0$ and λ constants, then there exists a positive and increasing function ϕ_ε such that the function $V_\varepsilon(x)=\phi_\epsilon(r(x))$, defined on the geodesic ball $B(\varepsilon)\subset H^n(-1)$ satisfies*

$$\Delta_pV_\varepsilon+\lambda V_\varepsilon^{p-1}-\beta V_\varepsilon^q\le 0,$$
$$V_\varepsilon\big|_{\partial B(\varepsilon)}=\infty.$$

Here $r(x)$ is the distance function on the ball $B(\varepsilon)$

Proof. In polar coordinates, the metric of $H^n(-1)$ is

$$ds^2=dr^2+\sinh^2(r)W^2$$

where W^2 is the metric on the sphere S^{n-1}. We get easily

$$\Delta_{H^n(-1)}=\frac{\partial^2}{\partial r^2}+(n-1)\coth(r)\frac{\partial}{\partial r}+\frac{1}{\sinh^2(r)}\Delta_{S^{n-1}}$$

where Δ_M is the Laplace-Beltrami operator on the manifold M. and

$$\Delta_pu=|\nabla u|^{p-2}\Delta_Mu+\big\langle\nabla u,\nabla|\nabla u|^{p-2}\big\rangle.$$

For $p\in(1,n)$, let $\Delta_p^Mu=\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ be the p-Laplacian operator of u on the manifold M. For $q>p-1$ we consider the function $\phi:(0,\varepsilon)\to R$,

$$\phi(r)=\Big(\sinh^2(\frac{\varepsilon}{2})-\sinh^2(\frac{r}{2})\Big)^{-\alpha},$$

with $\alpha = \frac{p}{q-p+1}$. Setting

$$a(r) = \sinh^2(\frac{\varepsilon}{2}) - \sinh^2(\frac{r}{2}), \quad V(x) = \phi\left(r(x)\right),$$

we obtain

$$\Delta_p^{H^n(-1)} V = \phi'^{p-2} \Delta_{H^n(-1)} V + (p-2)\phi'^{p-2}\phi''. \tag{2.2}$$

A direct computation shows that

$$\Delta_{H^n(-1)} V = \frac{1}{4}\alpha\left(\alpha+1\right) a(r)^{-(\alpha+2)} \sinh^2(r) + \frac{1}{2} n\alpha a(r)^{-(\alpha+1)} \cosh(r)$$

Therefore,

$$\begin{aligned}\Delta_p^{H^n(-1)} V + \lambda V^{p-1} = \Big(\frac{\alpha}{2}\Big)^{p-1} & a(r)^{-\alpha p+\alpha-p} \big[\frac{1}{2}\left(p-1\right)\left(\alpha+1\right)\sinh^p(r) \\ & + (n+p-2)a(r)\sinh^{p-2}(r)\cosh(r) + \lambda a(r)^p\big].\end{aligned}$$

Taking

$$\begin{aligned}C\left(\varepsilon, \lambda, p, q\right) = & \frac{1}{2}(p-1)\left(\alpha+1\right)\Big(\frac{\alpha}{2}\Big)^{p-1} \sinh^p\left(\varepsilon\right) \\ & + (n+p-2)\Big(\frac{\alpha}{2}\Big)^{p-1} a\left(0\right)\cosh\left(\varepsilon\right) + \lambda\left(a(0)\right)^p,\end{aligned}$$

we obtain $\Delta_p^{H^n(-1)} V + \lambda V^{p-1} \le C V^q$ and putting

$$\psi = \big(\frac{C}{\beta}\big)^{1/(q-p+1)} \phi, \tag{2.3}$$

we obtain the desired function. $\square$

Lemma 2.6. *Let Ω be a bounded domain. Suppose that there exists a compact domain $X \subset \Omega$ such that $K\big|_{\partial X} > 0$, then there exists a constant $C > 0$ such that for any positive regular solution u of* (1.1) *on Ω, we have $u\big|_{\partial X} \le C$, where ∂X is the boundary of X.*

Proof. Since $X \subset \Omega$ is compact, it follows that there exist a positive constant $\varepsilon > 0$ less than the injectivity radius of X and a positive constant $\beta > 0$ such that the ε-neighborhood of ∂X, $U_\varepsilon(\partial X)$ is contained in Ω and

$$K\big|_{U_\varepsilon(\partial X)} \ge \beta > 0. \tag{2.4}$$

Let $x_0 \in \partial X$ and let $r_o(x) = \operatorname{dist}(x_0, x)$ be the distance function on the geodesic ball $B(x_0, \varepsilon)$. Let Δ_p^M be the p-lapalcian operator on the manifold M. Let $\lambda = \sup_{x\in\Omega} k(x)$. By Lemma 2.5, there exists a positive and increasing function $V(x) = \phi_\varepsilon(r_o(x))$ defined on the geodesic ball $B(\varepsilon) \subset H^n(-1)$ satisfying

$$\Delta_p^{H^n(-1)} V_\varepsilon + \lambda V_\varepsilon^{p-1} \le \beta V_\varepsilon^q. \tag{2.5}$$

Since Ω is bounded, by rescaling the metric if necessary, we can assume that

$$\operatorname{Ricci}\big|_\Omega \ge -(n-1).$$

Knowing that the gradient of the distance function satisfies $|\nabla r| = 1$, we have

$$\Delta_p^M r = \Delta^M r\,.$$

By a geometric comparison argument, we have

$$\Delta_p^M r \le \Delta_p^{H^n(-1)} r. \tag{2.6}$$

On the other hand,

$$\Delta_M V_\varepsilon = \operatorname{div}\left(\nabla \phi_\varepsilon(r(x))\right) = \phi'_\varepsilon \Delta_M r + \phi''_\varepsilon.$$

Then

$$\Delta_p^M V_\varepsilon = {\phi_\varepsilon}'^{p-2} \Delta_M V_\varepsilon + (p-2)\, {\phi'_\varepsilon}^{p-2} \phi''_\varepsilon$$

and

$$\Delta_p^M V_\varepsilon = {\phi_\varepsilon'}^{p-1} \Delta_M r + (p-1)\, {\phi_\varepsilon'}^{p-2} \phi''_\varepsilon.$$

By the inequality (2.6), we have

$$\Delta_p^M V_\varepsilon \le \Delta_p^{H^n(-1)} V_\varepsilon$$

and from the inequalities (2.4) and (2.5), we deduce that

$$\Delta_p^M V_\varepsilon + k V_\varepsilon^{p-1} - K V_\varepsilon^q \le \Delta_p^{H^n(-1)} V_\varepsilon + \lambda V_\varepsilon^{p-1} - \beta V_\varepsilon^q \le 0.$$

which implies that V_ε is a positive supersolution of the equation(1.1) on $B(x_0,\varepsilon)$. Since $V_\varepsilon\big|_{\partial B(x_0,\varepsilon)} = \infty$, Lemma 2.4 shows that for any solution u of the equation (1.1), we have

$$u(x) \le V_\varepsilon(x) \ \forall x \in B(x_0,\varepsilon)$$

hence

$$u(x_0) \le V_\varepsilon(x_0) = \phi_\varepsilon(0) = C,$$

where C is a positive constant independent of x_0 and u. □

Lemma 2.7. *Let $\Omega \subset M$ be a bounded domain. Suppose that $K\big|_{\partial\Omega} > 0$ and there is a positive and bounded solution $v \in H_1^p(\Omega) \cap L^\infty(\Omega)$ of the equation (1.1) such that v is bounded from below by a positive constant. Then there exists a positive weak solution u of the boundary-value problem*

$$\begin{gathered} \Delta_p u + k u^{p-1} - K u^q = 0 \quad \text{on } \Omega \\ u = \infty \quad \text{on } \partial\Omega \end{gathered}$$

and $u \ge v$ on Ω. Moreover $u \in C^{1,\alpha}(X)$ on each compact $X \subset \Omega$, and some $\alpha \in (0,1)$.

Proof. Let $C = \inf_\Omega v$ (which is positive by hypothesis). Since v is bounded from above on Ω then there exists $n_0 \in N^*$ such that $\sup_\Omega v \le n_0 C$. Consider the boundary-value problem

$$\begin{gathered} \Delta_p u + k u^{p-1} - K u^q = 0 \quad \text{on } \Omega \\ u = nC\,, \quad n \ge n_0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.7}$$

Obviously, $v \in H_1^p(\Omega) \cap L^\infty(\Omega)$ and $nv \in H_1^p(\Omega) \cap L^\infty(\Omega)$ are respectively positive sub and supersolutions of problem (2.7), and hence by the sub and supersolutions method, the problem (2.7) has for each $n \ge n_0$ a positive solution $v_n \in H_1^p(\Omega) \cap L^\infty(\Omega)$ such that $v \le v_n \le nv$. Since $(v_{n+1} - v_n)\big|_{\partial\Omega} = C > 0$, it follows from Lemma 2.4 that $\{v_n\}_{n \ge n_0}$ in an increasing sequence of positive solutions of the equation (1.1) on Ω. Consider the set

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

and setting $X = \overline{\Omega}_\varepsilon \subset \Omega$, which is compact, then by Lemma 2.6 there exists for each $\varepsilon > 0$ (small enough) a constant $C(\varepsilon) > 0$ such that

$$\sup_{\partial\Omega_\varepsilon} v_n \le C(\varepsilon) \ \forall n \ge n_0. \tag{2.8}$$

Consider the function $u = C(\varepsilon)\, C^{-1} v$ and take $C(\varepsilon)$ such that $C(\varepsilon)\, C^{-1} > 1$, so that u is a positive supersolution of the equation (1.1). Since $(u - v_n)\,|_{\partial\Omega_\varepsilon} \geq 0$, it follows from Lemma 2.4 that $v_n \leq C(\varepsilon)\, C^{-1} v$ on Ω_ε for all $n \geq n_0$, and then $\{v_n\}_{n \geq n_0}$ is uniformly bounded on compact subsets of Ω. Hence$\{v_n\}_{n \geq n_0}$, converges in the distribution sense to a weak positive solution u of the equation (1.1) on Ω. By the regularity theorem $u \in C^{1,\alpha}(\Omega_\epsilon)$ for some $\alpha \in (0,1)$. It obvious that $u|_{\partial\Omega} = \infty$. ☐

Proof of Theorem 2.3. Let $\underline{u} \in H^p_{1,loc}(M) \cap L^\infty(M) \cap C^o(M)$ a positive subsolution of the equation (1.1) on M. Since K is permissible then there exists an increasing sequence of compact domains $\{\Omega_i\}_{i \geq 0}$ such that $M = \cup_i \Omega_i$ and $K\big|_{\partial\Omega_i} > 0$ for all $i \geq 0$ and a positive supersolution $\overline{u}_i \in H^p_1(\Omega_i) \cap C^o(\Omega_i)$ on each Ω_i. Since $\alpha\overline{u}$ (where α is a constant greater than 1) is again a positive supersolution of the equation (1.1) on Ω_i, we can assume that $\overline{u}_i \geq \underline{u}$ on Ω_i. Hence by the method of sub and supersolutions there exists a positive solution $u_i \in C^{1,\alpha}(\Omega_i)$ of the equation (1.1) such that $\underline{u} \leq u_i \leq \overline{u}_i$. Since u_i is bounded from below by $\underline{u}$ and Ω_i is compact, then u_i is bounded from below by a positive constant, thus it follows from Lemma 2.7 that there exists a positive $C^{1,\alpha}(\Omega_i)$-solution still denoted by u_i of the boundary-value problem

$$\begin{gathered}\Delta_p u_i + k u_i^{p-1} - K u_i^q = 0 \quad \text{in } \Omega_i \\ u_i = \infty \quad \text{on } \partial\Omega_i\,.\end{gathered}$$

Since for each $i_0 \geq 1$ we have $(u_{i+1} - u_i)\big|_{\partial\Omega_{i_0}} \leq 0$, Lemma 2.4 implies that $\{u_i\}_{i \geq i_0}$ is a decreasing sequence of positive solutions of the equation (1.1) on Ω_{i_0}. Moreover, all u_i are bounded from below by $\underline{u}$, thus the sequence $\{u_i\}_{i \geq i_0}$ converges in distribution sense to a weak solution of (1.1). By regularity theorem $u \in C^{1,\alpha}(\Omega_i)$ for some $\alpha \in (0,1)$.

Now, if v is an other solution of the equation (1.1) on $M = \underset{i}{\cup}\Omega_i$, then for $x_0 \in M$ there exist $i_0 \geq 1$ such that $x_0 \in \Omega_i$ for all $i \geq i_0$, as $u_i|_{\partial\Omega_i} = \infty$, Lemma 2.4 implies that $v \leq u_i$ for all $i \geq i_0$. In particular $v \leq \lim_{i\to\infty} u_i = u$. Thus u is maximal. ☐

3. Existence of supersolution

Let $K \geq 0$ and k be smooth functions on the manifold M. In this section we show that the existence or the nonexistence of a positive supersolution on a bounded domain $\Omega \subset M$ is completely determined by the sign of the first eigenvalue of the non linear operator $L_p u = -\Delta_p u - k|u|^{p-2}u$ on the zero set $Z = \{x \in \Omega : K(x) = 0\}$ of the function K. Let us recall some definitions first.

Definition 3.1. Let $\Omega \subset M$ be a bounded and smooth open set. The first eigenvalue of the non linear operator $L_p u = -\Delta_p u - k|u|^{p-2}u$ on Ω is

$$\lambda^\Omega_{1,p} = \inf \Big(\int_\Omega (|\nabla u|^p - k|u|^p)\, dv_g \Big) \tag{3.1}$$

where the infimum is taken over all functions $u \in H^p_{1,0}(\Omega)$ such that $\int_\Omega |u|^p dv_g = 1$.

Definition 3.2. Let $S \subset M$ be a bounded subset. The first eigenvalue of the non linear operator $L_p u = -\Delta_p u - k|u|^{p-2}u$ on Ω is

$$\lambda^S_{1,p} = \sup \lambda^\Omega_{1,p} \tag{3.2}$$

where the *sup* is taken over all smooth open sets Ω containing S. In particular $\lambda_{1,p}^{\phi} = +\infty$.

Definition 3.3. Let $S \subset M$ be an unbounded subset. The first eigenvalue of the non-linear operator $L_p u = -\Delta_p u - k|u|^{p-2}u$ on Ω is

$$\lambda_{1,p}^{S} = \lim_{r \to +\infty} \lambda_{1,p}^{\Omega_r} \tag{3.3}$$

where $\Omega_r = S \cap \overline{B}\,(o, r)$ for all $r > 0$ and $o \in M$ a fixed point.

Let Ω be a bounded domain. It is known that there exists a unique $C^{1,\alpha}(\Omega)$-eigenfunction satisfying

$$\begin{gathered}
\Delta_p \phi + k\phi^{p-1} + \lambda_{1,p}^{\Omega_0}\phi^{p-1} = 0 \quad \text{in } \Omega \\
\phi > 0 \quad \text{in } \Omega \\
\phi = 0 \quad \text{on } \partial\Omega \\
\frac{\partial \phi}{\partial \nu} < 0 \quad \text{on } \partial\Omega\,.
\end{gathered}$$

Let $Z = \{x \in M : K(x) = 0\}$ the zero set of the smooth function K and $\lambda_{1,p}^{Z\cap\Omega}$ be the first eigenvalue of the non-linear operator $L_p u = -\Delta_p u - k|u|^{p-2}u$ on $\Omega \cap Z$.

Theorem 3.4. *Let $K \geq 0$ be a smooth function on a bounded domain Ω. If $\lambda_{1,p}^{Z\cap\Omega} > 0$, then there exists a positive supersolution $\overline{u} \in H_1^p(\Omega) \cap L^\infty(\Omega)$ of the equation* (1.1) *on Ω. Conversely if there exists a positive supersolution $\overline{u} \in H_1^p(\Omega) \cap L^\infty(\Omega)$ of the equation* (1.1) *then $\lambda_{1,p}^{Z\cap\Omega} \geq 0$.*

Proof. Let $\Omega \subset M$ be a bounded domain. Suppose that $\lambda_{1,p}^{Z\cap\Omega} > 0$, it follows from the continuity of the first eigenvalue with respect to C^0 deformation of the domain that there exists a bounded domain Ω_0 such that $Z \cap \Omega \subset \Omega_0 \subset \Omega$ and $\lambda_{1,p}^{\Omega_0} > 0$. On Ω_0 there exists a unique positive eigenfunction $\phi \in C^{1,\alpha}\left(\overline{\Omega}_0\right)$ such that

$$\begin{gathered}
\Delta_p \phi + k\phi^{p-1} + \lambda_{1,p}^{\Omega_0}\phi^{p-1} = 0 \quad \text{in } \Omega_0 \\
\phi > 0 \quad \text{in } \Omega_0 \\
\phi = 0 \quad \text{on } \partial\Omega_0 \\
\frac{\partial \phi}{\partial \nu} < 0 \quad \text{on } \partial\Omega_0\,.
\end{gathered}$$

Writting $\Omega = (\Omega \backslash \Omega_0) \cup (\Omega \cap \Omega_0)$ and setting

$$\overline{u} = \chi_{\Omega_0}\phi + C\left(1 - \chi_{\Omega_0}\right)$$

where χ_Ω is the characteristic function,

$$\chi_\Omega = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

and C is a positive constant large enough so that $\overline{u} = C$, on $\Omega - \Omega_o$, is a positive supersolution of (1.1). On $\Omega \cap \Omega_0$, $\overline{u} = \phi$, but

$$\Delta_p \overline{u} + k\overline{u}^{p-1} - K\overline{u}^q = -\lambda_{1,p}^{\Omega_0}\overline{u}^{p-1} - K\overline{u}^q \leq 0$$

because $\lambda_{1,p}^{\Omega_0} > 0$. Therefore, $\overline{u} \in H_1^p(\Omega) \cap L^\infty(\Omega)$ is positive supersolution of the equation (1.1) on Ω.

Conversely, suppose that there exists a positive supersolution $\overline{u} \in H_1^p(\Omega) \cap L^\infty(\Omega)$ of (1.1) and $\lambda_{1,p}^{Z\cap\Omega} < 0$. It follows again from the continuity of the first eigenvalue with respect to C^0-deformation of the domain that there exists a bounded domain Ω_1 such that $Z \cap \Omega \subset \Omega_1 \subset \Omega$ and $\lambda_{1,p}^{\Omega_1} < 0$. By the same way as above, we can find a decreasing sequence $\{\Omega_i\}_{i\geq 0}$ of bounded domains such that $\Omega_i \subset \Omega$, $Z \cap \Omega = \cap_i \Omega_i$ and $\lambda_{1,p}^{\Omega_i} < 0$. On Ω_i there exists a positive eigenfunction $\phi_i \in C^{1,\alpha}(\overline{\Omega}_i)$ and $\frac{\partial \phi_i}{\partial \nu} < 0$ on $\partial\Omega_i$ satisfying

$$\begin{gathered}\Delta_p \phi_i + k\phi_i^{p-1} + \lambda_{1,p}^{\Omega_i}\phi_i^{p-1} = 0 \quad \text{in } \Omega_i \\ \phi_i = 0 \quad \text{on } \partial\Omega_i\,.\end{gathered}$$

Consider the boundary-value problem, with $q > p-1$,

$$\begin{gathered}\Delta_p u_i + ku_i^{p-1} - Ku_i^{q-1} = 0 \quad \text{in } \Omega_i \\ u_i = 0 \quad \text{on } \partial\Omega_i\,.\end{gathered} \tag{3.4}$$

One can check that for $\varepsilon > 0$ small and $C > 0$ large, $\varepsilon\phi_i$ and $C\overline{u}$ are respectively positive sub and supersolutions of the boundary-value problem (3.4) and $\varepsilon\phi_i \leq C\overline{u}$. Therefore, by the sub and supersolutions method there exists a positive $C^{1,\alpha}$ solution u_i of the problem (3.4) such that $\varepsilon\phi_i \leq u_i \leq C\overline{u}$, we have also $\frac{\partial u_i}{\partial \nu} < 0$ on $\partial\Omega_i$. Thus $\frac{\phi_i}{u_i}$ and $\frac{u_i}{\phi_i} \in L^\infty(\Omega_i)$. Consider now the set $\Omega_{i,C} = \{x \in \Omega_i : C\phi_i(x) < u_i(x)\}$, It follows from [1, Lemma 2] that

$$\begin{aligned}0 &\leq \int_{\Omega_{i,C}} \Big(\frac{\Delta_p(C\phi_i)}{(C\phi_i)^{p-1}} + \frac{-\Delta_p u_i}{u_i^{p-1}}\Big)\,(u_i^p - (C\phi_i)^p)\,dv_g \\ &= -\int_{\Omega_{i,C}} \Big(\lambda_{1,p}^{\Omega_i} + Ku_i^{q-p+1}\Big)\,(u_i^p - (C\phi_i)^p)\,dv_g\,.\end{aligned}$$

For i large enough this contradicts the fact that $\lambda_{1,p}^{\Omega_i} + K < 0$ and completes the proof. □

Theorem 3.5. *Let $K \geq 0$ be a smooth function on a bounded domain Ω. If $\lambda_{1,p}^{Z\cap\Omega} > 0$, then there exists a positive supersolution $\overline{u} \in C^{1,\alpha}(\Omega)$ of the equation (1.1) on Ω for some $\alpha \in (0,1)$.*

Proof. Let Ω_o, Ω_1 be bounded domains such that $Z \cap \Omega \subset \Omega_o \subset \Omega_1$ such that $\lambda_{1,p}^{Z\cap\Omega} > 0$. Let $v \in C^{1,\alpha}(\Omega_1)$ be the first eigenfunction on Ω_1 and $0 \leq \phi \leq 1$ a smooth function such that $\phi = 1$ on Ω_o, 0 outside Ω_1. We can check easily as in [2, Theorem 2.1] that the function $u = \phi v + (1-\phi)C$, where C is a suitably chosen constant, is a positive $C^{1,\alpha}(\Omega)$ supersolution of the equation (1.1). □

Corollary 3.6. *Let Z be the zero set of the function K. Suppose that the first eigenvalue $\lambda_{1,p}^{Z}$ of the operator $L_p u = -\Delta_p u - k(x)|u|^{p-2}u$ is strictly positive. Then the function K is permissible. In particular if $K > 0$ on M, K is permissible*

4. Example

Consider the cylinder $M = R^+ \times N$ where (N,h) is a compact manifold with Riemannian metric h and of scalar curvature $S_h \geq 0$. We endows M with the metric

$$g = dr^2 + f(r)^2 h$$

where f is smooth positive function. Denote by $\Gamma^l_{i,j}, S_g, \overline{R}^k_{ijl}$ and R^k_{ijl} $1 \le i,j,k,l \le n$ respectively the Christofell symbols, the scalar curvature, the curvature tensor on M and the curvature tensor on N.

¿From the local expression of Γ^α_{ij},

$$\Gamma^\alpha_{ij} = \frac{1}{2} g^{\alpha l}\big(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\big),$$

we have

$$\begin{gathered}
\Gamma^1_{ij} = -f(r)f'(r)h_{ij}, \quad 2 \le i,j \le n \\
\Gamma^1_{i1} = 0, \quad 1 \le i \le n \\
\Gamma^\alpha_{11} = 0, \quad 1 \le \alpha \le n \\
\Gamma^\alpha_{1j} = -f(r)/f'(r)\delta^\alpha_j, \quad 2 \le \alpha,\ j \le n \\
\Gamma^\alpha_{ij} = \frac{1}{2} g^{\alpha l}(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}), \quad 2 \le i,j,\alpha \le n\,.
\end{gathered}$$

A direct computation gives

$$\begin{gathered}
\overline{R}^\alpha_{1\alpha 1} = -f''(r)/f(r), \quad 2 \le \alpha \le n \\
\overline{R}^1_{i1j} = -f(r)f''(r)h_{ij}, \quad 2 \le i,j \le n \\
\overline{R}^\alpha_{i\alpha\alpha} = 0, \quad 1 \le i,\alpha \le n \\
\overline{R}^\alpha_{i\alpha j} = R^\alpha_{i\alpha j} - f'(r)^2 h_{ij}, \quad 2 \le i,j,\alpha \le n,\ j \ne \alpha
\end{gathered}$$

so

$$S_g = -2(n-1)f''(r)/f(r) - (n-1)(n-2)f'(r)^2/f(r)^2 + \frac{S_h}{f(r)^2}\,.$$

When we take $f(r) = \exp r^2$,

$$f'(r) > 0 \tag{4.1}$$

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} \lim f'(r)/f(r) = \lim_{r\to\infty} f''(r)/f(r) = \infty \tag{4.2}$$

For $r > 0$ large enough, by inequalities (4.1) and (4.2) we obtain $S_g \le -\varepsilon$. By re-parametrizing, we can assume that

$$S_g \le -\varepsilon \quad \text{for any } r > 0\,. \tag{4.3}$$

Let $K = \varepsilon + 4(n-1)(1+nr^2)$ then $k = -S_g \le K$. Now consider on M the equation

$$\Delta_p u - S_g u^{p-1} - K u^{p^*-1} = 0 \tag{4.4}$$

with $2 < p < n$ and $p^* = (pn/(n-p)$. Note that the positive function K is permissible by Corollary 3.6. Let

$$\phi = \begin{cases} (\delta r/r_1^2)^\alpha & \text{if } 0 < r < r_1 \\ (\delta/r)^\alpha & \text{if } r \ge r_1, \end{cases}$$

where $\alpha \geq 2/(p^* - p)$, δ and r_1 are constants which will be suitably chosen. For $0 < r < r_1$ we have

$$\begin{aligned}
&\Delta_p \phi - S_g \phi^{p-1} - [4(n-1)(1+4n^2r^2) + \varepsilon]\phi^{p^*-1} \\
&\geq \left(\frac{\delta r}{r_1^2}\right)^{(p-1)\alpha} \big[\varepsilon + (\frac{\alpha}{r})^{p-1}\Delta r + (p-1)(\alpha-1)\alpha^{p-1}(\frac{1}{r})^p \\
&\quad - (4(n-1)(1+nr^2) + \varepsilon)(\frac{\delta r}{r_1^2})^{(p^*-p)\alpha}\big] .
\end{aligned}$$

Letting δ be small, and r_1 large, and using that

$$\Delta r = f'(r)/f(r) = 2(n-1)r \,,$$

we obtain that the left-hand side of (4.4) is positive. In the case $r \geq r_1$, the same computations yield

$$\begin{aligned}
&\Delta_p \phi - S_g \phi^{p-1} - (4(n-1)(1+4n^2r^2) + \varepsilon)\phi^{p^*-1} \\
&\geq (\frac{\delta}{r})^{\alpha(p-1)} \big[\varepsilon - 2(n-1)\alpha^{p-1}(\frac{1}{r})^{p-2} + (p-1)(\alpha+1)\alpha^{p-1}(\frac{1}{r})^p \\
&\quad - (4(n-1)(1+4n^2r^2) + \varepsilon))(\frac{\delta}{r})^{\alpha(p^*-p)}\big] .
\end{aligned}$$

The same arguments as above show that the left-hand side of (4.4) is positive in this case. Therefore, ϕ is a positive subsolution of the equation (4.4) and by Theorem 2.3, there exists a positive weak solution to this equation on the manifold M.

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