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ON INTEGRAL INEQUALITIES FOR FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES

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Abstract. This paper presents some non-linear integral inequalities for functions of n independent variables. These results extend the Gronwall type inequalities obtained for two variables by Dragomir and Kim [2]

1. Introduction

Integral inequalities play a significant role in the study of differential and integral equations. One of the most useful inequalities of Gronwall type is given in the following lemma (see [1, 2]).

Lemma 1.1. *Let $u(t)$ and $k(t)$ be continuous, $a(t)$ and $b(t)$ Riemann integrable function on $J = [\alpha, \beta] \subset \mathbb{R}$ and $t \in \mathbb{R}$ with $b(t)$ and $k(t)$ nonnegative on J. If $u(t) \le a(t) + b(t) \int_\alpha^t k(s)u(s)ds$ for $t \in J$, then*

$$u(t) \le a(t) + b(t) \int_\alpha^t a(s)k(s) \exp\Big(\int_s^t b(\tau)k(\tau)d\tau\Big)ds, \quad t \in J, \tag{1.1}$$

If $u(t) \le a(t) + b(t) \int_t^\beta k(s)u(s)ds$ for $t \in J$, then

$$u(t) \le a(t) + b(t) \int_t^\beta a(s)k(s) \exp\Big(\int_t^s b(\tau)k(\tau)d\tau\Big)ds, \quad t \in J. \tag{1.2}$$

In the past few years, these inequalities have been generalized to more than one variable. Many authors have established Gronwall type integral inequalities in two or more independent variables; see for example [3, 4, 5, 6, 7]. The results obtained have generated a lot of research interests due to its usefulness in the theory of differential and integral equations. Dragomir and Kim [2] considered integral inequalities for functions with two independent variables. The purpose of this paper is to generalize their results by obtaining new integral inequalities in n independent variables.

In what follows we denote by $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. All the functions appearing in the inequalities are assumed to be real valued of n-variables which are nonnegative and continuous. All integrals exist on their domains of definitions.

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Throughout this paper, we shall assume that $x = (x_1, x_2, \dots x_n)$ and $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ are in $\mathbb{R}_+^n$. We shall denote

$$\int_{x^0}^{x} dt = \int_{x_1^0}^{x_1} \int_{x_2^0}^{x_2} \dots \int_{x_n^0}^{x_n} \dots dt_n \dots dt_1$$

and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, \dots, n$. For $x, t \in \mathbb{R}_+^n$, we shall write $t \le x$ whenever $t_i \le x_i$, $i = 1, 2, \dots, n$.

2. Results

Lemma 2.1. *Let $u(x), a(x)$ and $b(x)$ be nonnegative continuous functions, defined for $x \in \mathbb{R}_+^n$.*
(1) Assume that $a(x)$ is positive, continuous function, nondecreasing in each of the variables $x \in \mathbb{R}_+^n$. Suppose that

$$u(x) \le a(x) + \int_{x^0}^{x} b(t)u(t)dt \tag{2.1}$$

holds for all $x \in \mathbb{R}_+^n$ with $x \ge x^0$, then

$$u(x) \le a(x) \exp\Big(\int_{x^0}^{x} b(t)dt\Big), \tag{2.2}$$

(2) Assume that $a(x)$ is positive, continuous function, non-increasing in each of the variables $x \in \mathbb{R}_+^n$. Suppose that

$$u(x) \le a(x) + \int_{x}^{x^0} b(t)u(t)dt \tag{2.3}$$

holds for all $x \in \mathbb{R}_+^n$ with $x \le x^0$, then

$$u(x) \le a(x) \exp\Big(\int_{x}^{x^0} b(t)dt\Big). \tag{2.4}$$

Proof. The proof of (1) is similar to the proof of (2), so we present the proof of (2) and refer the reader to [1, p. 112] for more details.
(2) Since $a(x)$ is positive, non-increasing in each of the variables $x \in \mathbb{R}_+^n$, with $x \le x^0$, then

$$\frac{u(x)}{a(x)} \le 1 + \int_{x}^{x^0} b(t)\frac{u(t)}{a(t)}dt, \tag{2.5}$$

Setting

$$v(x) = \frac{u(x)}{a(x)}, \tag{2.6}$$

we have

$$v(x) \le 1 + \int_{x}^{x^0} b(t)v(t)dt, \tag{2.7}$$

Let

$$r(x) = 1 + \int_{x}^{x^0} b(t)v(t)dt, \tag{2.8}$$

Then $r(x_1^0, x_2, \dots, x_n) = 1$, and $v(x) \le r(x)$, $r(x)$ is positive and nonincreasing in each of the variables $x_2, \dots, x_n \in \mathbb{R}_+$. Hence

$$\begin{aligned} D_1 r(x) &= \int_{x_2}^{x_2^0} \int_{x_3}^{x_3^0} \dots \int_{x_n}^{x_n^0} b(x_1, t_2, \dots, t_n) v(x_1, t_2, \dots, t_n) dt_n \dots dt_2 \\ &\le \int_{x_2}^{x_2^0} \int_{x_3}^{x_3^0} \dots \int_{x_n}^{x_n^0} b(x_1, t_2, \dots, t_n) r(x_1, t_2, \dots, t_n) dt_n \dots dt_2 \\ &\le r(x) \int_{x_2}^{x_2^0} \int_{x_3}^{x_3^0} \dots \int_{x_n}^{x_n^0} b(x_1, t_2, \dots, t_n) dt_n \dots dt_2, \end{aligned} \tag{2.9}$$

Dividing both sides of (2.9) by $r(x)$ we get

$$\frac{D_1 r(x)}{r(x)} \le \int_{x_2}^{x_2^0} \int_{x_3}^{x_3^0} \dots \int_{x_n}^{x_n^0} b(x_1, t_2, \dots, t_n) dt_n \dots dt_2. \tag{2.10}$$

Integrating with respect to t_1 from x_1 to x_1^0, we have

$$r(x) \le \exp\Big(\int_{x^0}^{x^0} b(t) dt \Big), \tag{2.11}$$

Hence

$$v(x) \le \exp\Big(\int_{x}^{x_0} b(t) dt \Big). \tag{2.12}$$

Substituting (2.12) into (2.6), we have the result (2.4). $\square$

Theorem 2.2. *Let $u(x)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, $f(x)$ be real-valued non-negative continuous functions defined for $x \in \mathbb{R}_+^n$. Let $W(u(x))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive, and submultiplicative function for $u(x) \ge 0$, and let $H(u(x))$ be real-valued, positive, continuous, and non-decreasing function defined for $x \in \mathbb{R}_+^n$. Assume that $a(x)$, $f(x)$ are nondecreasing in the first variable x_1 for $x_1 \in \mathbb{R}_+$. If*

$$\begin{aligned} u(x) \le a(x) &+ b(x) \int_{\alpha}^{x_1} c(s, x_2, \dots, x_n) u(s, x_2, \dots, x_n) ds \\ &+ f(x) H\Big(\int_{x^0}^{x} d(t) W(u(t)) dt \Big), \end{aligned} \tag{2.13}$$

for $\alpha \ge 0$, $x, t \in \mathbb{R}_+^n$ with $\alpha \le x_1$ and $x^o \le t \le x$, then

$$u(x) \le p(x) \Big\{ a(x) + f(x) H\Big[G^{-1}\Big(G(A(t)) + \int_{x^0}^{x} d(t) W(p(t) f(t)) dt \Big) \Big] \Big\}, \tag{2.14}$$

for $\alpha \ge 0$, $x \in \mathbb{R}_+^n$ with $\alpha \le x_1$, where

$$p(x) = 1 + b(x) \int_{\alpha}^{x_1} c(s, x_2, \dots, x_n) \exp\Big(\int_{\alpha}^{x_1} b(\tau, x_2, \dots, x_n) c(\tau, x_2, \dots, x_n) d\tau \Big) ds, \tag{2.15}$$

$$A(t) = \int_{x^0}^{\infty} d(t) W(a(t) p(t)) dt, \tag{2.16}$$

$$G(z) = \int_{z^0}^{z} \frac{ds}{W(H(s))}, \quad z \ge z^0 > 0\,. \tag{2.17}$$

Here G^{-1} is the inverse function of G and

$$G\Big(\int_{x^0}^{\infty} d(t)W(a(t)p(t))dt\Big) + \int_{x^0}^{x} d(t)W(p(t)f(t))dt,$$

is in the domain of G^{-1} for $x \in \mathbb{R}^n_+$.

Proof. Define a function

$$z(x) = a(x) + f(x)H\Big(\int_{x^0}^{x} d(t)W(u(t))dt\Big), \tag{2.18}$$

Then (2.13) can be restated as

$$u(x) \le z(x) + b(x)\int_{\alpha}^{x_1} c(s, x_2, \dots, x_n)u(s, x_2, \dots, x_n)ds. \tag{2.19}$$

Clearly $z(x)$ is a nonnegative and continuous in $x_1 \in \mathbb{R}_+$. $x_2, x_3, \dots x_n \in \mathbb{R}_+$fixed in (2.19) and using (1) of lemma 1.1 to (2.19), we get

$$\begin{aligned} u(x) \le z(x) + b(x)\int_{\alpha}^{x_1} & z(s, x_2, \dots, x_n)c(s, x_2, \dots, x_n) \\ & \times \exp\Big(\int_{\alpha}^{x_1} b(\tau, x_2, \dots, x_n)c(\tau, x_2, \dots, x_n)d\tau\Big)ds, \end{aligned}$$

Moreover, $z(x)$ is nondecreasing in $x_1, x_1 \in R_+$, we obtain

$$u(x) \le z(x)p(x), \tag{2.20}$$

where $p(x)$ is defined by (2.15). From (2.18) we have

$$u(x) \le (a(x) + f(x)H(v(x)))\, p(x), \tag{2.21}$$

where $v(x) = \int_{x^0}^{x} d(t)W(u(t))dt$. From (2.21), we observe that

$$\begin{aligned} v(x) &\le \int_{x^0}^{x} d(t)W\left(\left(a(t) + f(t)H(v(t))\right)p(t)\right)dt \\ &\le \int_{x^0}^{x} d(t)W(a(t)p(t))dt + \int_{x^0}^{x} d(t)W\left(p(t)f(t)\right)W\left(H(v(t))\right)dt, \\ &\le \int_{x^0}^{\infty} d(t)W(a(t)p(t))dt + \int_{x^0}^{x} d(t)W\left(p(t)f(t)\right)W\left(H(v(t))\right)dt, \end{aligned} \tag{2.22}$$

Since W is subadditive and submultiplicative function. Define $r(x)$ as the right side of (2.22), then $r(x_0^1, x_2, \dots, x_n) = \int_{x^0}^{\infty} d(t)W(a(t)p(t))dt$, $v(x) \le r(x)$, $r(x)$ is positive nondecreasing in each of the variables $x_2, \dots, x_n \in \mathbb{R}_+$ and

$$\begin{aligned} D_1 r(x) = & \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\dots\int_{x_n^0}^{x_n} d(x_1, t_2, \dots, t_n) \\ & \times W\left(p(x_1, t_2, \dots, t_n)f(x_1, t_2, \dots, t_n)\right)W\left(H(v(x_1, t_2, \dots, t_n))\right)dt_n \dots dt_2 \\ \le & \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\dots\int_{x_n^0}^{x_n} d(x_1, t_2, \dots, t_n) \\ & \times W\left(p(x_1, t_2, \dots, t_n)f(x_1, t_2, \dots, t_n)\right)W\left(H(r(x_1, t_2, \dots, t_n))\right)dt_n \dots dt_2 \\ \le & W\left(H(r(x))\right)\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\dots\int_{x_n^0}^{x_n} d(x_1, t_2, \dots, t_n) \\ & \times W\left(p(x_1, t_2, \dots, t_n)f(x_1, t_2, \dots, t_n)\right)dt_n \dots dt_2. \end{aligned} \tag{2.23}$$

Dividing both sides of (2.23) by $W(H(r(x)))$ we get

$$\begin{aligned}\frac{D_1 r(x)}{W(H(r(x)))} \le \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\dots\int_{x_n^0}^{x_n} & d(x_1,t_2,\dots,t_n) \\ & \times W\left(p(x_1,t_2,\dots,t_n)f(x_1,t_2,\dots,t_n)\right) dt_n \dots dt_2,\end{aligned} \tag{2.24}$$

Note that for

$$G(z) = \int_{z^0}^{z} \frac{ds}{W(H(s))}, \quad z \ge z^0 > 0 \tag{2.25}$$

it follows that

$$D_1 G(r(x)) = \frac{D_1 r(x)}{W(H(r(x)))}, \tag{2.26}$$

From (2.25) , (2.26) and (2.24), we have

$$\begin{aligned}D_1 G(r(x)) \le \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\dots\int_{x_n^0}^{x_n} & d(x_1,t_2,\dots,t_n) \\ & \times W\left(p(x_1,t_2,\dots,t_n)f(x_1,t_2,\dots,t_n)\right) dt_n \dots dt_2,\end{aligned} \tag{2.27}$$

Now setting $x_1 = s$ in (2.27) and then integrating with respect to x_1^0 to x_1, we obtain

$$G(r(x)) \le G(r(x_1^0, x_2, \dots, x_n)) + \int_{x_0}^{x} d(t)W(p(t)f(t))dt \tag{2.28}$$

Noting that $r(x_1^0, x_2, \dots, x_n) = \int_{x_0}^{\infty} d(t)W(a(t)p(t))dt$, we have

$$r(x) \le G^{-1}\Big[G\Big(\int_{x^0}^{\infty} d(t)W(a(t)p(t))dt\Big) + \int_{x^0}^{x} d(t)W(p(t)f(t))dt\Big]. \tag{2.29}$$

The required inequality in (2.14) follows from the fact $v(x) \le r(x)$, (2.19) and (2.29) □

Theorem 2.3. *Let $u(x)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, $f(x)$, $W(u(x))$, and $H(u(x))$ be as defined in theorem 2.2. Assume that $a(x), f(x)$ are non-increasing in the first variable x_1, for $x_1 \in \mathbb{R}_+$. If*

$$\begin{aligned}u(x) \le a(x) + b(x) & \int_{x_1}^{\beta} c(s, x_2, \dots, x_n)u(s, x_2, \dots, x_n)ds \\ & + f(x)H\left(\int_{x}^{x_0} d(t)W(u(t))dt\right),\end{aligned} \tag{2.30}$$

for $\beta \ge 0$, $x \in \mathbb{R}_+^n$ with $\beta \ge x_1$ and $x \le x^0$. Then

$$u(x) \le \overline{p}(x)\Big\{a(x) + f(x)H\Big(G^{-1}\Big[G(\overline{A}(t)) + \int_{x}^{x_0} d(t)W(p(t)f(t))dt\Big]\Big)\Big\},$$

for $\beta \ge 0$, $x \in \mathbb{R}_+^n$ with $\beta \ge x_1$, where

$$\overline{p}(x) = 1 + b(x)\int_{x_1}^{\beta} c(s, x_2, \dots, x_n) \exp\Big(\int_{x_1}^{s} b(\tau, x_2, \dots, x_n)c(\tau, x_2, \dots, x_n)d\tau\Big)ds,$$

$$\overline{A}(t) = \int_{0}^{x^0} d(t)W(a(t)\overline{p}(t))dt,$$

$$G(z) = \int_{z^0}^{z} \frac{ds}{W(H(s))}, \quad z \ge z^0 > 0\,.$$

Here G^{-1} is the inverse function of G and

$$G\Big(\int_0^{x^0} d(t)W(a(t)p(t))dt\Big) + \int_x^{x^0} d(t)W(p(t)f(t))dt,$$

is in the domain of G^{-1} for $x \in \mathbb{R}^n_+$.

The proof is similar to the proof of Theorem 2.2 and so it is omitted.

Remark 2.4. *We note that in the special case $n = 2$ (integral inequalities in two independent variables) $x \in \mathbb{R}^2_+$ and $x_0 = (x^0_1, x^0_2) = (\infty, \infty)$ in theorem 2.3. our estimate reduces to Theorem 2.4 obtained by S. S. Dragomir and Y. H. Kim* [2].

Theorem 2.5. *Let $u(x), a(x), b(x), c(x)$ and $f(x)$ be real-valued nonnegative continuous functions defined for $x \in \mathbb{R}^n_+$ and $L : \mathbb{R}^{n+1}_+ \to \mathbb{R}^*_+$ be a continuous functions which satisfies the condition*

$$0 \le L(x, u) - L(x, v) \le M(x, v)\Phi^{-1}(u - v), \tag{2.31}$$

for $u \ge v \ge 0$, where $M(x, v)$ is a real-valued nonnegative continuous function defined for $x \in \mathbb{R}^n_+, v \in \mathbb{R}_+$. Assume that $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\Phi(0) = 0, \Phi^{-1}$ is the inverse function of Φ and

$$\Phi^{-1}(uv) \le \Phi^{-1}(u)\Phi^{-1}(v), \tag{2.32}$$

for $u, v \in \mathbb{R}_+$, Assume that $a(x), f(x)$ are nondecreasing in the first variable x_1 for $x_1 \in \mathbb{R}_+$. If

$$u(x) \le a(x)+b(x)\int_\alpha^{x_1} c(s, x_2, \dots, x_n)u(s, x_2, \dots, x_n)ds+f(x)\Phi\Big(\int_{x_0}^x L(t, u(t))dt\Big), \tag{2.33}$$

for $\alpha \ge 0,\ x \in \mathbb{R}^n_+$ with $\alpha \le x_1$ and $x^0 < x$. Then

$$u(x) \le p(x)\Big\{a(x) + f(x)\Phi\Big[e(x)\exp\Big(\int_{x^0}^x M(t, p(t)a(t))\Phi^{-1}\left(p(t)f(t)\right)dt\Big)\Big]\Big\} \tag{2.34}$$

for $\alpha \ge 0,\ x \in \mathbb{R}^n_+$ with $\alpha \le x_1$ and $x^0 < x$, where

$$p(x) = 1 + b(x)\int_\alpha^{x_1} c(s, x_2, \dots, x_n)\exp\Big(\int_s^{x_1} b(\tau, x_2, \dots, x_n)c(\tau, x_2, \dots, x_n)d\tau\Big)ds, \tag{2.35}$$

$$e(x) = \int_{x^0}^x L(t, p(t)a(t))dt. \tag{2.36}$$

Proof. Define the function

$$z(x) = a(x) + f(x)\Phi\Big(\int_{x_0}^x L(t, u(t))dt\Big), \tag{2.37}$$

Then (2.33) can be restated as

$$u(x) \le z(x) + b(x)\int_\alpha^{x_1} c(s, x_2, x_3, \dots, x_n)u(s, x_2, x_3, \dots, x_n)ds. \tag{2.38}$$

Clearly $z(x)$ is nonnegative and continuous in $x_1 \in \mathbb{R}_+$, where $x_2, x_{3,} \dots x_n \in \mathbb{R}_+$fixed in (2.38) and using 1 of lemma 1.1 to (2.38), we get

$$u(x) \le z(x) + b(x)\int_\alpha^{x_1} z(s, x_2, \dots, x_n)c(s, x_2, \dots, x_n)$$

$$\times \exp\Big(\int_s^{x_1} b(\tau, x_2, \dots, x_n)c(\tau, x_2, \dots, x_n)d\tau\Big)ds$$

Moreover, $z(x)$ is nondecreasing in $x_1, x_1 \in \mathbb{R}_+$, we obtain

$$u(x) \le z(x)p(x), \tag{2.39}$$

Where $p(x)$ is defined by (2.35). From (2.37) and (2.39) we have

$$u(x) \le p(x)\left[a(x) + f(x)\Phi(v(x))\right], \tag{2.40}$$

where

$$v(x) = \int_{x^0}^{x} L(t, u(t))dt,$$

From (2.40), and the hypotheses on L and Φ, we observe that

$$\begin{aligned} v(x) &\le \int_{x^0}^{x} \left(L\left(t, p(t)\left[a(t) + f(t)\Phi(v(t))\right]\right) - L\left(t, p(t)a(t)\right) + L\left(t, p(t)a(t)\right)\right)dt, \\ &\le \int_{x^0}^{x} L\left(t, p(t)a(t)\right)dt + \int_{x^0}^{x} M(t, p(t)a(t))\Phi^{-1}\left(p(t)f(t)\Phi(v(t))\right)dt, \\ &\le e(x) + \int_{x^0}^{x} M(t, p(t)a(t))\Phi^{-1}\left(p(t)f(t)\right)v(t)dt, \end{aligned} \tag{2.41}$$

where $e(x)$ is defined by (2.36). Clearly, $e(x)$ is positive, continuous, nondecreasing in each of the variables $x, x \in \mathbb{R}^n_+$. Now, by part (1) of lemma 2.1,

$$v(x) \le e(x)\exp\Big(\int_{x_0}^{x} M(t, p(t)a(t))\Phi^{-1}\left(p(t)f(t)\right)dt\Big). \tag{2.42}$$

Using (2.40) in (2.42), we get the required inequality in (2.34). □

Theorem 2.6. *Let $u(x)$, $a(x)$, $b(x)$, $c(x)$, $f(x)$, L, M, Φ, and Φ^{-1} be as defined in theorem 2.5. Assume that $a(x), f(x)$ are non-increasing in the first variable x_1 for $x_1 \in \mathbb{R}_+$. If*

$$u(x) \le a(x) + b(x)\int_{x_1}^{\beta} c(s, x_2, \dots, x_n)u(s, x_2, \dots, x_n)ds + f(x)\Phi\Big(\int_{x}^{x^0} L(t, u(t))dt\Big), \tag{2.43}$$

for $\beta \ge 0$, $x \in \mathbb{R}^n_+$ with $\beta \ge x_1$, $x < x^0$. Then

$$u(x) \le \overline{p}(x)\Big\{a(x) + f(x)\Phi\Big[\overline{e}(x)\exp\Big(\int_{x}^{x_0} M(t, p(t)a(t))\Phi^{-1}\big(p(t)f(t)\big)dt\Big)\Big]\Big\},$$

for $\beta \ge 0$, $x \in \mathbb{R}^n_+$ with $\beta \ge x_1$, $x < x^0$, where

$$\overline{p}(x) = 1 + b(x)\int_{x_1}^{\beta} c(s, x_2, \dots, x_n)\exp\Big(\int_{x_1}^{s} b(\tau, x_2, \dots, x_n)c(\tau, x_2, \dots, x_n)d\tau\Big)ds$$

$$\overline{e}(x) = \int_{x}^{x^0} L(t, \overline{p}(t)a(t))dt. \tag{2.44}$$

The proof is similar to the proof of Theorem 2.5 and so it is omitted.

Remark 2.7. *We note that in the special case $n = 2$, $x \in \mathbb{R}^2_+$ and $x^0 = (x^0_1, x^0_2) = (\infty, \infty)$ in theorem 2.6. Our estimate reduces to Theorem 2.6 obtained by Dragomir and Kim* [2].

Remark 2.8. *(1) The preceding results remaining valid if we replace* $b(x)\int_{\alpha}^{x_1} c(s,x_2,\dots,x_n)u(s,x_2,\dots,x_n)ds$ *by the general case* $b_i(x)\int_{\alpha_i}^{x_i} c_i(x_{1,}\dots,x_{i-1},s_i,x_{i+1},\dots,x_n)u(x_{1,}\dots,x_{i-1},s_i,x_{i+1},\dots,x)ds_i$*, for any* $i=2,\dots,n$ *fixed , and* $\alpha_i\ge 0$, $x=(x_1,\dots x_n)\in\mathbb{R}_+^n$ *with* $\alpha_i\le s_i\le x_i$, $x_i,s_i\in\mathbb{R}_+$.

(2) The preceding results are also valid if $b(x)\int_{x_1}^{\beta} c(s,x_2,\dots,x_n)u(s,x_2,\dots,x_n)ds$ *is replaced by the general case* $b_i(x)\int_{x_i}^{\beta_i} c_i(x_{1,}\dots,x_{i-1},s_i,x_{i+1},\dots,x_n)g(u(x_{1,}\dots,x_{i-1},s_i,x_{i+1},\dots,x_n))ds_i$*, for any* $i=2,\dots,n$ *fixed , and* $\alpha_i\ge 0$, $x=(x_1,\dots x_n)\in\mathbb{R}_+^n$ *with* $\alpha_i\le s_i\le x_i$, $x_i,s_i\in\mathbb{R}_+$*. where* $b_i(x)$ *and* $c_i(x)$ *be real-valued nonnegative continuous function defined for* $x\in\mathbb{R}_+^n$*, For any* $i=2,\dots,n$.

3. Further Inequalities

In this section we require the class of function S as defined in [2]. A function $g:\mathbb{R}_+\to\mathbb{R}_+$ is said to belong to the class S if it satisfies the following conditions:

(1) $g(u)$ is positive, nondecreasing and continuous for $u\ge 0$
(2) $(1/v)g(u)\le g(u/v)$, $u>0$, $v\ge 1$.

Theorem 3.1. *Let* $u(x)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, $f(x)$ *be real-valued nonnegative continuous function defined for* $x\in\mathbb{R}_+^n$ *and let* $g\in S$*. Also let* $W(u(x))$ *be real-valued, positive, continuous, strictly nondecreasing, subadditive, and submultiplicative function for* $u(x)\ge 0$ *and let* $H(u(x))$ *be a real-valued, continuous, positive, and nondecreasing function defined for* $x\in\mathbb{R}_+^n$*,and* $b(x)$ *nonincreasing in the first variable* x_1*. Assume that a function* $m(x)$ *is nondecreasing in the first variable* x_1 *and* $m(x)\ge 1$*, which is defined by*

$$m(x)=a(x)+f(x)H\Big(\int_{x^0}^{x} d(t)W(u(t))dt\Big), \tag{3.1}$$

for $x\in\mathbb{R}_+^n$, $x>x^0\ge 0$*. If*

$$u(x)\le m(x)+b(x)\int_{\alpha}^{x_1} c(s,x_2,\dots,x_n)g(u(s,x_2,\dots,x_n))ds, \tag{3.2}$$

for $\alpha\ge 0$, $x\in\mathbb{R}_+^n$ *with* $\alpha\le x_1$*, then*

$$u(x)\le F(x)\Big\{a(x)+f(x)H\Big[G^{-1}\Big(G(B(t))+\int_{x^0}^{x} d(t)W(F(t)f(t))dt\Big)\Big]\Big\}, \tag{3.3}$$

for $x\in\mathbb{R}_+^n$ *, where*

$$F(x)=\Omega^{-1}\Big(\Omega(1)+\int_{\alpha}^{x_1} b(s,x_2,\dots,x_n)c(s,x_2,\dots,x_n)ds\Big), \tag{3.4}$$

$$B(t)=\int_{x^0}^{\infty} d(t)W(a(t)F(t))dt, \tag{3.5}$$

$$\Omega(\delta)=\int_{\varepsilon}^{\delta}\frac{ds}{g(s)},\quad \delta\ge\varepsilon>0. \tag{3.6}$$

Here Ω^{-1} *is the inverse function of* Ω*, and* G,G^{-1} *are defined in Theorem 2.2, and* $\Omega(1)+\int_{\alpha}^{x_1} b(s,x_2,\dots,x_n)c(s,x_2,\dots,x_n)ds$ *is in the domain of* Ω^{-1}*, and*

$$G\Big(\int_{x^0}^{\infty} d(t)W(a(t)F(t))dt)+\int_{x^0}^{x} d(t)W(F(t)f(t)dt\Big),$$

is in the domain of G^{-1} *for* $x \in \mathbb{R}^n_+$.

Proof. We have $m(x)$ be a positive, continuous, nondecreasing in x_1 and $g \in S$, and $b(x)$ non-increasing in the first variable x_1. Then can be restated as

$$\frac{u(x)}{m(x)} \le 1 + \int_\alpha^{x_1} b(s, x_2, x_3, \dots, x_n) c(s, x_2, x_3, \dots, x_n) g\Big(\frac{u(s, x_2, x_3, \dots, x_n)}{m(s, x_2, x_3, \dots, x_n)}\Big) ds \tag{3.7}$$

The inequality (3.7) may be treated as one-dimensional Bihari-Lasalle inequality the inequality type was given by Gyori [3] (see [1]), for any fixed $x_2, x_3, \dots, x_n$, which implies

$$u(x) \le F(x) m(x). \tag{3.8}$$

Here $F(x)$ is defined by (3.4), by (3.1) and (3.8) we get

$$u(x) \le F(x) \left\{ a(x) + f(x) H(v(x)) \right\}, \tag{3.9}$$

where $v(x)$ is defined by

$$v(x) = \int_{x^0}^{x} d(t) W(u(t)) dt.$$

Using the last argument in the proof of Theorem 2.2, we obtain desired inequality in (3.3). $\square$

Theorem 3.2. *Let* $u(x)$, $a(x)$, $c(x)$, $d(x)$, $f(x)$, $W(u(x)$, *and* $H(u(x))$ *be as defined in the theorem 3.1 and let* $g \in S$ *and* $b(x)$ *be nonnegative continuous functions, non-decreasing in the first variable* x_1. *Assume that a function* $\overline{m}(x)$ *is non-increasing in the first variable* x_1 *and* $\overline{m}(x) \ge 1$, *which is defined by*

$$\overline{m}(x) = a(x) + f(x) H\Big(\int_x^{x^0} d(t) W(u(t)) dt \Big) \tag{3.10}$$

for $x \in \mathbb{R}^n_+$, $x^0 \ge x$. *If*

$$u(x) \le \overline{m}(x) + b(x) \int_{x_1}^{\beta} c(s, x_2, \dots, x_n) g(u(s, x_2, \dots, x_n)) ds, \tag{3.11}$$

for $\beta \ge 0$, $x \in \mathbb{R}^n_+$ *with* $\beta \ge x_1$, *then*

$$u(x) \le \overline{F}(x) \Big\{ a(x) + f(x) H\Big[G^{-1}\Big(G(\overline{B}(t)) + \int_x^{x^0} d(t) W(\overline{F}(t) f(t)) dt \Big) \Big] \Big\}, \tag{3.12}$$

for $x \in \mathbb{R}^n_+$. *Here*

$$\overline{F}(x) = \Omega^{-1}\Big(\Omega(1) + \int_{x_1}^{\beta} b(s, x_2, \dots, x_n) c(s, x_2, \dots, x_n) ds \Big), \tag{3.13}$$

$$\overline{B}(t) = \int_0^{x^0} d(t) W(a(t) \overline{F}(t)) dt, \tag{3.14}$$

and Ω *is defined in (3.6). Here* Ω^{-1} *is the inverse function of* Ω, *and* G, G^{-1} *are defined in theorem 2.2, and* $\Omega(1) + \int_{x_1}^{\beta} b(s, x_2, \dots, x_n) c(s, x_2, \dots, x_n) ds$ *is in the domain of* Ω^{-1}, *and*

$$G\Big(\int_0^{x^0} d(t) W(a(t) \overline{F}(t)) dt\Big) + \int_x^{x^0} d(t) W(\overline{F}(t) f(t)) dt$$

is in the domain of G^{-1} *for* $x \in \mathbb{R}^n_+$.

Proof. We have $\overline{m}(x)$ positive, continuous, nonincreasing in x_1. Also $g \in S$ and $b(x)$ nondecreasing in the first variable x_1. Then (3.11) can be restated as

$$\frac{u(x)}{\overline{m}(x)} \leq 1+\int_{x_1}^{\beta} b(s,x_2,x_3,\dots,x_n)c(s,x_2,x_3,\dots,x_n)g\Big(\frac{u(s,x_2,\dots,x_n)}{\overline{m}(s,x_2,\dots,x_n)}\Big)ds \quad (3.15)$$

This inequality can be treated as one-dimensional Bihari-Lasalle inequality [3] for a fixed $x_2, x_3, \dots, x_n$, which implies

$$u(x) \leq \overline{F}(x)\overline{m}(x) \quad (3.16)$$

where $\overline{F}(x)$ is defined by (3.13). Now , by following last argument as in the proof of Theorem 2.3 , we obtain desired inequality in (3.12) □

Corollary 3.3. *If $b(x) = 1$ for $x \in R^n_+$, then from*

$$u(x) \leq \overline{m}(x) + \int_{x_1}^{\beta} c(s,x_2,\dots,x_n)g(u(s,x_2,\dots,x_n))ds$$

with $\beta \geq x_1$, it follows that

$$u(x) \leq \overline{F}(x)\Big\{a(x) + f(x)H\Big[G^{-1}\Big(G(\overline{B}(t)) + \int_x^{x^0} d(t)W(\overline{F}(t)f(t))dt\Big)\Big]\Big\}$$

for $x \in \mathbb{R}^n_+$, where

$$\overline{F}(x) = \Omega^{-1}\Big(\Omega(1) + \int_{x_1}^{\beta} c(s,x_2,\dots,x_n)ds\Big)$$
$$\overline{B}(t) = \int_0^{x^0} d(t)W(a(t)\overline{F}(t))dt$$

Remark 3.4. *We note that in the special case $n = 2$,$x = (x_1, x_2) \in \mathbb{R}^2_+$, and $x^0 = (\infty, \infty)$ in corollary 3.3. Our estimate reduces to Theorem 3.2 obtained by Dragomir and Kim* [2].

Theorem 3.5. *Let $u(x)$, $a(x)$, $b(x)$, $c(x)$, $f(x)$, L, M, Φ, and Φ^{-1} be as defined in theorem 2.5. Let $g \in S$ and $b(x)$ nonincreasing in the first variable x_1. Assume that a function $n(x)$ is nondecreasing in the first variable x_1 and $n(x) \geq 1$ which is defined by*

$$n(x) = a(x) + f(x)\Phi\Big(\int_{x_0}^{x} L(t,u(t))dt\Big) \quad (3.17)$$

for $x \in \mathbb{R}^n_+$, $x \geq x_0 \geq 0$. If

$$u(x) \leq n(x) + b(x)\int_{\alpha}^{x_1} c(s,x_2,x_3,\dots,x_n)g(u(s,x_2,x_3,\dots,x_n))ds \quad (3.18)$$

for $\alpha \geq 0$, $x \in \mathbb{R}^n_+$ with $\alpha \leq x_1$, then

$$u(x) \leq F(x)\Big\{a(x) + f(x)\Phi\Big[e(x)\exp\Big(\int_{x^0}^{x} M(t,a(t)F(t))\Phi^{-1}\big(f(t)F(t)\big)dt\Big)\Big]\Big\} \quad (3.19)$$

for $x \in \mathbb{R}^n_+$, where $F(x)$ is defined in (3.4), $e(x)$ is defined in (2.36), Ω is defined in (3.6), Here Ω^{-1} is the inverse function of Ω, and
$\Omega(1) + \int_{\alpha}^{x_1} b(s,x_2,\dots,x_n)c(s,x_2,\dots,x_n)ds$ is in the domain of Ω for $x \in \mathbb{R}^n_+$.

Proof. We follow an argument similar to that of Theorem 3.1. We have $n(x)$ be a positive, continuous, nondecreasing in x_1 and $g \in S$, and $b(x)$ nonincreasing in the first variable x_1. Then can (3.18) be restated as

$$\frac{u(x)}{n(x)} \le 1 + \int_{\alpha}^{x_1} b(s, x_2, x_3, \dots, x_n) c(s, x_2, x_3, \dots, x_n) g\Big(\frac{u(s, x_2, \dots, x_n)}{n(s, x_2, \dots, x_n)}\Big) ds. \quad (3.20)$$

The inequality (3.20) may be treated as one-dimensional Bihari-Lasalle inequality, for any fixed $x_2, x_3, \dots, x_n$, which implies

$$u(x) \le F(x) n(x) \quad (3.21)$$

where $F(x)$ is defined by (3.4). From (3.17) and (3.21) we get

$$u(x) \le F(x) \Big[a(x) + f(x) H\Big(\int_{x^0}^{x} L(t, u(t)) dt \Big) \Big] \quad (3.22)$$

Following the last argument in the proof of Theorem 2.5, we obtain the desired inequality in (3.19). □

Theorem 3.6. *Let $u(x)$, $a(x)$, $b(x)$, $c(x)$, $f(x)$, L, M, Φ, and Φ^{-1} be as defined in theorem 2.5. Let $g \in S$ and $b(x)$ be nondecreasing in the first variable x_1. Assume that a function $\overline{n}(x)$ is nonincreasing in the first variable x_1 and $\overline{n}(x) \ge 1$, which is defined by*

$$\overline{n}(x) = a(x) + f(x) \Phi\Big(\int_{x}^{x^0} L(t, u(t)) dt \Big) \quad (3.23)$$

for $x \in \mathbb{R}^n_+$, $x^0 \ge x \ge 0$. If

$$u(x) \le \overline{n}(x) + b(x) \int_{x_1}^{\beta} c(s, x_2, \dots, x_n) g(u(s, x_2, \dots, x_n)) ds \quad (3.24)$$

for $\beta \ge 0$, $x \in \mathbb{R}^n_+$ with $\beta \ge x_1$, then

$$u(x) \le \overline{F}(x) \Big\{ a(x) + f(x) \Phi\Big[\overline{e}(x) \exp\Big(\int_{x}^{x^0} M(t, a(t)\overline{F}(t)) \Phi^{-1}\big(f(t)\overline{F}(t)\big) dt \Big) \Big] \Big\}$$

for $x \in \mathbb{R}^n_+$, where $\overline{F}(x)$ is defined in (3.13), $\overline{e}(x)$ is defined in (2.44), Ω is defined in (3.6). Here Ω^{-1} is the inverse function of Ω, and $\Omega(1) + \int_{x_1}^{\beta} b(s, x_2, \dots, x_n) c(s, x_2, \dots, x_n) ds$ is in the domain of Ω for $x \in \mathbb{R}^n_+$.

The proof of this theorem follows by an argument similar to that of Theorem 3.5; therefore, we omit it.

Corollary 3.7. *if $b(x) = 1$ for $x \in R^n_+$, then from*

$$u(x) \le \overline{n}(x) + \int_{x_1}^{\beta} c(s, x_2, \dots, x_n) g(u(s, x_2, \dots, x_n)) ds,$$

for $\beta \ge 0$ with $\beta \ge x_1$, then it follows that

$$u(x) \le \overline{F}(x) \Big\{ a(x) + f(x) \Phi\Big[\overline{e}(x) \exp\Big(\int_{x}^{x^0} M(t, a(t)\overline{F}(t)) \Phi^{-1}\big(f(t)\overline{F}(t)\big) dt \Big) \Big] \Big\}$$

for $x \in \mathbb{R}^n_+$, where

$$\overline{F}(x) = \Omega^{-1}\Big(\Omega(1) + \int_{x_1}^{\beta} c(s, x_2, \dots, x_n) ds \Big),$$

$$\overline{e}(x) = \int_{x}^{x^0} L(t, \overline{p}(t)a(t))dt,$$

$$\overline{p}(x) = 1 + \int_{x_1}^{\beta} c(s, x_2, \dots, x_n) \exp\Big(\int_{x_1}^{s} c(\tau, x_2, \dots, x_n)d\tau\Big)ds,$$

for $x \in \mathbb{R}_+^n$.Ω is defined in (3.6) , where Ω^{-1} is the inverse function of Ω, and $\Omega(1) + \int_{x_1}^{\beta} c(s, x_2, \dots, x_n)ds$ is in the domain of Ω for $x \in \mathbb{R}_+^n$.

Remark 3.8. *We note that in the special case $n = 2$, $x = (x_1, x_2) \in \mathbb{R}_+^2$, and $x^0 = (\infty, \infty)$ in corollary 3.7. our estimate reduces to Theorem 3.4 obtained by Dragomir and Kim* [2].

Remark 3.9. (1) All the preceding results remain valid when
$b(x)\int_{\alpha}^{x_1} c(s, x_2, \dots, x_n)g(u(s, x_2, \dots, x_n))ds$ is replaced by the general function
$b_i(x)\int_{\alpha_i}^{x_i} c_i(x_{1,.}dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n)g(u(x_{1,.}\dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n))ds_i$,
with $i = 2, \dots, n$ fixed, and $\alpha_i \geq 0$, $x = (x_1, \dots x_n) \in \mathbb{R}_+^n$ and with $\alpha_i \leq s_i \leq x_i$, $x_i, s_i \in \mathbb{R}_+$,
(2) The above results remain valid when
$b(x)\int_{x_1}^{\beta} c(s, x_2, \dots, x_n)g(u(s, x_2, \dots, x_n))ds$ is replaced by the general function
$b_i(x)\int_{x_i}^{\beta_i} c_i(x_{1,.}\dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n)g(u(x_{1,.}\dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n))ds_i$,
with $i = 2, \dots, n$ fixed, and $\alpha_i \geq 0$, $x = (x_1, \dots x_n) \in \mathbb{R}_+^n$ and with $\alpha_i \leq s_i \leq x_i$, $x_i, s_i \in \mathbb{R}_+$, where $b_i(x)$ and $c_i(x)$ be real-valued nonnegative continuous function defined for $x \in \mathbb{R}_+^n$, for all $i = 2, \dots, n$.

In a future work, we will present some applications for the results obtained in this work.

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