

TWO CONVERGENCE RESULTS FOR CONTINUOUS DESCENT METHODS

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ABSTRACT. We consider continuous descent methods for the minimization of convex functionals defined on general Banach space. We establish two convergence results for methods which are generated by regular vector fields. Since the complement of the set of regular vector fields is σ -porous, we conclude that our results apply to most vector fields in the sense of Baire's categories.

1. INTRODUCTION

The study of discrete and continuous descent methods is an important topic in optimization theory and in dynamical systems. See, for example, [3, 7, 9, 10]. Given a continuous convex function f on a Banach space X , we associate with f a complete metric space of vector fields $V : X \rightarrow X$ such that $f^0(x, Vx) \leq 0$ for all $x \in X$. Here $f^0(x, h)$ is the right-hand derivative of f at x in the direction $h \in X$. To each such vector field there correspond two gradient-like iterative processes. In our recent papers [10, 11] we show that for most of the vector fields in this space, both iterative processes generate sequences $\{x_n\}_{n=1}^\infty$ such that the sequences $\{f(x_n)\}_{n=1}^\infty$ tend to $\inf(f)$ as $n \rightarrow \infty$. In the present paper, we first briefly review these results and then study the convergence of the values of the function f to its infimum along the trajectories of an analogous continuous dynamical system governed by such vector fields.

When we say that most of the elements of a complete metric space Y enjoy a certain property, we mean that the set of points which have this property contains a G_δ everywhere dense subset of Y . In other words, this property holds generically. Such an approach, when a certain property is investigated for the whole space Y and not just for a single point in Y , has already been successfully applied in many areas of Analysis [1, 4-6, 8, 13, 16].

Now we recall the concept of porosity [2, 5, 6, 11, 14, 17] which enables us to obtain even more refined results. Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. We say that a subset $E \subset Y$ is porous in (Y, d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each

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$r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

A subset of the space Y is called σ -porous in (Y, d) if it is a countable union of porous subsets in (Y, d) . Other notions of porosity have been used in [2, 14]. We use the rather strong notion which appears in [5, 6, 11, 17].

Since porous sets are nowhere dense, all σ -porous sets are of the first category. If Y is a finite-dimensional Euclidean space \mathbb{R}^n , then σ -porous sets are of Lebesgue measure 0. The existence of a non- σ -porous set $P \subset \mathbb{R}^n$, which is of the first Baire category and of Lebesgue measure 0, was established in [14]. It is easy to see that for any σ -porous set $A \subset \mathbb{R}^n$, the set $A \cup P \subset \mathbb{R}^n$ also belongs to the family \mathcal{E} consisting of all the non- σ -porous subsets of \mathbb{R}^n which are of the first Baire category and have Lebesgue measure 0. Moreover, if $Q \in \mathcal{E}$ is a countable union of sets $Q_i \subset \mathbb{R}^n$, $i = 1, 2, \dots$, then there is a natural number j for which the set Q_j is non- σ -porous. Evidently, this set Q_j also belongs to \mathcal{E} . Thus one sees that the family \mathcal{E} is quite large. Also, every complete metric space without isolated points contains a closed nowhere dense set which is not σ -porous [15].

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E .

This paper is organized as follows. In the second section we briefly review our work on discrete descent methods. In the third section we present our two theorems (Theorems 3.2 and 3.3) on continuous descent methods. The first one is concerned with infinite horizon trajectories, while the second deals with finite horizon perturbed trajectories. We prove Theorem 3.2 in Section 4. Theorem 3.3 is proved in Section 5.

2. DISCRETE DESCENT METHODS

Let $(X^*, \|\cdot\|_*)$ be the dual space of the Banach space $(X, \|\cdot\|)$, and let $f : X \rightarrow \mathbb{R}^1$ be a convex continuous function which is bounded from below. Recall that for each pair of sets $A, B \subset X^*$,

$$H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|_*, \sup_{y \in B} \inf_{x \in A} \|x - y\|_*\}$$

is the Hausdorff distance between A and B .

For each $x \in X$, let

$$\partial f(x) = \{l \in X^* : f(y) - f(x) \geq l(y - x) \text{ for all } y \in X\}$$

be the subdifferential of f at x . It is well known that the set $\partial f(x)$ is nonempty and norm-bounded. Set

$$\inf(f) = \inf\{f(x) : x \in X\}.$$

Denote by \mathcal{A} the set of all mappings $V : X \rightarrow X$ such that V is bounded on every bounded subset of X (i.e., for each $K_0 > 0$ there is $K_1 > 0$ such that $\|Vx\| \leq K_1$ if $\|x\| \leq K_0$), and for each $x \in X$ and each $l \in \partial f(x)$, $l(Vx) \leq 0$. We denote by \mathcal{A}_c the set of all continuous $V \in \mathcal{A}$, by \mathcal{A}_u the set of all $V \in \mathcal{A}$ which are uniformly

continuous on each bounded subset of X , and by \mathcal{A}_{au} the set of all $V \in \mathcal{A}$ which are uniformly continuous on the subsets

$$\{x \in X : \|x\| \leq n \text{ and } f(x) \geq \inf(f) + 1/n\}$$

for each integer $n \geq 1$. Finally, let $\mathcal{A}_{auc} = \mathcal{A}_{au} \cap \mathcal{A}_c$.

Next we endow the set \mathcal{A} with a metric ρ : For each $V_1, V_2 \in \mathcal{A}$ and each integer $i \geq 1$, we first set

$$\rho_i(V_1, V_2) = \sup\{\|V_1x - V_2x\| : x \in X \text{ and } \|x\| \leq i\}$$

and then define

$$\rho(V_1, V_2) = \sum_{i=1}^{\infty} 2^{-i} [\rho_i(V_1, V_2)(1 + \rho_i(V_1, V_2))^{-1}].$$

Clearly, (\mathcal{A}, ρ) is a complete metric space. It is also not difficult to see that the collection of the sets

$$E(N, \epsilon) = \{(V_1, V_2) \in \mathcal{A} \times \mathcal{A} : \|V_1x - V_2x\| \leq \epsilon, x \in X, \|x\| \leq N\},$$

where $N, \epsilon > 0$, is a base for the uniformity generated by the metric ρ . Evidently, \mathcal{A}_c , \mathcal{A}_u , \mathcal{A}_{au} and \mathcal{A}_{auc} are all closed subsets of the metric space (\mathcal{A}, ρ) . In the sequel we assign to all these spaces the same metric ρ .

To compute $\inf(f)$, we associate in [10, 11] with each vector field $W \in \mathcal{A}$ two gradient-like iterative processes to be defined below.

Note that the counterexample studied in Section 2.2 of Chapter VIII of [7] shows that, even for two-dimensional problems, the simplest choice for a descent direction, namely the normalized steepest descent direction,

$$V(x) = \operatorname{argmin} \left\{ \max_{l \in \partial f(x)} \langle l, d \rangle : \|d\| = 1 \right\},$$

may produce sequences the functional values of which fail to converge to the infimum of f . This vector field V belongs to \mathcal{A} and the Lipschitzian function f attains its infimum. The steepest descent scheme (Algorithm 1.1.7) presented in Section 1.1 of Chapter VIII of [7] corresponds to any of the two iterative processes we consider below.

In infinite dimensions the minimization problem is even more difficult and less understood. Moreover, positive results usually require special assumptions on the space and the functions. However, as shown in [10] (under certain assumptions on the function f), for an arbitrary Banach space X and a generic vector field $V \in \mathcal{A}$, the values of f tend to its infimum for both processes.

In [11] we introduced the class of regular vector fields $V \in \mathcal{A}$ and established three main results. The first one, Theorem 2.2 below, shows (under the two mild assumptions A1 and A2 on f stated below) that the complement of the set of regular vector fields is not only of the first category, but also σ -porous in each of the spaces \mathcal{A} , \mathcal{A}_c , \mathcal{A}_u , \mathcal{A}_{au} and \mathcal{A}_{auc} . We then show (Theorem 2.3) that for any regular vector field $V \in \mathcal{A}_{au}$, the values of the function f tend to its infimum for both processes. If, in addition to A1 and A2, f also satisfies the assumption A3, then this convergence result is valid for any regular $V \in \mathcal{A}$. The last result in [11], Theorem 2.4 below, is a stability theorem for regular vector fields.

These results are valid in any Banach space and for those convex functions which satisfy the following two assumptions.

A1 There exists a bounded set $X_0 \subset X$ such that

$$\inf(f) = \inf\{f(x) : x \in X\} = \inf\{f(x) : x \in X_0\};$$

A2 for each $r > 0$ the function f is Lipschitzian on the ball $\{x \in X : \|x\| \leq r\}$.

Note that assumption A1 clearly holds if $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$.

We say that a mapping $V \in \mathcal{A}$ is regular if for any natural number n there exists a positive number $\delta(n)$ such that for each $x \in X$ satisfying

$$\|x\| \leq n \text{ and } f(x) \geq \inf(f) + 1/n,$$

and each $l \in \partial f(x)$, we have

$$l(Vx) \leq -\delta(n).$$

Denote by \mathcal{F} the set of all regular vector fields $V \in \mathcal{A}$. It is not difficult to verify the following property of regular vector fields. It means, in particular, that $\mathcal{A} \setminus \mathcal{F}$ is a face of the convex cone \mathcal{A} .

Proposition 2.1. *Assume that $V_1, V_2 \in \mathcal{A}$, V_1 is regular, $\phi : X \rightarrow [0, 1]$, and that for each integer $n \geq 1$,*

$$\inf\{\phi(x) : x \in X \text{ and } \|x\| \leq n\} > 0.$$

Then the mapping $x \rightarrow \phi(x)V_1x + (1 - \phi(x))V_2x$, $x \in X$, also belongs to \mathcal{F} .

The first result of [11] shows that in a very strong sense most of the vector fields in \mathcal{A} are regular.

Theorem 2.2. *Assume that both A1 and A2 hold. Then $\mathcal{A} \setminus \mathcal{F}$ (respectively, $\mathcal{A}_c \setminus \mathcal{F}$, $\mathcal{A}_{au} \setminus \mathcal{F}$ and $\mathcal{A}_{auc} \setminus \mathcal{F}$) is a σ -porous subset of the space \mathcal{A} (respectively, \mathcal{A}_c , \mathcal{A}_{au} and \mathcal{A}_{auc}). Moreover, if f attains its infimum, then the set $\mathcal{A}_u \setminus \mathcal{F}$ is also a σ -porous subset of the space \mathcal{A}_u .*

Now let $W \in \mathcal{A}$. We associate with W the following two gradient-like iterative processes.

For $x \in X$ we denote by $P_W(x)$ the set of all $y \in \{x + \alpha Wx : \alpha \in [0, 1]\}$ such that

$$f(y) = \inf\{f(x + \beta Wx) : \beta \in [0, 1]\}.$$

Given any initial point $x_0 \in X$, one can construct a sequence $\{x_i\}_{i=0}^\infty \subset X$ such that for $i = 0, 1, \dots$,

$$x_{i+1} \in P_W(x_i). \quad (2.1)$$

This is our first iterative process.

Next we describe the second iterative process. Given a sequence $\mathbf{a} = \{a_i\}_{i=0}^\infty \subset (0, 1]$ such that

$$\lim_{i \rightarrow \infty} a_i = 0 \quad \text{and} \quad \sum_{i=0}^{\infty} a_i = \infty, \quad (2.2)$$

we construct for each initial point $x_0 \in X$ a sequence $\{x_i\}_{i=0}^\infty \subset X$ according to the following rule: For $i = 0, 1, \dots$,

$$x_{i+1} = \begin{cases} x_i + a_i W(x_i) & \text{if } f(x_i + a_i W(x_i)) < f(x_i), \\ x_i & \text{otherwise.} \end{cases} \quad (2.3)$$

In the sequel we will also make use of the assumption:

A3 For each integer $n \geq 1$, there exists $\delta > 0$ such that for each $x_1, x_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leq n, f(x_i) \geq \inf(f) + 1/n, i = 1, 2, \text{ and } \|x_1 - x_2\| \leq \delta,$$

we have $H(\partial f(x_1), \partial f(x_2)) \leq 1/n$.

This assumption is certainly satisfied if f is differentiable and its derivative is uniformly continuous on those bounded subsets of X over which the infimum of f is larger than $\inf(f)$.

The next result of [11] is a convergence theorem for those iterative processes associated with regular vector fields. It is of interest to note that we obtain convergence when either the regular vector field W or the subdifferential ∂f enjoys a certain uniform continuity property.

Theorem 2.3. *Assume that $W \in \mathcal{A}$ is regular, A1, A2 are valid and that at least one of the following two conditions holds: 1. $W \in \mathcal{A}_{au}$, or 2. A3 is valid. Then the following two assertions are true*

- (i) *Let $\{x_i\}_{i=0}^\infty \subset X$ satisfy (2.1) for all $i = 0, 1, \dots$. If $\liminf_{i \rightarrow \infty} \|x_i\| < \infty$, then $\lim_{i \rightarrow \infty} f(x_i) = \inf(f)$.*
- (ii) *Let the sequence $\mathbf{a} = \{a_i\}_{i=0}^\infty \subset (0, 1]$ satisfy (2.2) and let the sequence $\{x_i\}_{i=0}^\infty \subset X$ satisfy (2.3) for all $i = 0, 1, \dots$. If $\{x_i\}_{i=0}^\infty$ is bounded, then $\lim_{i \rightarrow \infty} f(x_i) = \inf(f)$.*

Finally, we impose an additional coercivity condition on f , and establish the following stability theorem. Note that this coercivity condition implies A1.

Theorem 2.4. *Assume that $f(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, $V \in \mathcal{A}$ is regular, A2 is valid and that at least one of the following two conditions holds: 1. $V \in \mathcal{A}_{au}$, or 2. A3 is valid.*

Let $K, \epsilon > 0$ be given. Then there exist a neighborhood \mathcal{U} of V in \mathcal{A} and a natural number N_0 such that the following two assertions are true:

- (i) *For each $W \in \mathcal{U}$ and each sequence $\{x_i\}_{i=0}^{N_0} \subset X$ which satisfies $\|x_0\| \leq K$ and (2.1) for all $i = 0, \dots, N_0 - 1$, the inequality $f(x_{N_0}) \leq \inf(f) + \epsilon$ holds.*
- (ii) *For each sequence of numbers $\mathbf{a} = \{a_i\}_{i=0}^\infty \subset (0, 1]$ satisfying (2.2), there exists a natural number N such that for each $W \in \mathcal{U}$ and each sequence $\{x_i\}_{i=0}^N \subset X$ which satisfies $\|x_0\| \leq K$ and (2.3) for all $i = 0, \dots, N - 1$, the inequality $f(x_N) \leq \inf(f) + \epsilon$ holds.*

A partial extension of the results reviewed in this section to functions $f : X \rightarrow \mathbb{R}$ which are not necessarily convex can be found in [12].

3. CONTINUOUS DESCENT METHODS

Let $T > 0$, $x_0 \in X$ and let $u : [0, T] \rightarrow X$ be a Bochner integrable function. Set

$$x(t) = x_0 + \int_0^t u(s) ds, \quad t \in [0, T].$$

Then $x : [0, T] \rightarrow X$ is differentiable and $x'(t) = u(t)$ for a.e. $t \in [0, T]$. Recall that the function $f : X \rightarrow \mathbb{R}$ is assumed to be convex and continuous and therefore it is, in fact, locally Lipschitzian. It follows that its restriction to the set $\{x(t) : t \in [0, T]\}$ is Lipschitzian. Indeed, since the set $\{x(t) : t \in [0, T]\}$ is compact, the closure of its convex hull C is both compact and convex, and so the restriction of f to C

is Lipschitzian. Hence the function $(f \cdot x)(t) := f(x(t))$, $t \in [0, T]$, is absolutely continuous. It follows that for almost every $t \in [0, T]$, both the derivatives $x'(t)$ and $(f \cdot x)'(t)$ exist:

$$\begin{aligned} x'(t) &= \lim_{h \rightarrow 0} h^{-1}[x(t+h) - x(t)], \\ (f \cdot x)'(t) &= \lim_{h \rightarrow 0} h^{-1}[f(x(t+h)) - f(x(t))]. \end{aligned}$$

Proposition 3.1. *Assume that $t \in [0, T]$ and that both the derivatives $x'(t)$ and $(f \cdot x)'(t)$ exist. Then*

$$(f \cdot x)'(t) = \lim_{h \rightarrow 0} h^{-1}[f(x(t) + hx'(t)) - f(x(t))]. \quad (3.1)$$

Proof. There exist a neighborhood \mathcal{U} of $x(t)$ in X and a constant $L > 0$ such that

$$|f(z_1) - f(z_2)| \leq L\|z_1 - z_2\| \quad \text{for all } z_1, z_2 \in \mathcal{U}. \quad (3.2)$$

Let $\epsilon > 0$. There exists $\delta > 0$ such that

$$x(t+h), x(t) + hx'(t) \in \mathcal{U} \quad \text{for each } h \in [-\delta, \delta] \cap [-t, T-t], \quad (3.3)$$

and such that for each $h \in [(-\delta, \delta) \setminus \{0\}] \cap [-t, T-t]$,

$$\|x(t+h) - x(t) - hx'(t)\| < \epsilon|h|. \quad (3.4)$$

Let

$$h \in [(-\delta, \delta) \setminus \{0\}] \cap [-t, T-t]. \quad (3.5)$$

It follows from (3.3), (3.2) and (3.4) that

$$|f(x(t+h)) - f(x(t) + hx'(t))| \leq L\|x(t+h) - x(t) - hx'(t)\| < L\epsilon|h|. \quad (3.6)$$

Clearly,

$$\begin{aligned} [f(x(t+h)) - f(x(t))]h^{-1} &= [f(x(t+h)) - f(x(t) + hx'(t))]h^{-1} \\ &\quad + [f(x(t) + hx'(t)) - f(x(t))]h^{-1}. \end{aligned} \quad (3.7)$$

The relations (3.6) and (3.7) imply

$$\begin{aligned} &|[f(x(t+h)) - f(x(t))]h^{-1} - [f(x(t) + hx'(t)) - f(x(t))]h^{-1}| \\ &\leq |f(x(t+h)) - f(x(t) + hx'(t))| |h^{-1}| \leq L\epsilon. \end{aligned}$$

Since ϵ is an arbitrary positive number, we conclude that (3.1) holds. \diamond

Now assume that $V \in \mathcal{A}$ and that the differentiable function $x : [0, T] \rightarrow X$ satisfies

$$x'(t) = V(x(t)) \quad \text{a.e. } t \in [0, T]. \quad (3.8)$$

Then by Proposition 3.1, $(f \cdot x)'(t) \leq 0$ for a.e. $t \in [0, T]$, and $f(x(t))$ is decreasing on $[0, T]$.

In the sequel we denote by $\mu(E)$ the Lebesgue measure of $E \subset \mathbb{R}$.

Theorem 3.2. *Let $V \in \mathcal{A}$ be regular, let $x : [0, \infty) \rightarrow X$ be differentiable and suppose that*

$$x'(t) = V(x(t)) \quad \text{for a.e. } t \in [0, \infty). \quad (3.9)$$

Assume that there exists a positive number r such that

$$\mu(\{t \in [0, T] : \|x(t)\| \leq r\}) \rightarrow \infty \quad \text{as } T \rightarrow \infty. \quad (3.10)$$

Then $\lim_{t \rightarrow \infty} f(x(t)) = \inf(f)$.

Note that in this theorem the requirements imposed on the vector field V and the function f are considerably less stringent than those of Theorem 2.3.

Theorem 3.3. *Let $V \in \mathcal{A}$ be regular, let f be Lipschitzian on bounded subsets of X , and assume that $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$. Let K_0 and ϵ be positive. Then there exist $N_0 > 0$ and $\delta > 0$ such that for each $T \geq N_0$ and each differentiable mapping $x : [0, T] \rightarrow X$ satisfying*

$$\|x(0)\| \leq K_0 \quad \text{and} \quad \|x'(t) - V(x(t))\| \leq \delta \quad \text{for a.e. } t \in [0, T],$$

the following inequality holds for all $t \in [N_0, T]$:

$$f(x(t)) \leq \inf(f) + \epsilon.$$

4. PROOF OF THEOREM 3.2

Assume the contrary. Since $f(x(t))$ is decreasing on $[0, \infty)$, this means that there exists $\epsilon > 0$ such that

$$\lim_{t \rightarrow \infty} f(x(t)) > \inf(f) + \epsilon. \quad (4.1)$$

Then by Proposition 3.1 and (3.9), for each $T > 0$,

$$\begin{aligned} f(x(T)) - f(x(0)) &= \int_0^T (f \cdot x)'(t) dt \\ &= \int_0^T f^0(x(t), x'(t)) dt \\ &= \int_0^T f^0(x(t), V(x(t))) dt \\ &\leq \int_{\Omega_T} f^0(x(t), V(x(t))) dt, \end{aligned} \quad (4.2)$$

where

$$\Omega_T = \{t \in [0, T] : \|x(t)\| \leq r\}. \quad (4.3)$$

Since V is regular, there exists $\delta > 0$ such that for each $x \in X$ satisfying

$$\|x\| \leq r + 1 \quad \text{and} \quad f(x) \geq \inf(f) + \epsilon/2, \quad (4.4)$$

and each $l \in \partial f(x)$, we have

$$l(Vx) \leq -\delta. \quad (4.5)$$

It follows from (4.2), (4.3), (4.1), the definition of δ (see (4.4), (4.5)) and (3.10) that for each $T > 0$,

$$f(x(T)) - f(x(0)) \leq \int_{\Omega_T} f^0(x(t), V(x(t))) dt \leq -\delta \mu(\Omega_T) \rightarrow -\infty$$

as $T \rightarrow \infty$, which is a contradiction completing the proof.

5. PROOF OF THEOREM 3.3

We assume that $\epsilon < 1/2$ and choose

$$K_1 > \sup\{f(x) : x \in X \text{ and } \|x\| \leq K_0 + 1\}. \quad (5.1)$$

The set

$$\{x \in X : f(x) \leq K_1 + |\inf(f)| + 4\} \quad (5.2)$$

is bounded. Therefore there exists $K_2 > K_0 + K_1$ such that

$$\text{if } f(x) \leq K_1 + |\inf(f)| + 4, \quad \text{then } \|x\| \leq K_2. \quad (5.3)$$

There exists a number $K_3 > K_2 + 1$ such that

$$\sup\{f(x) : x \in X \text{ and } \|x\| \leq K_2 + 1\} + 2 < \inf\{f(x) : x \in X \text{ and } \|x\| \geq K_3\}. \quad (5.4)$$

There exists a number $L_0 > 0$ such that

$$|f(x_1) - f(x_2)| \leq L_0 \|x_1 - x_2\| \quad (5.5)$$

for each $x_1, x_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leq K_3 + 1. \quad (5.6)$$

Fix an integer

$$n > K_3 + 8/\epsilon. \quad (5.7)$$

There exists a positive number $\delta(n) < 1$ such that

(P1) for each $x \in X$ satisfying $\|x\| \leq n$ and $f(x) \geq \inf(f) + 1/n$, and each $l \in \partial f(x)$, we have $l(Vx) \leq -\delta(n)$.

Choose a natural number $N_0 > 8$ such that

$$8^{-1}\delta(n)N_0 > |\inf(f)| + \sup\{|f(z)| : z \in X \text{ and } \|z\| \leq K_2\} + 4 \quad (5.8)$$

and a positive number δ which satisfies

$$8\delta(N_0 + 1)(L_0 + 1) < \epsilon \quad \text{and} \quad (1 + L_0)\delta < \delta(n)/2. \quad (5.9)$$

Let $T \geq N_0$ and let $x : [0, T] \rightarrow X$ be a differentiable function such that

$$\|x(0)\| \leq K_2 \quad (5.10)$$

and

$$\|x'(t) - V(x(t))\| \leq \delta \text{ for a.e. } t \in [0, T]. \quad (5.11)$$

We will show that

$$\|x(t)\| \leq K_3, \quad t \in [0, \min\{2N_0, T\}]. \quad (5.12)$$

Assume the contrary. Then there exists $t_0 \in (0, \min\{2N_0, T\}]$ such that

$$\|x(t)\| \leq K_3, \quad t \in [0, t_0) \text{ and } \|x(t_0)\| = K_3. \quad (5.13)$$

It follows from Proposition 3.1, the convexity of directional derivatives, the inequality $f^0(x(t), Vx(t)) \leq 0$ which holds for all $t \in [0, T]$, (5.13), the definition of L_0 (see

(5.5), (5.6)), and (5.11) that

$$\begin{aligned}
 f(x(t_0)) - f(x(0)) &= \int_0^{t_0} (f \cdot x)'(t) dt \\
 &= \int_0^{t_0} f^0(x(t), x'(t)) dt \\
 &\leq \int_0^{t_0} f^0(x(t), V(x(t))) dt + \int_0^{t_0} f^0(x(t), x'(t) - V(x(t))) dt \\
 &\leq \int_0^{t_0} f^0(x(t), x'(t) - V(x(t))) dt \\
 &\leq \int_0^{t_0} L_0 \|x'(t) - V(x(t))\| dt \leq t_0 L_0 \delta.
 \end{aligned}$$

Thus by (5.9),

$$f(x(t_0)) \leq f(x(0)) + 2N_0 L_0 \delta < f(x(0)) + 1.$$

Since $\|x(0)\| \leq K_2$ (see (5.10)) and $\|x(t_0)\| = K_3$, the inequality just obtained contradicts (5.4). The contradiction we have reached proves (5.12).

We will now show that there exists

$$t_0 \in [1, N_0] \tag{5.14}$$

such that

$$f(x(t_0)) \leq \inf(f) + \epsilon/8. \tag{5.15}$$

Assume the contrary. Then

$$f(x(t)) > \inf(f) + \epsilon/8 \text{ and } \|x(t)\| \leq K_3, \quad t \in [1, N_0]. \tag{5.16}$$

It follows from (5.16), Property (P1) and (5.7) that

$$f^0(x(t), V(x(t))) \leq -\delta(n), \quad t \in [1, N_0]. \tag{5.17}$$

By (5.17), (5.16), (5.11), (5.9), the convexity of the directional derivatives of f , and the definition of L_0 (see (5.5), (5.6)), we have, for a.e. $t \in [1, N_0]$,

$$\begin{aligned}
 f^0(x(t), x'(t)) &\leq f^0(x(t), V(x(t))) + f^0(x(t), x'(t) - V(x(t))) \\
 &\leq -\delta(n) + L_0 \|x'(t) - V(x(t))\| \\
 &\leq -\delta(n) + L_0 \delta \leq -\delta(n)/2.
 \end{aligned} \tag{5.18}$$

It follows from the convexity of the directional derivatives of f , the inclusion $V \in \mathcal{A}$, (5.11), (5.12) and the definition of L_0 (see (5.5), (5.6)), that for a.e. $t \in [0, 1]$,

$$\begin{aligned}
 f^0(x(t), x'(t)) &\leq f^0(x(t), V(x(t))) + f^0(x(t), x'(t) - V(x(t))) \\
 &\leq f^0(x(t), x'(t) - V(x(t))) \\
 &\leq L_0 \|x'(t) - V(x(t))\| \leq L_0 \delta.
 \end{aligned} \tag{5.19}$$

The inequalities (5.10), (5.18) and (5.19) imply that

$$\begin{aligned} & \inf(f) - \sup \{f(z) : z \in X, \|z\| \leq K_2\} \\ & \leq f(x(N_0)) - f(x(0)) \\ & = \int_0^{N_0} f^0(x(t), x'(t)) dt \\ & = \int_0^1 f^0(x(t), x'(t)) dt + \int_1^{N_0} f^0(x(t), x'(t)) dt \\ & \leq -2^{-1} \delta(n) N_0 / 2 + 1. \end{aligned}$$

This contradicts (5.8). The contradiction we have obtained yields the existence of a point t_0 which satisfies both (5.14) and (5.15). Clearly, $\|x(t_0)\| \leq K_2$. Having established (5.12) and the existence of such a point t_0 for an arbitrary mapping x satisfying both (5.10) and (5.11), we now consider the mapping $x_0(t) = x(t + t_0)$, $t \in [0, T - t_0]$. Evidently, (5.10) and (5.11) hold true with x replaced by x_0 and T replaced by $T - t_0$. Hence, if $T - t_0 \geq N_0$, then we have

$$\|x(t)\| = \|x_0(t - t_0)\| \leq K_3, \quad t \in [t_0, t_0 + \min\{2N_0, T\}],$$

and there exists $t_1 \in [t_0 + 1, t_0 + N_0]$ for which

$$f(x(t_1)) \leq \inf(f) + \epsilon/8.$$

Repeating this procedure, we obtain by induction a finite sequence of points $\{t_i\}_{i=0}^q$ such that

$$\begin{aligned} t_0 & \in [1, N_0], \quad t_{i+1} - t_i \in [1, N_0], \quad i = 0, \dots, q-1, \quad T - t_q < N_0, \\ f(x(t_i)) & \leq \inf(f) + \epsilon/8, \quad i = 0, \dots, q, \\ \|x(t)\| & \leq K_3, \quad t \in [t_0, T]. \end{aligned}$$

Let $i \in \{0, \dots, q\}$, $t \leq T$, and $0 < t - t_i \leq N_0$. Then by Proposition 3.1, the convexity of the directional derivative of f , the inclusion $V \in \mathcal{A}$, the definition of L_0 (see (5.5), (5.6)), (5.9) and (5.11), we have

$$\begin{aligned} f(x(t)) - f(x(t_i)) & = \int_{t_i}^t f^0(x(t), x'(t)) dt \\ & \leq \int_{t_i}^t f^0(x(t), V(x(t))) dt + \int_{t_i}^t f^0(x(t), x'(t) - V(x(t))) dt \\ & \leq \int_{t_i}^t f^0(x(t), x'(t) - V(x(t))) dt \\ & \leq \int_{t_i}^t L_0 \|x'(t) - V(x(t))\| dt \\ & \leq L_0 \delta (t - t_i) \leq 2N_0 L_0 \delta < \epsilon/4 \end{aligned}$$

and hence

$$f(x(t)) \leq f(x(t_i)) + \epsilon/4 \leq \inf(f) + \epsilon/2.$$

This completes the proof of Theorem 3.3. ◇

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