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UNIFORM STABILIZATION FOR THE TIMOSHENKO BEAM BY A LOCALLY DISTRIBUTED DAMPING

ABDELAZIZ SOUFYANE & ALI WEHBE

Abstract. We study the uniform stabilization of a Timoshenko beam by one control force. We prove that under, one locally distributed damping, the exponential stability for this system is assured if and only if the wave speeds are the same.

1. Introduction

A basic linear model, developed in [19], for describing the transverse vibration of beams is given by two coupled partial differential equations

$$\begin{aligned} \rho w_{tt} &= (K(w_x - \varphi))_x, \\ I_\rho \varphi_{tt}(x,t) &= (EI\varphi_x)_x + K(w_x - \varphi). \end{aligned} \quad \text{on } (0,l) \times \mathbb{R}^+ \tag{1.1}$$

Here, t is the time variable and x the space coordinate along the beam, the length of which is l, in its equilibrium position. The function w is the transverse displacement of the beam and φ is the rotation angle of a filament of the beam. The coefficients ρ, I_ρ, E, I and K are the mass per unit length, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus respectively. The natural energy of the beam is

$$\mathcal{E}(t) = \frac{1}{2}\int_0^l \rho \left|\partial_t w\right|^2 + I_\rho \left|\partial_t \varphi\right|^2 + EI \left|\partial_x \varphi\right|^2 + K \left|\partial_x w - \varphi\right|^2)dx. \tag{1.2}$$

The aim of this paper is to study two types of stabilization of this system, the internal and the boundary stabilization. Let us mention some results about the stabilizability of the Timoshenko beam. The case of two boundary control force has already been considered by Kim and Renardy [4] for the Timoshenko beam. They proved the exponential decay of the energy $\mathcal{E}(t)$ by using a multiplier technique and provided numerical estimates of the eigenvalues of the operator associated to this system, and by Lagnese & Lions [11] for the study of the exact controllability, Taylor [7] studied the boundary control of system (1.1) with variable physical characteristics. Recently Shi & Feng [16] established the exponential decay of the energy $\mathcal{E}(t)$ with locally distributed feedback (two feedback).We will first prove that it is possible to stabilize uniformly (with respect to the initial data) this beam,

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which is assumed to be clamped at the two ends of $(0,l)$, by using a unique locally distributed feedback

$$\begin{gathered}\rho w_{tt} = (K(w_x - \varphi))_x,\\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(w_x - \varphi) - b(x)\varphi_t, \quad \text{on } (0,l)\times\mathbb{R}^+ \\ w(0,t) = w(l,t) = 0; \quad \varphi(0,t) = \varphi(l,t) = 0.\end{gathered} \tag{1.3}$$

where b is a positive continuous function of the space variable. Indeed, we prove the uniform stability holds for system (1.3) if and only if the wave speeds $\frac{K}{\rho}$ and $\frac{EI}{I_\rho}$ are the same. If not, the asymptotic stability for this system is proved. The techniques we use consist in the computation of the essential type (see the definition in the next section) of the associated semigroups, thanks to the result of Neves et al. [6].

Our paper is organized as follows. In section 2, we give a result of a well-posedness of the solution of the system and we study the strong asymptotic stability and the nonuniform stability of (1.3) under the assumption $\frac{K}{\rho} \neq \frac{EI}{I_\rho}$. In section 3, we study the uniform stability of (1.3) under the assumption $\frac{K}{\rho} = \frac{EI}{I_\rho}$.

2. Well-posedness, asymptotic and nonuniform stability

Well-posedness. Here we outline the existence theory of the solution of the Timoshenko beam. To establish the existence and uniqueness of the solution of the studied model we use the semigroup theory, we suppose that $b \in C([0,l])$. The energy space associated to system (1.3) is

$$H := (H_0^1(]0,l[) \times L^2(]0,l[))^2$$

The inner product in the energy space is defined as follows:

$$(Y_1, Y_2) := \int_0^l (K(\partial_x u_1 - w_1)(\partial_x u_2 - w_2) + \rho v_1 v_2 + I_\rho f_1 f_2 + EI(\partial_x w_1 \partial_x w_2))dx,$$

where $Y_k = \begin{pmatrix} u_k \\ v_k \\ w_k \\ f_k \end{pmatrix} \in H$, $k = 1,2$. In the sequel we will denote by $\|Y\|^2 := (Y,Y)$, the norm in the energy space. The system 1.3 can be written as

$$\partial_t Y(t) = LY(t),$$

where

$$Y(t) := \begin{pmatrix} w(t) \\ \partial_t w(t) \\ \varphi(t) \\ \partial_t \varphi(t) \end{pmatrix}, \quad L := \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{K}{\rho}\partial_{xx} & 0 & -\frac{K}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & I \\ \frac{K}{I_\rho}\partial_x & 0 & \frac{EI}{I_\rho}\partial_{xx} - \frac{K}{I_\rho} & -b(x) \end{pmatrix}$$

with $D(L) := ((H^2(]0,l[) \cap H_0^1(]0,l[)) \times H_0^1(]0,l[))^2$.

Proposition 2.1 ([17],[16]). *The operator $(L, D(L))$ generates a C_0-semigroup of contractions $(e^{Lt})_{t\geq 0}$ on H.*

Asymptotic strong Stability and nonuniform stability. Before giving our first result, we need to recall some results and the following definitions:

e^{Lt} is asymptotically stable if, for any $Y_0 \in H$ $\lim_{t\to\infty} e^{Lt}Y_0 = 0$.

e^{Lt} is uniformly (or exponentially) stable if there exist $\omega < 0$ and $M > 0$ such that

$$\|e^{Lt}\| \le M e^{\omega t} \quad t \in \mathbb{R}^+$$

For a continuous linear operator from a Banach space into itself, we define its essential spectral radius $r_e(L)$ as

$$r_e(L) = \inf \Big\{ R > 0 : \mu \in \sigma(L), |\mu| > R \text{ implies } \mu \text{ is an isolated eigenvalue of finite multiplicity} \Big\} \tag{2.1}$$

It is well-known (see Gohberg-Krein [3]) that, if $r(L)$ is the spectral radius of L

$$r_e(L) \le r(L);$$
$$r_e(L+K) = r_e(L) \quad \forall K \in L(X),\ K \text{ compact}$$

If e^{Lt} is a C_0-semigroup generated by L, then (see for instance [5]) there exist two real numbers $\omega = \omega(L)$ and $\omega_e = \omega_e(L)$ such that

$$r(e^{Lt}) = e^{\omega t}, \tag{2.2}$$
$$r_e(e^{Lt}) = e^{\omega_e t} \quad \forall t \in \mathbb{R}^+. \tag{2.3}$$

ω is often called the *type* and ω_e the *essential type* of the semigroup. A third real number which plays an essential role in stability theory is the *spectral abscissa* $s(L)$ of L defined by

$$s(L) = \sup \{\operatorname{Re}\lambda : \lambda \in \sigma(L)\}.$$

These three real numbers are related as follows [5, Theorem 3.6.1., p. 107]:

$$\omega(L) = \max(\omega_e(L), s(L)).$$

Clearly, the uniform stability of e^{Lt} is equivalent to $\omega(L) < 0$. We are now ready to state our first result.

Theorem 2.2. *Assume that $b \in C([0,l])$ and*

$$b(x) \ge \overline{b} > 0 \quad \text{on } [b_0, b_1] \subset [0,l]. \tag{2.4}$$

Then:

- *e^{Lt} is asymptotically stable.*
- *If $\frac{K}{\rho} \neq \frac{EI}{I_\rho}$, then e^{Lt} is non uniformly stable.*

Before giving the proof of this theorem we need to recall the following result.

Theorem 2.3 (Benchimol [2]). *Let L be a maximal linear operator in a complex Hilbert space H and assume that:*

(a) *L has a compact resolvent.*
(b) *L does not have purely imaginary eingenvalues.*

Then e^{Lt} is strongly stable.

Theorem 2.2. To prove strong stability of e^{Lt}, it remains to verify the properties (a) and (b) of theorem 2.3.

Property (a) follows at once from the compactness of the imbedding $D(L)$ in H, a consequence of Rellich's theorem.

Suppose the conclusion of (b) is false, then L have a purely imaginary eigenvalues $i\omega$. Let U_1 the eigenvectors associated to $i\omega$. Using the definition of L it follows that $LU_1 = i\omega U_1$ if and only if

$$\begin{gathered} K\partial_{xx}u - K\partial_x v = -\rho\omega^2 u, \\ EI.\partial_{xx}v + K\partial_x u - Kv - ib(x)\omega v = -I_\rho\omega^2 v, \\ u\big|_{0,l} = v\big|_{0,l} = 0. \end{gathered} \tag{2.5}$$

We multiply the second equation of this system by v to obtain

$$\int_0^l (EI|\partial_x v|^2 - K\partial_x u.v + (K - I_\rho\omega^2)|v|^2)dx = 0 \quad \text{and} \quad \int_0^l \omega b(x)|v|^2 dx = 0.$$

Then $v = 0$, on $[b_0, b_1]$, and

$$EI.\partial_{xx}v + K\partial_x u - Kv = -I_\rho\omega^2 v,$$

which implies $\partial_x u = 0$ on $]b_0, b_1[$. From

$$K\partial_{xx}u - K\partial_x v = -\rho\omega^2 u,$$

it follows that $u = 0$ on $]b_0, b_1[$.

Now, it is trivial to see that $u = v = 0$ is the solution of the system

$$\begin{gathered} K\partial_{xx}u - K\partial_x v = -\rho\omega^2 u, \\ EI.\partial_{xx}v + K\partial_x u - Kv = -I_\rho\omega^2 v, \\ v = u = 0 \quad \text{on }]b_0, b_1[, \\ u\big|_{0,l} = v\big|_{0,l} = 0. \end{gathered} \tag{2.6}$$

This complete the proof of the strong stability of $(e^{Lt})_{t\geq 0}$.

The proof of the nonuniform stability for e^{Lt} when $\frac{K}{\rho} \neq \frac{EI}{I\rho}$ is based on the computation of the essential type of $(e^{Lt})_{t\geq 0}$. Let

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{K}{I_\rho} & 0 \end{pmatrix}$$

This operator is compact in the energy space H. Then $r_e(L) = r_e(L - D)$. Now we calculate the essential spectral radius of $L_1 := L - D$, for this reason we transform the equations by introducing the following variables:

$$\begin{gathered} p = -\sqrt{\frac{K}{\rho}}\partial_x u + \partial_t u, \quad q = \sqrt{\frac{K}{\rho}}\partial_x u + \partial_t u, \\ \varphi = -\sqrt{\frac{EI}{I_\rho}}\partial_x v + \partial_t v, \quad \psi = \sqrt{\frac{EI}{I_\rho}}\partial_x v + \partial_t v. \end{gathered}$$

which are solutions of the system

$$\partial_t \begin{pmatrix} p \\ \varphi \\ q \\ \psi \end{pmatrix} + K\partial_x \begin{pmatrix} p \\ \varphi \\ q \\ \psi \end{pmatrix} + C \begin{pmatrix} p \\ \varphi \\ q \\ \psi \end{pmatrix} = 0, \tag{2.7}$$

with the boundary conditions

$$(p+q)(0,t) = (\varphi+\psi)(0,t) = (p+q)(l,t) = (\varphi+\psi)(l,t) = 0,$$

where

$$K = \begin{pmatrix} \sqrt{K/\rho} & 0 & 0 & 0 \\ 0 & \sqrt{EI/I_\rho} & 0 & 0 \\ 0 & 0 & -\sqrt{K/\rho} & 0 \\ 0 & 0 & 0 & -\sqrt{EI/I_\rho} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -\frac{K}{2\rho}\sqrt{\frac{I_\rho}{EI}} & 0 & \frac{K}{2\rho}\sqrt{\frac{I_\rho}{EI}} \\ \frac{\sqrt{K\rho}}{2I_\rho} & -\frac{b(x)}{2I_\rho} & -\frac{\sqrt{K\rho}}{2I_\rho} & -\frac{b(x)}{2I_\rho} \\ 0 & -\frac{K}{2\rho}\sqrt{\frac{I_\rho}{EI}} & 0 & \frac{K}{2\rho}\sqrt{\frac{I_\rho}{EI}} \\ \frac{\sqrt{K\rho}}{2I_\rho} & -\frac{b(x)}{2I_\rho} & -\frac{\sqrt{K\rho}}{2I_\rho} & -\frac{b(x)}{2I_\rho} \end{pmatrix}$$

When $\frac{K}{\rho} \neq \frac{EI}{I_\rho}$, we can apply a result in [6, p 324] which implies that $r_e(e^{L_1 t}) = r_e(e^{L_2 t}) = \exp(\alpha t)$ where

$$L_2 = \begin{pmatrix} \sqrt{K/\rho} & 0 & 0 & 0 \\ 0 & \sqrt{EI/I_\rho} & 0 & 0 \\ 0 & 0 & -\sqrt{K/\rho} & 0 \\ 0 & 0 & 0 & -\sqrt{EI/I_\rho} \end{pmatrix}\partial_x + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{b(x)}{2I_\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{b(x)}{2I_\rho} \end{pmatrix}$$

and $\alpha = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(L_2)\}$. To compute α we solve the system

$$\sqrt{\frac{K}{\rho}}\partial_x p = \lambda p, \quad \sqrt{\frac{K}{\rho}}\partial_x q = -\lambda q,$$
$$\sqrt{\frac{EI}{I_\rho}}\partial_x \varphi = (\lambda + \frac{b(x)}{2I_\rho})\varphi, \quad \sqrt{\frac{EI}{I_\rho}}\partial_x \psi = -(\lambda + \frac{b(x)}{2I_\rho})\psi,$$

with the boundary conditions

$$(p+q)(0,t) = (\varphi+\psi)(0,t) = (p+q)(l,t) = (\varphi+\psi)(l,t) = 0.$$

Then we have

$$p(x) = c_1 \exp(\lambda\sqrt{\frac{\rho}{K}}x), \quad q(x) = -c_1 \exp(-\lambda\sqrt{\frac{\rho}{K}}x),$$
$$\varphi(x) = c_2 \exp(\sqrt{\frac{I_\rho}{EI}}(\lambda x + \int_0^x \frac{b(y)}{2I_\rho}dy)),$$
$$\psi(x) = -c_2 \exp(-\sqrt{\frac{I_\rho}{EI}}(\lambda x + \int_0^x \frac{b(y)}{2I_\rho}dy)),$$

where c_1 and c_2 are in $\mathbb{R}\backslash\{0\}$. Using the boundary conditions, we see that one of the following two equations must hold for non-trivial solutions,

$$\exp(\lambda\sqrt{\frac{\rho}{K}}l) - \exp(-\lambda\sqrt{\frac{\rho}{K}}l) = 0,$$
$$\exp(\sqrt{\frac{I_\rho}{EI}}(\lambda l + \int_0^l \frac{b(y)}{2I_\rho}dy)) - \exp(-\sqrt{\frac{I_\rho}{EI}}(\lambda l + \int_0^l \frac{b(y)}{2I_\rho}dy) = 0,$$

these imply that

$$\operatorname{Re}(\lambda)=0 \quad or \quad \operatorname{Re}(\lambda)=-\frac{1}{l}\int_0^l \Big(\frac{b(x)}{2I_\rho}\Big)dx,$$

and thus

$$\alpha=\sup\{\operatorname{Re}(\lambda), \lambda\in\sigma(L_2)\}=0.$$

Then $r_e(e^{L_2 t})=1$. Using the property $r(e^{Lt})\geq r_e(e^{Lt})$ we deduce, the non uniform stability of the semigroup associated to the Timoshenko beam if $\frac{K}{\rho}\neq\frac{EI}{I\rho}$. □

3. Uniform stability

In this section our main result is the following theorem.

Theorem 3.1. *Assuming that $b(x)$ satisfies (2.4), e^{Lt} is exponentially stable if $\frac{K}{\rho}=\frac{EI}{I_\rho}$.*

The proof of the uniform stability is based on the construction of a Lyapunov function $\phi(Y(t))$ satisfying the following inequalities:

i) There exist two positives constants, d_0, d_1 such that

$$d_0\left\|Y\right\|^2\leq\phi(Y)\leq d_1\left\|Y\right\|^2 \quad \forall\, Y\,\epsilon\, D(L) \tag{3.1}$$

ii) There exist a positive constant d_2 such that

$$\partial_t\phi(Y(t))\leq -d_2\left\|Y(t)\right\|^2 \tag{3.2}$$

To construct this function we will use the multiplicative technique.

Lemma 3.2. *If $\phi(Y(t))$ satisfies (3.1), (3.2), then there exist m and e a positive constant such that*

$$\|Y(t)\|^2\leq m\exp(-et)\|Y(0)\|^2$$

Proof. It follows from (3.1) that $-\left\|Y(t)\right\|^2\leq-\frac{1}{d_1}\phi(Y(t))$. Using (3.2) then we have

$$\partial_t\phi(Y(t))\leq-\frac{d_2}{d_1}\phi(Y(t))$$

thus

$$\phi(Y(t))\leq\exp(-\frac{d_2}{d_1}t)\phi(Y)\,.$$

Using (3.1), we have $\|Y(t)\|^2\leq\frac{d_1}{d_0}\exp(-\frac{d_2}{d_1}t)\|Y\|^2$ □

Lemma 3.3. *For any positive constant ε_1 and scalar functions h, c such that: $h(0)>0$, $h(l)<0$, $h_x<0$ on $[b_0,b_1]$, $h_x>0$ on $[0,l]\backslash[b_0,b_1]$, and $c_{xx}<0$ on $[0,l]$, we have the following statements:*
i)

$$\begin{aligned}\partial_t\int_0^l I_\rho\partial_t\varphi.h(x)\varphi_x dx =&\frac{-EI}{2}\int_0^l h_x(\varphi_x)^2dx-\frac{I_\rho}{2}\int_0^l h_x(\partial_t\varphi)^2dx+\frac{EI}{2}[h.(\varphi_x)^2]_0^l\\ &-\int_0^l(b(x)\partial_t\varphi).h(x)\varphi_x dx+K\int_0^l(w_x-\varphi).h\varphi_x dx,\end{aligned}$$

ii)

$$\partial_t\int_0^l I_\rho\partial_t\varphi.c(x)\varphi dx=-EI\int_0^l c(\varphi_x)^2dx+\frac{EI}{2}\int_0^l c_{xx}(\varphi)^2dx+I_\rho\int_0^l c(\partial_t\varphi)^2dx.$$

$$+ K\int_0^l (w_x-\varphi).c\varphi dx - \int_0^l (b(x)\partial_t\varphi).c(x)\varphi dx.$$

iii)

$$\begin{aligned}
&\partial_t \int_0^l \rho\partial_t w.\varepsilon_1(-\partial_{xx})^{-1}\partial_x(h(x)\varphi_x + c(x)\varphi)dx\\
&= K\int_0^l \varepsilon_1 h(x)\varphi_x + \varepsilon_1 c(x)\varphi dx.\int_0^l \varphi dx + \int_0^l K(w_x-\varphi).(\varepsilon_1 h(x)\varphi_x + \varepsilon_1 c(x)\varphi)dx\\
&+ \int_0^l \rho\partial_t w.((-\partial_{xx})^{-1}\partial_x(\varepsilon_1 h(x)\partial_t\varphi_x + \varepsilon_1 c(x)\partial_t\varphi))dx,
\end{aligned}$$

where $(-\partial_{xx})^{-1}$ is taken in the sense of Dirichlet boundary conditions.

Proof. i) Multiplying the second equation in 1.3) by $h(x)\varphi_x$, we get

$$\begin{aligned}
&\partial_t \int_0^l I_\rho\partial_t\varphi.h(x)\varphi_x dx = \int_0^l I_\rho\partial_{tt}\varphi.h(x)\varphi_x dx + \int_0^l I_\rho\partial_t\varphi.h(x)\partial_t\varphi_x dx\\
&= \int_0^l (EI\varphi_{xx} + K(w_x-\varphi) - b(x)\partial_t\varphi).h(x)\varphi_x dx + \int_0^l I_\rho\partial_t\varphi.h(x)\partial_t\varphi_x dx\\
&= \int_0^l (-\frac{EI}{2}(\varphi_x)^2 h_x - \frac{I_\rho}{2}(\partial_t\varphi)^2 h_x)dx\\
&\quad + \int_0^l (K(w_x-\varphi) - b(x)\partial_t\varphi).h(x)\varphi_x dx + \frac{EI}{2}[h.(\varphi_x)^2]_0^l
\end{aligned}$$

ii) Multiplying the second equation of (1.3) by $c(x)\varphi$, we get

$$\begin{aligned}
&\partial_t \int_0^l I_\rho\partial_t\varphi.c(x)\varphi dx\\
&= \int_0^l I_\rho\partial_{tt}\varphi.c(x)\varphi dx + \int_0^l I_\rho\partial_t\varphi.c(x)\partial_t\varphi dx\\
&= \int_0^l (EI\varphi_{xx} + K(w_x-\varphi) - b(x)\partial_t\varphi).c(x)\varphi dx + \int_0^l I_\rho\partial_t\varphi.c(x)\partial_t\varphi dx\\
&= -EI\int_0^l c(\varphi_x)^2 dx + \frac{EI}{2}\int_0^l c_{xx}(\varphi)^2 dx + K\int_0^l (w_x-\varphi).c\varphi dx\\
&\quad - \int_0^l (b(x)\partial_t\varphi).c(x)\varphi dx + I_\rho\int_0^l c(\partial_t\varphi)^2 dx
\end{aligned}$$

iii) Multiplying the first equation of (1.3) by

$$f = (-\partial_{xx})^{-1}\partial_x(\varepsilon_1 h(x)\varphi_x + \varepsilon_1 c(x)\varphi), \quad f(0)=f(l)=0,$$

we obtain

$$\begin{aligned}
&\partial_t \int_0^l (\rho\partial_t w.f)dx\\
&= \int_0^l K\partial_x(w_x-\varphi).f\,dx + \int_0^l \rho\partial_t w.(\partial_t f)dx\\
&= -\int_0^l K(w_x-\varphi).f_x\,dx + \int_0^l \rho\partial_t w.(\partial_t f)dx
\end{aligned}$$

$$= -\int_0^l K(w_x-\varphi).\partial_x(-\partial_{xx})^{-1}\partial_x(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx + \int_0^l \rho\partial_t w.(\partial_t f)dx$$

Note that $-\partial_x(-\partial_{xx})^{-1}\partial_x v = v - \frac{1}{l}\int_0^l v\, dx$. Then

$$\begin{aligned}
&\partial_t \int_0^l (\rho\partial_t w.f)dx \\
&= \int_0^l K(w_x-\varphi).(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx \\
&\quad - \frac{1}{l}\int_0^l \varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi dx.\int_0^l K(w_x-\varphi)dx + \int_0^l \rho\partial_t w.(\partial_t f)dx \\
&= \int_0^l K(w_x-\varphi).(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx + \frac{K}{l}\int_0^l \varepsilon_1 h(x)\varphi_x \\
&\quad + \varepsilon_1 c(x)\varphi dx.\int_0^l \varphi dx - \int_0^l \rho\partial_t w.((-\partial_{xx})^{-1}\partial_x(\varepsilon_1 h(x)\partial_t\varphi_x+\varepsilon_1 c(x)\partial_t\varphi))dx.
\end{aligned}$$

□

Now we introduce the function

$$\phi_1(Y(t)) := \mathcal{E}(t) + \varepsilon_1\int_0^l I_\rho\partial_t\varphi.h(x)\varphi_x dx + \varepsilon_1\int_0^l I_\rho\partial_t\varphi.c(x)\varphi dx - \int_0^l (\rho\partial_t w.f)dx$$

Lemma 3.4. *For positive constants a_i $(i = 1, 4)$ and*

$$\alpha = |(h_x\partial_x(-\partial_{xx})^{-1} - h)|_{L^2(0,l)}\,, \quad \beta = |\partial_x(-\partial_{xx})^{-1}|_{L^2(0,l)}$$

we have

$$\begin{aligned}
&|\partial_t\phi_1(Y(t))| \\
&\le \int_0^l (-b(x) - \varepsilon_1\frac{I_\rho}{2}h_x + \varepsilon_1 I_\rho c + \varepsilon_1\frac{a_1 b^2}{2} + \varepsilon_1\frac{a_2 b^2}{2} + \frac{\varepsilon_1\alpha}{2a_3} + \frac{\varepsilon_1\beta c^2}{2a_4})(\partial_t\varphi)^2 dx \\
&\quad + \int_0^l (-\varepsilon_1\frac{EI}{2}h_x - \varepsilon_1 EIc + \varepsilon_1\frac{h^2}{2a_1})(\varphi_x)^2 dx + \varepsilon_1\frac{EI}{2}[h.(\varphi_x)^2]_0^l \\
&\quad + \int_0^l (\varepsilon_1\frac{EI}{2}c_{xx} + \varepsilon_1\frac{c^2}{2a_2})(\varphi)^2 dx \\
&\quad + \big|K\int_0^l (\varepsilon_1 h(x)\varphi_x + \varepsilon_1 c(x)\varphi)dx.\int_0^l \varphi dx\big| \\
&\quad + (\frac{\varepsilon_1 a_3\alpha}{2} + \frac{\varepsilon_1 a_4\beta}{2})\int_0^l (\rho\partial_t w)^2 dx.
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\partial_t\phi_1(Y(t)) \\
&= \partial_t\mathcal{E}(t) + \varepsilon_1\partial_t\int_0^l I_\rho\partial_t\varphi.h(x)\varphi_x dx + \varepsilon_1\partial_t\int_0^l I_\rho\partial_t\varphi.c(x)\varphi dx - \partial_t\int_0^l (\rho\partial_t w.f)dx \\
&= -\int_0^l b(x)(\partial_t\varphi)^2 dx + \varepsilon_1(\int_0^l (-\frac{EI}{2}(\varphi_x)^2 h_x - \frac{I_\rho}{2}(\partial_t\varphi)^2 h_x)dx \\
&\quad + \int_0^l (K(w_x-\varphi) - b(x)\partial_t\varphi).h(x)\varphi_x dx + \frac{EI}{2}[h.(\varphi_x)^2]_0^l)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon_1(-EI\int_0^l c(\varphi_x)^2dx+\frac{EI}{2}\int_0^l c_{xx}(\varphi)^2dx+K\int_0^l (w_x-\varphi).c\varphi dx\\
&-\int_0^l (b(x)\partial_t\varphi).c(x)\varphi dx+I_\rho\int_0^l c(\partial_t\varphi)^2dx)\\
&-\int_0^l K(w_x-\varphi).(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx\\
&-\frac{K}{l}\int_0^l(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx.\int_0^l\varphi dx\\
&-\int_0^l\rho\partial_t w.((-\partial_{xx})^{-1}\partial_x(\varepsilon_1 h(x)\partial_t\varphi_x+\varepsilon_1 c(x)\partial_t\varphi))dx\\
=&\int_0^l(-b(x)-\varepsilon_1\frac{I_\rho}{2}h_x+\varepsilon_1 I_\rho c)(\partial_t\varphi)^2dx+\int_0^l(-\varepsilon_1\frac{EI}{2}h_x-\varepsilon_1 EIc)(\varphi_x)^2dx\\
&-\varepsilon_1\int_0^l(b(x)\partial_t\varphi).h(x)\varphi_x dx-K\int_0^l(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx.\int_0^l\varphi dx\\
&+\varepsilon_1\frac{EI}{2}[h.(\varphi_x)^2]_0^l+\varepsilon_1\frac{EI}{2}\int_0^l c_{xx}(\varphi)^2dx-\varepsilon_1\int_0^l(b(x)\partial_t\varphi).c(x)\varphi dx\\
&-\int_0^l\rho\partial_t w.((-\partial_{xx})^{-1}\partial_x(\varepsilon_1 h(x)\partial_t\varphi_x+\varepsilon_1 c(x)\partial_t\varphi))dx.\\
=&\int_0^l(-b(x)-\varepsilon_1\frac{I_\rho}{2}h_x+\varepsilon_2 I_\rho c)(\partial_t\varphi)^2dx+\int_0^l(-\varepsilon_1\frac{EI}{2}h_x-\varepsilon_1 EIc)(\varphi_x)^2dx\\
&-\varepsilon_1\int_0^l(b(x)\partial_t\varphi).h(x)\varphi_x dx-\frac{K}{l}\int_0^l(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx.\int_0^l\varphi dx\\
&+\varepsilon_1\frac{EI}{2}[h.(\varphi_x)^2]_0^l+\varepsilon_1\frac{EI}{2}\int_0^l c_{xx}(\varphi)^2dx-\varepsilon_1\int_0^l(b(x)\partial_t\varphi).c(x)\varphi dx\\
&-\int_0^l\rho\varepsilon_1(h_x\partial_x(-\partial_{xx})^{-1}-h)\partial_t w.\partial_t\varphi dx+\int_0^l\rho\varepsilon_1\partial_x(-\partial_{xx})^{-1}\partial_t w.c(x)\partial_t\varphi dx.
\end{aligned}
$$

Using Young's inequality we obtain

$$
\begin{gathered}
\Big|\int_0^l(b(x)\partial_t\varphi).h(x)\varphi_x dx\Big|\le\frac{a_1}{2}\int_0^l(b(x)\partial_t\varphi)^2dx+\frac{1}{2a_1}\int_0^l(h(x)\varphi_x)^2dx,\\
\Big|\int_0^l(b(x)\partial_t\varphi).c(x)\varphi dx\Big|\le\frac{a_2}{2}\int_0^l(b(x)\partial_t\varphi)^2dx+\frac{1}{2a_2}\int_0^l(c(x)\varphi)^2dx,\\
\Big|\int_0^l\rho\varepsilon_1(h_x\partial_x(-\partial_{xx})^{-1}-h)\partial_t w.\partial_t\varphi dx\Big|\le\frac{\varepsilon_1 a_3\alpha}{2}\int_0^l(\rho\partial_t w)^2dx+\frac{\varepsilon_1\alpha}{2a_3}\int_0^l(\partial_t\varphi)^2dx\,,\\
\Big|\int_0^l\rho\varepsilon_1\partial_x(-\partial_{xx})^{-1}\partial_t w.c(x)\partial_t\varphi dx\Big|\le\frac{\varepsilon_1 a_4\beta}{2}\int_0^l(\rho\partial_t w)^2dx+\frac{\varepsilon_1\beta}{2a_4}\int_0^l(c(x)\partial_t\varphi)^2dx.
\end{gathered}
$$

Then

$$
\begin{aligned}
&|\partial_t\phi_1(Y(t))|\\
&\le\int_0^l(-b(x)-\varepsilon_1\frac{I_\rho}{2}h_x+\varepsilon_1 I_\rho c+\varepsilon_1\frac{a_1b^2}{2}+\varepsilon_1\frac{a_2b^2}{2}+\frac{\varepsilon_1\alpha}{2a_3}+\frac{\varepsilon_1\beta c^2}{2a_4})(\partial_t\varphi)^2dx
\end{aligned}
$$

$$+\int_0^l(-\varepsilon_1\frac{EI}{2}h_x-\varepsilon_1 EIc+\varepsilon_1\frac{h^2}{2a_1})(\varphi_x)^2dx$$
$$+\varepsilon_1\frac{EI}{2}[h.(\varphi_x)^2]_0^l+\int_0^l(\varepsilon_1\frac{EI}{2}c_{xx}+\varepsilon_1\frac{c^2}{2a_2})(\varphi)^2dx$$
$$+\big|\frac{K}{l}\int_0^l(\varepsilon_1 h(x)\varphi_x+\varepsilon_1 c(x)\varphi)dx.\int_0^l\varphi dx\big|+(\frac{\varepsilon_1 a_3\alpha}{2}+\frac{\varepsilon_1 a_4\beta}{2})\int_0^l(\rho\partial_t w)^2dx.$$

□

Lemma 3.5. *For scalar positive constants a_i ($i=5,6$), and scalar functions k, d such that $k(0)>0$, $k(l)<0$, $k_x<0$ on $[b_0,b_1]$, $k_x>0$ on $[0,l]\backslash[b_0,b_1]$, and $d>0$ on $[0,l]$, we have: i)*

$$\big|\partial_t\int_0^l\rho\partial_t w.k(x)w_xdx\big|\le\int_0^l(-\frac{K}{2}k_x+K\frac{a_5}{2})(w_x)^2dx+K\frac{1}{2a_5}\int_0^l(k(x)\varphi_x)^2dx$$
$$-\int_0^l\frac{\rho}{2}k_x(\partial_t w)^2dx+\frac{K}{2}[k.(w_x)^2]_0^l,$$

ii)

$$-\partial_t\int_0^l\rho\partial_t w.d(x)wdx=\int_0^l K(w_x-\varphi).\partial_x(d(x)w)dx-\int_0^l\rho d(x)(\partial_t w)^2dx,$$

and iii)

$$\big|\partial_t\int_0^l(\partial_t\varphi.(\varphi-w_x)-\partial_t w.\varphi_x)dx\big|$$
$$\le\int_0^l(-\frac{K}{I_\rho}+\frac{a_6}{2I_\rho})(\varphi-w_x)^2dx+\int_0^l(1+\frac{I_\rho b^2}{2a_6})(\partial_t\varphi)^2dx+\big|\frac{K}{\rho}[\varphi_x.w_x]_0^l\big|.$$

Proof. i) Multiplying the first equation in (1.3) by $k(x)w_x$, we obtain

$$\begin{aligned}&\partial_t\int_0^l\rho\partial_t w.k(x)w_xdx\\&=\int_0^l\rho\partial_{tt}w.k(x)w_xdx+\int_0^l\rho\partial_t w.k(x)\partial_t w_xdx\\&=\int_0^l K\partial_x(w_x-\varphi).k(x)w_xdx+\int_0^l\rho\partial_t w.k(x)\partial_t w_xdx\\&=-\int_0^l\frac{K}{2}k_x(w_x)^2dx-\int_0^l K\varphi_x.k(x)w_xdx-\int_0^l\frac{\rho}{2}k_x(\partial_t w)^2dx+\frac{K}{2}[k.(w_x)^2]_0^l\end{aligned}$$

Using Young's inequality we obtain

$$\int_0^l\varphi_x.k(x)w_xdx\le\frac{a_5}{2}\int_0^l(w_x)^2dx+\frac{1}{2a_5}\int_0^l(k(x)\varphi_x)^2dx.$$

Then

$$\partial_t\int_0^l I_\rho\partial_t\varphi.h(x)\varphi_xdx\le\int_0^l(-\frac{K}{2}k_x+K\frac{a_5}{2})(w_x)^2dx+K\frac{1}{2a_5}\int_0^l(k(x)\varphi_x)^2dx$$
$$-\int_0^l\frac{\rho}{2}k_x(\partial_t w)^2dx+\frac{K}{2}[k.(w_x)^2]_0^l\,.$$

ii) Multiplying the first equation in (1.3) by $d(x)w$, we obtain

$$\begin{aligned}-\partial_t\int_0^l \rho\partial_t w.d(x)wdx &= -\int_0^l \rho\partial_{tt}w.d(x)wdx - \int_0^l \rho\partial_t w.d(x)\partial_t wdx\\ &= -\int_0^l K\partial_x(w_x-\varphi).d(x)wdx - \int_0^l \rho d(x)(\partial_t w)^2dx\\ &= \int_0^l K(w_x-\varphi).\partial_x(d(x)w)dx - \int_0^l \rho d(x)(\partial_t w)^2dx\end{aligned}$$

iii) Multiplying the first equation in (1.3) by φ_x and the second equation of system (1.3) by $\varphi - w_x$, we obtain

$$\begin{aligned}&\partial_t\int_0^l (\partial_t\varphi.(\varphi-w_x) - \partial_t w.\varphi_x)dx\\ &= \int_0^l \partial_{tt}\varphi.(\varphi-w_x)dx + \int_0^l \partial_t\varphi.(\partial_t\varphi - \partial_t w_x)dx\\ &\quad - \int_0^l \partial_{tt}w.\varphi_x dx - \int_0^l \partial_t w\partial_t\varphi_x dx\\ &= \int_0^l \frac{EI}{I_\rho}\varphi_{xx}.(\varphi-w_x)dx - \int_0^l \frac{K}{I_\rho}(\varphi-w_x)^2dx - \int_0^l \frac{b(x)}{I_\rho}\partial_t\varphi.(\varphi-w_x)dx\\ &\quad + \int_0^l \partial_t\varphi.(\partial_t\varphi-\partial_t w_x)dx - \int_0^l \frac{K}{\rho}\partial_x(w_x-\varphi).\varphi_x dx - \int_0^l \partial_t w\partial_t\varphi_x dx.\end{aligned}$$

Using the fact that $\frac{K}{\rho} = \frac{EI}{I_\rho}$, we have

$$\begin{aligned}&\partial_t\int_0^l (\partial_t\varphi.(\varphi-w_x) - \partial_t w.\varphi_x)dx\\ &= -\int_0^l \frac{K}{I_\rho}(\varphi-w_x)^2dx - \int_0^l \frac{b(x)}{I_\rho}\partial_t\varphi.(\varphi-w_x)dx + \int_0^l (\partial_t\varphi)^2dx - \frac{K}{\rho}[\varphi_x.w_x]_0^l.\end{aligned}$$

Using Young's inequality we have

$$\Big|\int_0^l \frac{b(x)}{I_\rho}\partial_t\varphi.(\varphi-w_x)dx\Big| \le \frac{I_\rho}{2a_6}\int_0^l (b(x)\partial_t\varphi)^2dx + \frac{a_6}{2I_\rho}\int_0^l (\varphi-w_x)^2dx$$

Therefore,

$$\begin{aligned}&\Big|\partial_t\int_0^l (\partial_t\varphi.(\varphi-w_x) - \partial_t w.\varphi_x)dx\Big|\\ &\le \int_0^l \Big(-\frac{K}{I_\rho} + \frac{a_6}{2I_\rho}\Big)(\varphi-w_x)^2dx + \int_0^l \Big(1+\frac{I_\rho b^2}{2a_6}\Big)(\partial_t\varphi)^2dx + \Big|\frac{K}{\rho}[\varphi_x.w_x]_0^l\Big|.\end{aligned}$$

□

Now, we define the Lyapunov function associated to this problem.

$$\begin{aligned}\phi(Y(t)) =& \phi_1(Y(t)) + \varepsilon_2\int_0^l (\partial_t\varphi.(\varphi-w_x) - \partial_t w.\varphi_x)dx\\ &+ \varepsilon_3\int_0^l \rho\partial_t w.(k(x)w_x - d(x)w)dx.\end{aligned} \tag{3.3}$$

Lemma 3.6. *For ε_i $(i = 1, 2, 3)$ sufficiently small and c, h, k, d satisfying*

$$-\frac{h_x(x)}{2} + c(x) < 0 \quad \forall\, x \in]0, l[\setminus]b_0, b_1[$$
$$\frac{h_x(x)}{2} + c(x) > 0 \quad \forall\, x \in]0, l[$$
$$\frac{k_x(x)}{2} + d(x) > 0 \quad \forall\, x \in]0, l[,$$

the function $\phi(Y(t))$ satisfies (3.1) and (3.2).

Proof. Note that

$$\begin{aligned}\partial_t\phi(Y(t)) =& \partial_t\phi_1(Y(t)) + \varepsilon_2\partial_t\int_0^l(\partial_t\varphi.(\varphi - w_x) - \partial_t w.\varphi_x)dx\\ &+ \varepsilon_3\partial_t\int_0^l \rho\partial_t w.(k(x)w_x - d(x)w)dx.\end{aligned}$$

Using the Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned}&|\partial_t\phi(Y(t))|\\ &\le \int_0^l(-b(x) - \varepsilon_1\frac{I_\rho}{2}h_x + \varepsilon_1 I_\rho c(x) + \varepsilon_1\frac{a_1b^2}{2} + \varepsilon_1\frac{a_2b^2}{2} + \frac{\varepsilon_1\alpha}{2a_3}\\ &\quad + \frac{\varepsilon_1\beta c^2}{2a_4} + \varepsilon_2(1 + \frac{I_\rho b^2}{2a_6}))(\partial_t\varphi)^2dx\\ &\quad + \int_0^l(-\varepsilon_1\frac{EI}{2}h_x - \varepsilon_1 EIc + \varepsilon_1\frac{(h(x))^2}{2a_1} + K\frac{\varepsilon_3}{2a_5}(k(x))^2)(\varphi_x)^2dx\\ &\quad + \int_0^l(\varepsilon_1\frac{EI}{2}c_{xx} + \varepsilon_1\frac{(c(x))^2}{2a_2})(\varphi)^2dx + \varepsilon_2\int_0^l(-\frac{K}{I_\rho} + \frac{a_6}{2I_\rho})(\varphi - w_x)^2dx\\ &\quad + \varepsilon_3\Big|\int_0^l K(w_x - \varphi).\partial_x(d(x)w)dx\Big| + \Big|\frac{K}{l}\int_0^l(\varepsilon_1 h(x)\varphi_x + \varepsilon_1 c(x)\varphi)dx.\int_0^l\varphi dx\Big|\\ &\quad + \varepsilon_2\Big|\frac{K}{\rho}[\varphi_x.w_x]_0^l\Big| + \varepsilon_1\frac{EI}{2}[h.(\varphi_x)^2]_0^l + \frac{K\varepsilon_3}{2}[k.(w_x)^2]_0^l\\ &\quad + \varepsilon_3\int_0^l(-\frac{K}{2}k_x + K\frac{a_5}{2})(w_x)^2dx\\ &\quad + \int_0^l(-\frac{\varepsilon_3\rho}{2}k_x - \varepsilon_3\rho d(x) + (\frac{\varepsilon_1 a_3\alpha}{2} + \frac{\varepsilon_1 a_4\beta}{2})\rho^2)(\partial_t w)^2dx\,.\end{aligned}$$

Using Cauchy Shwartz's inequality and Young's inequality, we obtain

$$\Big|\int_0^l K(w_x - \varphi).(d(x)w_x)dx\Big| \le \frac{Ka_7}{2}\int_0^l(w_x - \varphi)^2dx + \frac{K}{2a_7}\int_0^l(d(x)w_x)^2dx$$

where $a_7 > 0$,

$$\Big|\int_0^l K(w_x - \varphi).d_x w dx\Big| \le \frac{Ka_7}{2}\int_0^l(w_x - \varphi)^2dx + \frac{K\overline{d}^2c_0}{2a_7}\int_0^l(w_x)^2dx$$

where $\overline{d} = \max(d_x)$,

$$\Big|\frac{K}{l}\int_0^l(\varepsilon_1 h(x)\varphi_x + \varepsilon_1 c(x)\varphi)dx.\int_0^l\varphi dx\Big|$$

$$\leq \frac{K}{l}\frac{l^2\varepsilon_1}{2}\int_0^l (h(x)\varphi_x)^2dx + \int_0^l (\frac{Kl^2\varepsilon_1(2+c^2(x))}{2})\varphi^2 dx.$$

Then

$$\begin{aligned}
|\partial_t\phi(Y(t))| \leq& \int_0^l (-b(x) - \varepsilon_1\frac{I_\rho}{2}h_x + \varepsilon_1 I_\rho c(x) + \varepsilon_1\frac{a_1b^2}{2} + \varepsilon_1\frac{a_2b^2}{2} + \frac{\varepsilon_1\alpha}{2a_3} \\
&+ \frac{\varepsilon_1\beta c^2}{2a_4} + \varepsilon_2(1+\frac{I_\rho b^2}{2a_6}))(\partial_t\varphi)^2dx \\
&+ \int_0^l (-\varepsilon_1\frac{EI}{2}h_x - \varepsilon_1 EIc + \varepsilon_1\frac{(h(x))^2}{2a_1} + K\frac{\varepsilon_3}{2a_5}(k(x))^2 \\
&+ K\frac{l^2\varepsilon_1}{2l}(h(x))^2 + c_0(\frac{Kl^2\varepsilon_1(2+\overline{c}^2)}{2}))(\varphi_x)^2dx \\
&+ \int_0^l (\varepsilon_1\frac{EI}{2}c_{xx} + \varepsilon_1\frac{(c(x))^2}{2a_2})(\varphi)^2dx \\
&+ \int_0^l (-\frac{\varepsilon_2K}{I_\rho} + \frac{\varepsilon_2a_6}{2I_\rho} + \varepsilon_3Ka_7)(\varphi - w_x)^2dx \\
&+ (\varepsilon_2\frac{K}{2\rho} + \varepsilon_1\frac{EI}{2}h(l))(\varphi_x(l))^2 + (\varepsilon_2\frac{K}{2\rho} - \varepsilon_1\frac{EI}{2}h(0))(\varphi_x(0))^2 \\
&+ (\varepsilon_2\frac{K}{2\rho} + \frac{K\varepsilon_3}{2}k(l))(w_x(l))^2 + (\varepsilon_2\frac{K}{2\rho} - \frac{K\varepsilon_3}{2}k(0))(w_x(0))^2 \\
&+ \varepsilon_3\int_0^l (-\frac{K}{2}k_x + K\frac{a_5}{2} + \frac{K\overline{d}^2c_0}{2a_7} + \frac{Kd^2}{2a_7})(w_x)^2dx \\
&+ \int_0^l (-\frac{\varepsilon_3\rho}{2}k_x - \varepsilon_3\rho d(x) + (\frac{\varepsilon_1a_3\alpha}{2} + \frac{\varepsilon_1a_4\beta}{2})\rho^2)(\partial_tw)^2dx,
\end{aligned}$$

where c_0 is the Poincarré constant and $\overline{c} = \max(c(x))$ on $[0,l]$, then we choose ε_i ($i = 1,2,3$) sufficiently small and we choose the constant a_i ($i = 1\dots 7$) such that

$$-b(x) + \varepsilon_1\frac{a_1b^2}{2} + \varepsilon_1\frac{a_2b^2}{2} + \varepsilon_2\frac{I_\rho b^2}{2a_6} < 0,$$
$$-\varepsilon_1\frac{I_\rho}{2}h_x + \varepsilon_1I_\rho c(x) + \frac{\varepsilon_1\alpha}{2a_3} + \frac{\varepsilon_1\beta c^2}{2a_4} + \varepsilon_2 < 0,$$
$$-\varepsilon_1\frac{EI}{2}h_x(x) - \varepsilon_1EIc(x) + \varepsilon_1\frac{(h(x))^2}{2a_1} + K\frac{\varepsilon_3}{2a_5}(k(x))^2 + K\frac{l^2\varepsilon_1}{2l}(h(x))^2 < 0,$$
$$-\frac{\varepsilon_2K}{I_\rho} + \frac{\varepsilon_2a_6}{2I_\rho} + \frac{\varepsilon_3Ka_7}{2} < 0,$$
$$-\frac{\varepsilon_3\rho}{2}k_x - \varepsilon_3\rho d(x) + (\frac{\varepsilon_1a_3\alpha}{2} + \frac{\varepsilon_1a_4\beta}{2})\rho^2 < 0,$$
$$\frac{EI}{2}c_{xx} + \frac{(c(x))^2}{2a_2} < 0,$$
$$-\frac{K}{2}k_x + K\frac{a_5}{2} + \frac{K\overline{d}^2c_0}{2a_7} + \frac{Kd^2}{2a_7} < 0,$$
$$\varepsilon_2 < \min(-\varepsilon_1EI\frac{\rho}{K}h(l), \varepsilon_1EI\frac{\rho}{K}h(0), -\rho\varepsilon_3k(l), \rho\varepsilon_3k(0))$$

Then $\phi(Y(t))$ satisfies (3.2). The inequality (3.1) follows from the Cauchy Schwartz inequality. Thus the system is exponentially stable. $\square$

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In response to a message from a reader, the authors want to post links to the concerns by Prof. Fatiha Alabau-Boussouira and to some corrections. See https://arxiv.org/pdf/2308.01611 and https://arxiv.org/pdf/2308.01625

Abdelaziz Soufyane
Math & Comp Sciences department, UAE University, PO Box 17551 Al Ain, UAE
Email address: Soufyane.Abdelaziz@uaeu.ac.ae

Ali Wehbe
Universite Libanaise Faculte des Sciences, Section 1, Hadath, Beyrouth, Lebanon
Email address: ali_wehbe@yahoo.fr